TOKYO J. MATH.
VOL. 13, NO. 1, 1990

Correspondences for Hecke Rings and (Co-)Homology Groups on Smooth Compactifications of Siegel Modular Varieties

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(Communicated by K. Katase)

Dedicated to Professor Tosihusa Kimura on his 60th birthday

Abstract. We show that the Hecke rings act on the l-adic cohomology groups of suitable non-singular projective toroidal compactifications of the higher dimensional modular varieties. We extend the fixed point theory of Lefschetz to the correspondences for the Hecke rings on those compactifications. We treat here the Siegel modular case.

Introduction and notations.

Let $g\geqq 1$, $w\geqq 0$, $j\geqq 1$, $k\geqq 1$, and $N\geqq 3$ be rational integers. Let $\mathscr{R}$ denote a ring. Write
$M_{j,k}(\mathscr{R})=$the set of $j\times k$ matrices with coefficients in $\mathscr{R}$; $\mathscr{R}^j=M_{1,j}(\mathscr{R})$;
$1_k=$the $k\times k$ unit matrix $\in M_{k,k}(\boldsymbol{Z})$; $\langle g\rangle=\dfrac{g(g+1)}{2}$; $J_g=\begin{bmatrix}0 & -1_g\\ 1_g & 0\end{bmatrix}\in M_{2g,2g}(\boldsymbol{Z})$;
$\mathfrak{H}_g=$the Siegel upper half plane of degree g
$=\{Z\in M_{g,g}(\boldsymbol{C})\mid Z={}^tZ.\ \ \operatorname{Im} Z$ is positive definite.$\}$;
$\mathrm{Sp}(g, \boldsymbol{Z})=$the full symplectic modular group $\subset M_{2g,2g}(\boldsymbol{Z})$;
$\Gamma_g(N)=\Gamma(N)=$the principal congruence subgroup of $\mathrm{Sp}(g, \boldsymbol{Z})$ of level N;
$\mathrm{GSp}^+(g, \boldsymbol{R})=\{\gamma\in \mathrm{GL}(2g, \boldsymbol{R})\mid {}^t\gamma J_g\gamma J_g^{-1}$ is a scalar matrix whose eigenvalue is positive.$\}$;
$r(\alpha)=$the eigenvalue of ${}^t\alpha J_g\alpha J_g^{-1}$ for $\alpha\in \mathrm{GSp}^+(g, \boldsymbol{R})$;
$\mathrm{GSp}^+(g, \boldsymbol{Q})=\mathrm{GSp}^+(g, \boldsymbol{R})\cap M_{2g,2g}(\boldsymbol{Q})$; $\mathrm{GSp}^+(g, \boldsymbol{Z})=\mathrm{GSp}^+(g, \boldsymbol{R})\cap M_{2g,2g}(\boldsymbol{Z})$;
$\mathrm{GSp}^+(g, \boldsymbol{R})\ltimes \boldsymbol{R}^{2gw}=$the semi-direct product of $\mathrm{GSp}^+(g, \boldsymbol{R})$ and $\boldsymbol{R}^{2gw}$ with $\boldsymbol{R}^{2gw}$ normal such that

$$(\alpha, \boldsymbol{m})\cdot(\beta, \boldsymbol{n})=\left(\alpha\cdot\beta,\ r(\beta)^{-1}\boldsymbol{m}\begin{bmatrix}\beta & & 0\\ & \beta\cdot\ \cdot & \\ 0 & & \cdot\ \beta\end{bmatrix}+\boldsymbol{n}\right)$$

Received April 15, 1989.

for all $\boldsymbol{m}$ and $\boldsymbol{n}\in \boldsymbol{R}^{2gw}$ and all α and $\beta\in \mathrm{GSp}^+(g, \boldsymbol{R})$. (In the right side the products are those for matrices.) We let $\mathrm{GSp}^+(g, \boldsymbol{R})\ltimes \boldsymbol{R}^{2gw}$ act on the complex analytic space $\mathfrak{H}_g\times \boldsymbol{C}^{gw}=\{(Z, \xi_1, \xi_2, \cdots, \xi_w)\mid Z\in\mathfrak{H}_g,\ \xi_j\in \boldsymbol{C}^g$ for any $j\in[1, w]\}$ to the left as follows. Write $\boldsymbol{m}=(\boldsymbol{m}_1, \boldsymbol{n}_1, \boldsymbol{m}_2, \boldsymbol{n}_2, \cdots, \boldsymbol{m}_w, \boldsymbol{n}_w)\in \boldsymbol{R}^{2gw}$ with $\boldsymbol{m}_j\in \boldsymbol{R}^g$ and $\boldsymbol{n}_j\in \boldsymbol{R}^g$ for any $j\in[1, w]$, and write $\alpha=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in \mathrm{GSp}^+(g, \boldsymbol{R})$ partitioned into blocks on dimension $g\times g$. Then

$$\left(\begin{pmatrix} A & B\\ C & D\end{pmatrix}, (\boldsymbol{m}_1, \boldsymbol{n}_1, \boldsymbol{m}_2, \boldsymbol{n}_2, \cdots, \boldsymbol{m}_w, \boldsymbol{n}_w)\right)(Z, \xi_1, \xi_2, \cdots, \xi_w)$$
$$=\left((AZ+B)(CZ+D)^{-1},\ r\left(\begin{pmatrix} A & B\\ C & D\end{pmatrix}\right)\left(\xi_1+(\boldsymbol{m}_1, \boldsymbol{n}_1)\begin{pmatrix} Z\\ 1_g\end{pmatrix}\right)(CZ+D)^{-1},\right.$$
$$r\left(\begin{pmatrix} A & B\\ C & D\end{pmatrix}\right)\left(\xi_2+(\boldsymbol{m}_2, \boldsymbol{n}_2)\begin{pmatrix} Z\\ 1_g\end{pmatrix}\right)(CZ+D)^{-1},\ \cdots,$$
$$\left. r\left(\begin{pmatrix} A & B\\ C & D\end{pmatrix}\right)\left(\xi_w+(\boldsymbol{m}_w, \boldsymbol{n}_w)\begin{pmatrix} Z\\ 1_g\end{pmatrix}\right)(CZ+D)^{-1}\right).$$

It is well known that any congruence subgroup Γ' of $\mathrm{Sp}(g, \boldsymbol{Z})$ is commensurable with $\alpha\Gamma'\alpha^{-1}$ for any $\alpha\in \mathrm{GSp}^+(g, \boldsymbol{Q})$. In this paper we assume $\Gamma=\Gamma_g(N)$ with $N\geqq 3$ for simplicity though our results hold good for any congruence subgroup of $\mathrm{Sp}(g, \boldsymbol{Z})$ acting on $\mathfrak{H}_g$ with no fixed points. Write $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))=$ the Hecke ring over $\boldsymbol{Z}$ with respect to the group Γ and the monoid $\mathrm{GSp}^+(g, \boldsymbol{Z})$, which contains Γ. (For the definition of the Hecke ring, see e.g. Shimura [18].) Let A_Γ be the universal principally polarized abelian variety with level N structure over the complex analytic quotient space $\Gamma\backslash\mathfrak{H}_g$. Let A_Γ^w be the w-fold fibred product of A_Γ over the $\Gamma\backslash\mathfrak{H}_g$, and let $E'\colon A_\Gamma^w\to\Gamma\backslash\mathfrak{H}_g$ be the canonical morphism. We may regard the complex analytic quotient space $(\Gamma\ltimes \boldsymbol{Z}^{2gw}\backslash\mathfrak{H}_g\times \boldsymbol{C}^{gw})$ as this A_Γ^w. By the theory of the toroidal compactification (cf. Ash et al. [2], Chai [4], Namikawa [14], [15]), the $(\Gamma\ltimes \boldsymbol{Z}^{2gw}\backslash\mathfrak{H}_g\times \boldsymbol{C}^{gw})$ has a non-singular projective toroidal compactification. It is not necessarily unique. Write A_Γ^0 for the $\Gamma\backslash\mathfrak{H}_g$. We consider simultaneous non-singular projective toroidal compactifications of A_Γ^w and A_Γ^0. Take a regular and projective $\mathrm{Sp}(g, \boldsymbol{Z})$-admissible family $\Sigma^{(0)}=\{\Sigma_b^{(0)}\}_{F_b:\text{ rational components}}$ of polyhedral cone decompositions for the toroidal compactification of A_Γ^0. e.g. Take a suitable refinement of the second Voronoi decomposition. Then take its mixed cone decomposition (cf. Namikawa [14] and [15]). By this we have a regular and projective $\mathrm{Sp}(g, \boldsymbol{Z})\ltimes \boldsymbol{Z}^{2gw}$-admissible family $\Sigma^{(1)}=\{\Sigma_a^{(1)}\}_{F_a:\text{ rational components}}$ of polyhedral cone decompositions for the toroidal compactification of A_Γ^w.

Then we get desired simultaneous compactifications $(A_\Gamma)^\sim$ and $(A_\Gamma^0)^\sim$ and a proper canonical morphism $E: (A_\Gamma)^\sim \to (A_\Gamma^0)^\sim$. For simplicity write $M = (A_\Gamma)^\sim$ and ${}_0M = (A_\Gamma^0)^\sim$ from now on throughout this paper. Let l be a prime number. Write

$Z_0(M, \boldsymbol{Z})$ = the group of 0-cycles on M with $\boldsymbol{Z}$-coefficients;

$H_n(M, \boldsymbol{Z})$ = the singular n-th homology group of M with $\boldsymbol{Z}$-coefficients;

$H^n(M, \boldsymbol{Q}_l)$ = the n-th l-adic cohomology group of M (cf. Artin et al. [1] and Deligne [5]);

$H^n({}_0M, (R^m E_* \boldsymbol{Q}_l)^{\otimes j})$ = the n-th l-adic cohomology group of ${}_0M$ with coefficients in the j-fold tensor product of the m-th l-adic direct image sheaf $R^m E_* \boldsymbol{Q}_l$ (cf. Artin et al. [1] and Deligne [5]).

To explain our results in this paper we need

THEOREM A (Hatada [8, Theorem 1]). *The Hecke ring* $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ *acts on* $H_n(M, \boldsymbol{Z})$, *i.e., there is a ring homomorphism* $f_n: \mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \to \mathrm{End}_{\boldsymbol{Z}}(H_n(M, \boldsymbol{Z}))$ *for each integer* $n \in [0, g(g+1)+2gw]$.

Write $\mathscr{F} = \{\Phi \ltimes L \mid \Phi$ is a subgroup of $\mathrm{GSp}^+(g, \boldsymbol{Q})$ which is commensurable with $\mathrm{Sp}(g, \boldsymbol{Z})$. Φ acts on $\mathfrak{H}_g$ with no fixed points. L is an additive subgroup of $\boldsymbol{Q}^{2gw}$ with $(d_1\boldsymbol{Z})^{2gw} \subset L \subset (d_2^{-1}\boldsymbol{Z})^{2gw}$ for some positive d_1 and $d_2 \in \boldsymbol{Z}$ which may depend on L. $\Phi \ltimes L$ is a subgroup of $\mathrm{GSp}^+(g, \boldsymbol{Q}) \ltimes \boldsymbol{Q}^{2gw}$.$\}$. For simplicity write $\mathfrak{D} = \mathfrak{H}_g \times \boldsymbol{C}^{gw}$. For any group $G \in \mathscr{F}$, we denote by $(G\backslash\mathfrak{D})^\sim$ the toroidal compactification with respect to the above $\Sigma^{(1)}$ in this paper. Standard facts about the toroidal compactification yield the following two propositions.

PROPOSITION B (Hatada [8, Proposition 1.1]). *Assume* $G_1 \subset G_2$ *where* G_1 *and* $G_2 \in \mathscr{F}$. *Let* φ *denote the canonical holomorphic map*: $G_1\backslash\mathfrak{D} \to G_2\backslash\mathfrak{D}$. *Then there exists a unique finite surjective holomorphic map* $\varphi^\sim: (G_1\backslash\mathfrak{D})^\sim \to (G_2\backslash\mathfrak{D})^\sim$ *whose restriction to* $G_1\backslash\mathfrak{D}$ *is* φ.

PROPOSITION C (Hatada [8, Proposition 1.2]). *Let* $G \in \mathscr{F}$ *and let* $a \in \mathrm{GSp}^+(g, \boldsymbol{Q}) \ltimes \boldsymbol{Q}^{2gw}$. *Let* $\bar{a}$ *denote the biholomorphic map*: $G\backslash\mathfrak{D} \to aGa^{-1}\backslash\mathfrak{D}$ *given by* $GP \mapsto aGa^{-1}(aP)$ $(P \in \mathfrak{D})$. *Then there exists a unique biholomorphic map* $a^\sim: (G\backslash\mathfrak{D})^\sim \to (aGa^{-1}\backslash\mathfrak{D})^\sim$ *whose restriction to* $G\backslash\mathfrak{D}$ *is* $\bar{a}$. *(Note that* $aGa^{-1} \in \mathscr{F}$ *if* $G \in \mathscr{F}$.*)*

Let $\alpha \in \mathrm{GSp}^+(g, \boldsymbol{Z})$ and write $c = r(\alpha)$. We have $\Gamma\alpha\Gamma = \cup_{i=1}^{\mu} \Gamma\alpha_i$ (disjoint) where $\mu = (\Gamma: \Gamma \cap \alpha^{-1}\Gamma\alpha)$. Note that $(\alpha_i, 0) \in \mathrm{GSp}^+(g, \boldsymbol{Q}) \ltimes \boldsymbol{Q}^{2gw}$. Look at the following commutative diagram.

(0.1)

$$\begin{array}{ccc}
(\Gamma_g(c^2N)\ltimes(cZ)^{2gw}\backslash\mathfrak{D})^{\sim} & & \\
\downarrow \pi_i & & \\
((\alpha_i,0)^{-1}(\Gamma_g(cN)\ltimes(cZ)^{2gw})(\alpha_i,0)\backslash\mathfrak{D})^{\sim} & \xrightarrow{(\alpha_i,0)^{\sim}} & (\Gamma_g(cN)\ltimes(cZ)^{2gw}\backslash\mathfrak{D})^{\sim} \\
\downarrow \pi^{(i)} & & \downarrow [\pi] \\
(\Gamma\ltimes Z^{2gw}\backslash\mathfrak{D})^{\sim}=M & & (\Gamma\ltimes Z^{2gw}\backslash\mathfrak{D})^{\sim}=M
\end{array}$$

Here $\Gamma=\Gamma_g(N)$, $\pi=\pi^{(i)}\circ\pi_i$, and the vertical lines denote the canonical maps given by Proposition B. For each closed point $P\in M$, there exists a closed point $P'\in(\Gamma_g(c^2N)\ltimes(cZ)^{2gw}\backslash\mathfrak{D})^{\sim}$ such that $\pi(P')=P$. We see that the points (0-cycle) $\sum_{i=1}^{\mu}[\pi]\circ(\alpha_i,0)^{\sim}\circ\pi_i(P')$ are determined only by P and do not depend on the choice of P' with $\pi(P')=P$ (cf. Hatada [8, §2]). Write $S(\Gamma\alpha\Gamma)_P=\bigcup_{i=1}^{\mu}[\pi]\circ(\alpha_i,0)^{\sim}\circ\pi_i(P')$. We define $\mathscr{X}_M(\Gamma\alpha\Gamma)$ to be the subset $\bigcup_{P\in M}\{P\}\times S(\Gamma\alpha\Gamma)_P$ in the product variety $M\times M$. We see that $\mathscr{X}_M(\Gamma\alpha\Gamma)$ is a locally analytic subvariety of the projective variety $M\times M$ and that it is a $\langle g\rangle+gw$ dimensional subscheme of $M\times M$ over C.

For an element $\mathscr{A}_1=\sum_j m_j\Gamma\alpha_j\Gamma\neq 0$ of $\mathrm{HR}(\Gamma,\mathrm{GSp}^+(g,Z))$ where each $m_j\neq 0\in Z$ and $\Gamma\alpha_j\Gamma\neq\Gamma\alpha_{j'}\Gamma$ if $j\neq j'$, define $\mathscr{X}_M(\mathscr{A}_1)$ to be $\bigcup_j\mathscr{X}_M(\Gamma\alpha_j\Gamma)$. Let $f_n(\mathscr{A}_1)\otimes_Z\mathrm{id.}$ be the element of $\mathrm{End}_Q(H_n(M,Q))$ under the isomorphism $H_n(M,Z)\otimes_Z Q\cong H_n(M,Q)$. Write $d=\langle g\rangle+gw=\dim_C M$. Let $D_n\colon H^{2d-n}(M,Q)\to H_n(M,Q)$ denote the Q-linear isomorphism by the Poincaré duality for each $n\in[0,2d]$. Let $\langle\ ,\ \rangle_n$ denote the Kronecker index: $H^n(M,Q)\times H_n(M,Q)\to Q$, which is non-degenerate, for each $n\in[0,2d]$. Let ${}^t(f_n(\mathscr{A}_1)\otimes_Z\mathrm{id.})$ denote the transposed map $\in\mathrm{End}_Q(H^n(M,Q))$ with respect to this $\langle\ ,\ \rangle_n$. Define *Coincidence Number* $L_M(\mathscr{A}_1,\mathscr{A}_2)$ on M for elements $\mathscr{A}_1$ and $\mathscr{A}_2\in\mathrm{HR}(\Gamma,\mathrm{GSp}^+(g,Z))$ by

$$L_M(\mathscr{A}_1,\mathscr{A}_2)=\sum_{n=0}^{2d}(-1)^n\mathrm{Tr}\{D_n\circ{}^t(f_{2d-n}(\mathscr{A}_2)\otimes_Z\mathrm{id.})\circ D_n^{-1}\circ(f_n(\mathscr{A}_1)\otimes_Z\mathrm{id.})\}\ .$$

Our main results in §1 - §3 are the following theorems 1, 2 and 3.

THEOREM 1. *There exists a certain natural Z-bilinear map F_n: $\mathrm{HR}(\Gamma,\mathrm{GSp}^+(g,Z))\times\mathrm{HR}(\Gamma,\mathrm{GSp}^+(g,Z))\to\mathrm{Hom}_Z(H_n(M,Z),H_n(M\times M,Z))$ for each integer $n\geqq 0$.* (For details see §1, Theorem 1.5 and Lemma 1.6.)

Write $\Delta(M)=\{(x,y)\in M\times M\mid x=y\}$. Let $\iota\colon(M\times M,\varnothing)\to(M\times M, M\times M-\Delta(M))$ be the inclusion map for pairs of topological spaces. Let ι_{*n} denote the induced Z-linear map: $H_n(M\times M,Z)\to H_n(M\times M, M\times M-\Delta(M),Z)$ for each $n\geqq 0$. Since M is a complex projective manifold, it is orientable. Let s be an orientation of M. Let z denote the fundamental class of M attached to s, and let U denote the Thom class of M attached to s. (Hence $z\in H_{2d}(M, \boldsymbol{Z})$; $U\in H^{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Z})$.) One has the isomorphism: $H_{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Z})\cong\boldsymbol{Z}$ given by $\zeta\mapsto\langle U, \zeta\rangle_{2d}$ under the Kronecker index. Define *Coincidence Index* $I_M(\mathscr{A}_1, \mathscr{A}_2)$ on M for elements $\mathscr{A}_1$ and $\mathscr{A}_2\in\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ to be the integer corresponding to the class

$$\iota_{*2d}\circ(F_{2d}((\mathscr{A}_1, \mathscr{A}_2))(z)\in H_{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Z})$$

under the above isomorphism. In §1 we also define *Coincidence Class* $\bar{\varepsilon}_M(\mathscr{A}_1, \mathscr{A}_2)\in H^{2d}(M, \boldsymbol{Q})$ for elements $\mathscr{A}_1$ and $\mathscr{A}_2\in\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$.

THEOREM 2. *$L_M(\mathscr{A}_1, \mathscr{A}_2)=I_M(\mathscr{A}_1, \mathscr{A}_2)$ for all $\mathscr{A}_1$ and $\mathscr{A}_2\in\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$.* (We call this Coincidence Theorem.)

THEOREM 3. *Let $\mathscr{A}_1$ and $\mathscr{A}_2\in\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. If $L_M(\mathscr{A}_1, \mathscr{A}_2)\neq 0$, then $\mathscr{X}_M(\mathscr{A}_1)\cap\mathscr{X}_M(\mathscr{A}_2)\neq\emptyset$.* (We call this Fixed Point Theorem.)

Let l be a prime number. In §4 and §5 we show the following two theorems.

THEOREM 4. *The Hecke ring $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l$ over $\boldsymbol{Q}_l$ acts on $H^n(M, \boldsymbol{Q}_l)$ as an anti-ring homomorphism, i.e., there exists a natural anti-ring homomorphism $f^{(n)}$: $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l\to\mathrm{End}_{\boldsymbol{Q}_l}H^n(M, \boldsymbol{Q}_l)$ for each integer $n\geqq 0$.*

THEOREM 5. *The Hecke ring $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l$ over $\boldsymbol{Q}_l$ acts on $H^n({}_0M, (R^mE_*\boldsymbol{Q}_l)^{\otimes j})$ as an anti-ring homomorphism, i.e., there exists a natural anti-ring homomorphism $\mathfrak{f}^{(n,m,j)}$: $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l\to\mathrm{End}_{\boldsymbol{Q}_l}H^n({}_0M, (R^mE_*\boldsymbol{Q}_l)^{\otimes j})$ for each integers $n\geqq 0$, $m\geqq 0$ and $j\geqq 1$.*

This action is compatible with the action in Theorem A through the comparison theorem of l-adic cohomology: $H^n(M, \boldsymbol{C})\cong H^n(M, \boldsymbol{Q}_l)\otimes_{\boldsymbol{Q}_l}\boldsymbol{C}$ and the Kronecker index.

The main results of the present paper were announced in Hatada [9].

The previous title of the present paper was "Correspondences for Hecke rings and (co-)homology groups on Siegel modular varieties".

§1. Some homomorphisms and coincidence index.

Recall $\Gamma=\Gamma_g(N)$ with $N\geqq 3$, $M=(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^\sim$ and $\mathfrak{D}=\mathfrak{H}_g\times\boldsymbol{C}^{gw}$. Let Δ: $M\to M\times M$ denote the diagonal map defined by $\Delta(x)=(x, x)$ for any $x\in M$. Then $\Delta(M)=\mathrm{Image}\,\Delta$.

Now let α and β be elements of $\mathrm{GSp}^+(g, \boldsymbol{Z})$. Write $r(\alpha)=c$ and $r(\beta)=b$. Let $\Gamma\alpha\Gamma=\bigcup_{i=1}^{\mu}\Gamma\alpha_i$ (disjoint) and let $\Gamma\beta\Gamma=\bigcup_{x=1}^{\mu'}\Gamma\beta_x$ (disjoint). We define holomorphic maps $K(\alpha_i)$ $(1\leqq i\leqq\mu)$ and $K(\beta_x)$ $(1\leqq x\leqq\mu')$ by the compositions of the maps in the following commutative diagrams. Use the relations of $(\alpha_i, 0)^{-1}(\Gamma_g(bcN)\ltimes(bc\boldsymbol{Z})^{2gw})(\alpha_i, 0)\supset\Gamma_g(bc^2N)\ltimes(bc\boldsymbol{Z})^{2gw}$ and $(\beta_x, 0)^{-1}(\Gamma_g(bcN)\ltimes(bc\boldsymbol{Z})^{2gw})(\beta_x, 0)\supset\Gamma_g(b^2cN)\ltimes(bc\boldsymbol{Z})^{2gw}$ (cf. Hatada [8, Lemma 2.1]).

(1.1.1)

$$\begin{array}{ccc}
(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim} & & \\
\downarrow & & \\
(\Gamma_g(bc^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim} & & \\
\downarrow & & \\
((\alpha_i, 0)^{-1}(\Gamma_g(bcN)\ltimes(bc\boldsymbol{Z})^{2gw})(\alpha_i, 0)\backslash\mathfrak{D})^{\sim} & \xrightarrow{(\alpha_i, 0)^{\sim}} & (\Gamma_g(bcN)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim} \\
\downarrow & & \downarrow \\
M & & M
\end{array}$$

$$\Pi:(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim}\to M,\qquad K(\alpha_i):(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim}\to M$$

(1.1.2)

$$\begin{array}{ccc}
(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim} & & \\
\downarrow & & \\
(\Gamma_g(b^2cN)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim} & & \\
\downarrow & & \\
((\beta_x, 0)^{-1}(\Gamma_g(bcN)\ltimes(bc\boldsymbol{Z})^{2gw})(\beta_x, 0)\backslash\mathfrak{D})^{\sim} & \xrightarrow{(\beta_x, 0)^{\sim}} & (\Gamma_g(bcN)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim} \\
\downarrow & & \downarrow \\
M & & M
\end{array}$$

$$\Pi:(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim}\to M,\qquad K(\beta_x):(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim}\to M$$

In (1.1.1) and (1.1.2) the vertical lines denote the canonical holomorphic maps given in Proposition B.

LEMMA 1.1. (i) *For each (closed) point* $P\in M$ *there exists a (closed) point* $P'\in(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim}$ *such that* $\Pi(P')=P$.

(ii) *The 0-cycle* $\sum_{i=1}^{\mu}K(\alpha_i)(P')$ *(resp.* $\sum_{x=1}^{\mu'}K(\beta_x)(P')$*) on* M *is determined by* $P\in M$ *and the double coset* $\Gamma\alpha\Gamma$ *(resp.* $\Gamma\beta\Gamma$*), and does neither depend on the choice of* $P'\in(\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim}$ *with* $\Pi(P')=P$ *nor on representatives* $\{\alpha_i\}_{i=1}^{\mu}$ *(resp.* $\{\beta_x\}_{x=1}^{\mu'}$*) for* $\Gamma\backslash\Gamma\alpha\Gamma$ *(resp.* $\Gamma\backslash\Gamma\beta\Gamma$*).*

The proof of Lemma 1.1 goes in the same way as in Hatada [8, §2, (DEF. 1)]. We may leave it to the reader.

By Lemma 1.1 we may write

$$(\Gamma\alpha\Gamma)(P)=\bigcup_{i=1}^{\mu}K(\alpha_i)(P')\ ; \qquad (\Gamma\beta\Gamma)(P)=\bigcup_{x=1}^{\mu'}K(\beta_x)(P')$$

for $P\in M$ with $\Pi(P')=P$. Note that $\mathscr{X}_M(\Gamma\alpha\Gamma)=\cup_{P\in M}\{P\}\times(\Gamma\alpha\Gamma)(P)$ in $M\times M$ and that $\mathscr{X}_M(\Gamma\beta\Gamma)=\cup_{P\in M}\{P\}\times(\Gamma\beta\Gamma)(P)$ in $M\times M$. (Here $\mathscr{X}_M(\Gamma\alpha\Gamma)$ and $\mathscr{X}_M(\Gamma\beta\Gamma)$ denote the subschemes defined in the introduction.)

Now let γ_1 and $\gamma_2\in\Gamma$, let $i\in[1, \mu]$, and let $x\in[1, \mu']$. We denote by $(K(\gamma_1\alpha_i), K(\gamma_2\beta_x))$ the holomorphic map defined by the following commutative diagram.

(1.2.1)

$$\begin{array}{ccc} & \xrightarrow{K(\gamma_1\alpha_i)} & M \\ & & \uparrow \text{proj}_1 \\ (\Gamma_g(b^2c^2N)\ltimes(bc\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^{\sim} & \xrightarrow{(K(\gamma_1\alpha_i),\ K(\gamma_2\beta_x))} & M\times M \\ & & \downarrow \text{proj}_2 \\ & \xrightarrow{K(\gamma_2\beta_x)} & M \end{array}$$

We obtain $K(\gamma_1\alpha_i)=K(\alpha_i)$ and $K(\gamma_2\beta_x)=K(\beta_x)$ since $(\Gamma_g(bcN)\ltimes(bc\boldsymbol{Z})^{2gw})\triangleleft(\Gamma\ltimes\boldsymbol{Z}^{2gw})$ and $(\gamma_v, 0)^{\sim}$: $M=(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^{\sim}\to M$ is the identity map for $v=1$ and 2.

Recall the following theorem and proposition.

THEOREM 1.2 (Łojasiewicz [12]). *Let X be a compact complex manifold and let A be an analytic subset of X. Then there exists a finite analytic (hence C^1-) triangulation of $|X|$ in which A appears as the support of a subcomplex.* (For details see [12, pp. 463-464].)

PROPOSITION D (Hatada [8, Proposition 1.3]). *Let $\mathscr{F}$ denote the set defined in the introduction. Let G_1 and $G_2\in\mathscr{F}$. Assume*

$$G_1\triangleleft G_2\subset \mathrm{Sp}(g, \boldsymbol{Z})\ltimes\boldsymbol{Z}^{2gw}\ .$$

Then one obtains:

(i) *The canonical morphism $\varphi^{\sim}$: $(G_1\backslash\mathfrak{D})^{\sim}\to(G_2\backslash\mathfrak{D})^{\sim}$, given by Proposition* B, *is a Galois covering with the Galois group G_2/G_1.*

(ii) *Let P_1 and P_2 be (closed) points in $(G_1\backslash\mathfrak{D})^{\sim}$. Then the following are equivalent.*

$$\varphi^{\sim}(P_1)=\varphi^{\sim}(P_2) \iff h^{\sim}(P_1)=P_2 \text{ for some } h\in G_2\ .$$

Here $h^{\sim}$ *denotes the biholomorphic map*: $(G_1\backslash\mathfrak{D})^{\sim}\to(G_1\backslash\mathfrak{D})^{\sim}$ *given by Proposition* C.

Setting $X=M$ and $A=$the branch locus for the Π in (1.1.1) and (1.1.2) we apply Theorem 1.2. Note $A\subset M$. Then we have a C^1-triangulation $t: |\mathfrak{K}|\to M$ enjoying the condition in Theorem 1.2, where $\mathfrak{K}$ denotes a finite ordered simplicial complex. Let $\mathfrak{z}$ (resp. σ) be an ordered n-cycle (resp. n-chain) with Z-coefficients of $\mathfrak{K}$. We may write uniquely

$$\mathfrak{z}\ (\text{resp. } \sigma)=\sum_{\substack{\text{a finite sum with}\\ \text{respect to } v}} l^{(v)}\langle P_1^{(v)}, P_2^{(v)}, \cdots, P_{n+1}^{(v)}\rangle$$

where $l^{(v)}\in Z$ and $P_j^{(v)}$ is a vertex of $\mathfrak{K}$. Let $\{e_1, e_2, \cdots, e_{n+1}\}$ be the standard basis of the $(n+1)$-dimensional Euclidean space R^{n+1}. Write Δ^n for the n-simplex with the vertices $e_1, e_2, \cdots, e_n$ and e_{n+1}. Now let $s^{(v)}$ denote the map: $|\Delta^n|\to|\langle P_1^{(v)}, P_2^{(v)}, \cdots, P_{n+1}^{(v)}\rangle|$ given by $e_j\mapsto P_j^{(v)}$ $(j=1, 2, \cdots, n+1)$ and extending this linearly to $|\Delta^n|$ for each v. For the image of $\mathfrak{z}$ (resp. σ) under these maps, we have a singular n-cycle (resp. n-chain) $s(\mathfrak{z})$ (resp. $s(\sigma))\underset{\text{def.}}{=}\sum_v l^{(v)}t\circ s^{(v)}$ with Z-coefficients on M. (It is well known that this correspondence induces an isomorphism: $H_n(\mathfrak{K}, Z)\to H_n(M, Z)$.) By our choice of a C^1-triangulation $t: |\mathfrak{K}|\to M$, if necessary, by taking a suitable subdivision of $\mathfrak{K}$, for the $\mathfrak{z}$ there is a singular n-chain V with Z-coefficients on $(\Gamma_g(b^2c^2N)\ltimes(bcZ)^{2gw}\backslash\mathfrak{D})^{\sim}$ such that (i) $\Pi(V)=s(\mathfrak{z})=\sum_v l^{(v)}t\circ s^{(v)}$ and (ii) $\partial V\underset{\text{def.}}{=}$the image of V under the boundary homomorphism, is written as $\sum_{u=1}^k A_u-\sum_{u=1}^k B_u$ for some positive $k\in Z$ where $(\gamma_u, m_u)^{\sim}(A_u)=B_u$ with some $\gamma_u\in\Gamma$ and some $m_u\in Z^{2gw}$ for each $u\in[1, k]$. Here $(\gamma_u, m_u)^{\sim}$: $(\Gamma_g(b^2c^2N)\ltimes(bcZ)^{2gw}\backslash\mathfrak{D})^{\sim}\to(\Gamma_g(b^2c^2N)\ltimes(bcZ)^{2gw}\backslash\mathfrak{D})^{\sim}$. As a singular n-chain with Z-coefficients on $M\times M$, we may define $(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)(s(\mathfrak{z}))$ by

$$\text{(DEF. (1.2.2))}\qquad (\Gamma\alpha\Gamma, \Gamma\beta\Gamma)(s(\mathfrak{z}))=\sum_{i=1}^{\mu}\sum_{x=1}^{\mu'}(K(\alpha_i), K(\beta_x))(V)\,.$$

This does neither depend on the choice of representatives $\{\alpha_i\}_{i=1}^{\mu}$ (resp. $\{\beta_x\}_{x=1}^{\mu'}$) for $\Gamma\backslash\Gamma\alpha\Gamma$ (resp. $\Gamma\backslash\Gamma\beta\Gamma$) nor on the choice of V enjoying (i). (Take a suitable subdivision of $\mathfrak{K}$ if necessary.) This proof is similar to the proof of (DEF. 1) in Hatada [8, § 2]. We obtain the following three Lemmas 1.3, 1.4 and 1.6 whose proofs are also similar to those for Lemmas 2.3, 2.4 and 2.5 in Hatada [8] respectively. We may leave their proofs to the reader.

LEMMA 1.3. *This* $(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)(s(\mathfrak{z}))$ *is a singular* n*-cycle with* Z*-coefficients on* $M\times M$.

(For this proof use the facts that $(\alpha_i, \mathbf{0})\cdot(\gamma_u, \boldsymbol{m}_u)\in(\Gamma\ltimes \boldsymbol{Z}^{2gw})(\alpha_{i'}, \mathbf{0})$ with $1\leqq i'\leqq\mu$, that $(\beta_x, \mathbf{0})\cdot(\gamma_u, \boldsymbol{m}_u)\in(\Gamma\ltimes \boldsymbol{Z}^{2gw})(\beta_{x'}, \mathbf{0})$ with $1\leqq x'\leqq\mu'$, and that the map: $(i, x)\mapsto(i', x')$ is a permutation of $[1, \mu]\times[1, \mu']\cap \boldsymbol{Z}\times\boldsymbol{Z}$ for each $(\gamma_u, \boldsymbol{m}_u)$.)

By the construction, this $(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)(s(\mathfrak{z}))$ is a singular n-cycle on $M\times M-\Delta(M)$ if $\mathscr{X}_M(\Gamma\alpha\Gamma)\cap\mathscr{X}_M(\Gamma\beta\Gamma)=\varnothing$.

LEMMA 1.4. *Let σ be an ordered $(n+1)$-chain with $\boldsymbol{Z}$-coefficients of $\mathfrak{K}$, and let $\partial\sigma$ be its boundary. The singular n-cycle $(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)(s(\partial\sigma))$ is a singular n-boundary with $\boldsymbol{Z}$-coefficients.*

From Lemmas 1.3 and 1.4 we obtain

THEOREM 1.5. (i) *The map $s(\mathfrak{z})\mapsto(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)(s(\mathfrak{z}))$ induces a unique $\boldsymbol{Z}$-linear map* $\in \mathrm{Hom}_{\boldsymbol{Z}}(H_n(M, \boldsymbol{Z}), H_n(M\times M, \boldsymbol{Z}))$.

(ii) *If $\mathscr{X}_M(\Gamma\alpha\Gamma)\cap\mathscr{X}_M(\Gamma\beta\Gamma)=\varnothing$, the induced map in this* (i) *is an element of* $\mathrm{Hom}_{\boldsymbol{Z}}(H_n(M, \boldsymbol{Z}), H_n(M\times M-\Delta(M), \boldsymbol{Z}))$.

We write $\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_n$ for the induced map $\in \mathrm{Hom}_{\boldsymbol{Z}}(H_n(M, \boldsymbol{Z}), H_n(M\times M, \boldsymbol{Z}))$ in Theorem 1.5. Here $n\in[0, 2d]$.

LEMMA 1.6. *This map $\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_n$ does not depend on a way in choosing a C^1-triangulation enjoying the conditions in Theorem* 1.2 *with respect to $A=$the branch locus for the Π and $X=M$.*

Let $n\in[0, 2d]$. Let us consider the map: a pair $(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)$ of double cosets $\mapsto\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_n$. We can extend it $\boldsymbol{Z}$-bilinearly to F_n: $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\times\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\to\mathrm{Hom}_{\boldsymbol{Z}}(H_n(M, \boldsymbol{Z}), H_n(M\times M, \boldsymbol{Z}))$ uniquely. Namely $F_n(\sum_j m_j\Gamma\alpha_j\Gamma, \sum_k l_k\Gamma\beta_k\Gamma)=\sum_j\sum_k m_jl_k\{\Gamma\alpha_j\Gamma, \Gamma\beta_k\Gamma\}_n$. Theorem 1 in the introduction is proved.

We obtain

THEOREM 1.7. *If $\mathscr{X}_M(\Gamma\alpha\Gamma)\cap\mathscr{X}_M(\Gamma\beta\Gamma)=\varnothing$ for α and $\beta\in \mathrm{GSp}^+(g, \boldsymbol{Z})$, then the coincidence index $I_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)=0$.*

PROOF. If $\mathscr{X}_M(\Gamma\alpha\Gamma)\cap\mathscr{X}_M(\Gamma\beta\Gamma)=\varnothing$, the map $\iota_{*2d}\circ\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_{2d}$ splits as follows by Theorem 1.5:

$$H_{2d}(M, \boldsymbol{Z})\longrightarrow H_{2d}(M\times M-\Delta(M), \boldsymbol{Z})$$
$$\longrightarrow H_{2d}(M\times M-\Delta(M), M\times M-\Delta(M), \boldsymbol{Z})=\{0\}\,.$$

Theorem 1.7 is proved.

We get readily

COROLLARY 1.8. *Let* $\mathscr{A}_1=\sum_j m_j\Gamma\alpha_j\Gamma$ *and* $\mathscr{A}_2=\sum_k l_k\Gamma\beta_k\Gamma\in \mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. *If* $I_M(\mathscr{A}_1, \mathscr{A}_2)\neq 0$, *then* $\mathscr{X}_M(\mathscr{A}_1)\cap\mathscr{X}_M(\mathscr{A}_2)\neq\emptyset$.

Consider $\iota_{*2d}\circ(F_{2d}(\mathscr{A}_1, \mathscr{A}_2))$: $H_{2d}(M, \boldsymbol{Z})\to H_{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Z})$ for $\mathscr{A}_1$ and $\mathscr{A}_2$ of $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. Tensoring this with $\boldsymbol{Q}$ over $\boldsymbol{Z}$ we have the $\boldsymbol{Q}$-linear map $\iota_{*2d}\circ(F_{2d}(\mathscr{A}_1, \mathscr{A}_2))\otimes_{\boldsymbol{Z}}\mathrm{id.}: H_{2d}(M, \boldsymbol{Q})\to H_{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Q})$ under $H_{2d}(M, \boldsymbol{Z})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}\cong H_{2d}(M, \boldsymbol{Q})$ and $H_{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Z})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}\cong H_{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Q})$. With respect to the complete duality of the Kronecker index, the transposed map ${}^t(\iota_{*2d}\circ(F_{2d}(\mathscr{A}_1, \mathscr{A}_2))\otimes_{\boldsymbol{Z}}\mathrm{id.})$ is a $\boldsymbol{Q}$-linear map from $H^{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Q})$ to $H^{2d}(M, \boldsymbol{Q})$. In the same way we have the $\boldsymbol{Q}$-linear map $\iota_{*2d}\otimes_{\boldsymbol{Z}}\mathrm{id.}: H_{2d}(M\times M, \boldsymbol{Q})\to H_{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Q})$ and its transposed map ${}^t(\iota_{*2d}\otimes_{\boldsymbol{Z}}\mathrm{id.}): H^{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Q})\to H^{2d}(M\times M, \boldsymbol{Q})$ with respect to the Kronecker index. Let U be the Thom class of M, let $\bar{U}$ denote the canonical image of U in $H^{2d}(M\times M, M\times M-\Delta(M), \boldsymbol{Q})$, and let $U^{\sim}={}^t(\iota_{*2d}\otimes_{\boldsymbol{Z}}\mathrm{id.})(\bar{U})$. Let $\bar{z}$ denote the canonical image of the fundamental class z of M in $H_{2d}(M, \boldsymbol{Q})$.

DEFINITION 1.9. Let $\mathscr{A}_1$ and $\mathscr{A}_2\in\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. We define *Coincidence Class* $\bar{\varepsilon}_M(\mathscr{A}_1, \mathscr{A}_2)$ on M of $\mathscr{A}_1$ and $\mathscr{A}_2$ to be

$${}^t(\iota_{*2d}\circ(F_{2d}(\mathscr{A}_1, \mathscr{A}_2))\otimes_{\boldsymbol{Z}}\mathrm{id.})(\bar{U})$$

which is an element of $H^{2d}(M, \boldsymbol{Q})$.

In §3 we prove

THEOREM 1.10. *Notations being as above, we obtain that*

$$\langle\bar{\varepsilon}_M(\mathscr{A}_1, \mathscr{A}_2), \bar{z}\rangle_{2d}=L_M(\mathscr{A}_1, \mathscr{A}_2)$$
for all $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *in* $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$.

Here $\langle\ ,\ \rangle_{2d}$ *denotes the Kronecker index.*

REMARK 1.11. Let $\alpha\in\mathrm{GSp}^+(g, \boldsymbol{Z})$. Recall $\mathfrak{D}=\mathfrak{H}_g\times\boldsymbol{C}^{gw}$. Set $X=$the complex analytic quotient space $((\Gamma\ltimes\boldsymbol{Z}^{2gw})\cap((\alpha, 0)^{-1}(\Gamma\ltimes\boldsymbol{Z}^{2gw})(\alpha, 0)))\backslash\mathfrak{D}$. Let $v_1: X\to(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})$ and $v_2: X\to((\alpha, 0)^{-1}(\Gamma\ltimes\boldsymbol{Z}^{2gw})(\alpha, 0)\backslash\mathfrak{D})$ be the canonical maps. Let $[(\alpha, 0)]$ be the map : $((\alpha, 0)^{-1}(\Gamma\ltimes\boldsymbol{Z}^{2gw})(\alpha, 0)\backslash\mathfrak{D})\to(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})$ induced from the map $(\alpha, 0):\mathfrak{D}\to\mathfrak{D}$. Write $v_3=[(\alpha, 0)]\circ v_2$. Consider the graph of $v_3\circ{}^tv_1$, for which we write $\mathrm{GR}(v_3\circ{}^tv_1)$, in the product $(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})\times(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})$. Then the $\mathscr{X}_M(\Gamma\alpha\Gamma)$ coincides with the closure of $\mathrm{GR}(v_3\circ{}^tv_1)$ in $M\times M$ with respect to the usual complex topology and also with respect to the Zariski topology.

§ 2. Some lemmas.

We need the following three lemmas. We use the Künneth formula for $H_j(M\times M, \boldsymbol{Q})$.

LEMMA 2.1. (Vick [21, p. 151, (5.21)]). *Let* $k\in[0, 2d]\cap\boldsymbol{Z}$ *and let* $\langle\ ,\ \rangle_k$ *denote the Kronecker index. Then*

$$\langle x, \zeta\rangle_k=(-1)^{2d(2d-k)}\langle U^{\sim}, D_{2d-k}(x)\otimes_{\boldsymbol{Q}}\zeta\rangle_{2d}$$

for all $x\in H^k(M, \boldsymbol{Q})$ *and all* $\zeta\in H_k(M, \boldsymbol{Q})$. (For the definition of D_{2d-k} see the introduction.)

Recall that Δ is the diagonal map : $M\to M\times M$. Let $\Delta_{*_k}: H_k(M, \boldsymbol{Z})\to H_k(M\times M, \boldsymbol{Z})$ be the linear map induced from the map Δ for each $k\in[0, 2d]\cap\boldsymbol{Z}$. Select a homogeneous basis $\{x_{\imath}\}_{\imath}$ for $\bigoplus_{q=0}^{2d} H^q(M, \boldsymbol{Q})$. Denote by $\{a_{\imath}\}_{\imath}$ the basis of $\bigoplus_{q=0}^{2l} H_q(M, \boldsymbol{Q})$ dual to $\{x_{\imath}\}_{\imath}$ under the Kronecker index. Define another basis $\{x'_{\imath}\}_{\imath}$ for $\bigoplus_{q=0}^{2d} H^q(M, \boldsymbol{Q})$ by $D_{(\dim a_{\imath})}(x'_{\imath})=a_{\imath}$ for every $\imath$ and let $\{a'_{\imath}\}_{\imath}$ be the basis for $\bigoplus_{q=0}^{2d} H_q(M, \boldsymbol{Q})$ dual to $\{x'_{\imath}\}_{\imath}$ under the Kronecker index.

LEMMA 2.2 (Vick [21, p. 186, Lemma 6.11]). *One has*

$$(\Delta_{*_{2d}}\otimes_{\boldsymbol{Z}}\mathrm{id.})(\bar{z})=\sum_{\imath=1}^{\rho}(-1)^{(\dim a_{\imath})(\dim a'_{\imath})}(a_{\imath}\otimes_{\boldsymbol{Q}} a'_{\imath})$$

where $\rho=\dim_{\boldsymbol{Q}}\bigoplus_{q=0}^{2} H_q(M, \boldsymbol{Q})$ *and* $(\Delta_{*_{2d}}\otimes_{\boldsymbol{Z}}\mathrm{id.}): H_{2d}(M, \boldsymbol{Q})\to H_{2d}(M\times M, \boldsymbol{Q})$.

Let $k\in[0, 2d]\cap\boldsymbol{Z}$. By the Künneth theorem we have the following split exact sequence.

$$\{0\}\longrightarrow\bigoplus_{p+q=k} H_p(M, \boldsymbol{Z})\otimes_{\boldsymbol{Z}} H_q(M, \boldsymbol{Z})\longrightarrow H_k(M\times M, \boldsymbol{Z})$$
$$\longrightarrow\bigoplus_{p+q=k-1}\mathrm{Tor}(H_p(M, \boldsymbol{Z}), H_q(M, \boldsymbol{Z}))\longrightarrow\{0\}\ .$$

Let $\mathscr{A}_1$ and $\mathscr{A}_2\in\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. Let f_n be the homomorphism in Theorem A for $n\in[0, 2d]$. Then we see that $\sum_{p+q=k}(f_p(\mathscr{A}_1)\otimes_{\boldsymbol{Z}} f_q(\mathscr{A}_2))$ (resp. the pair of $f_p(\mathscr{A}_1)$ and $f_q(\mathscr{A}_2)$) induces a unique (canonical) endomorphism of $\bigoplus_{p+q=k} H_p(M, \boldsymbol{Z})\otimes_{\boldsymbol{Z}} H_q(M, \boldsymbol{Z})$ (resp. $\mathrm{Tor}(H_p(M, \boldsymbol{Z}), H_q(M, \boldsymbol{Z}))$). From these we obtain the endomorphism $[\mathscr{A}_1, \mathscr{A}_2]_k$ of $H_k(M\times M, \boldsymbol{Z})$ given by:

(2.3.1)

$$\begin{array}{ccc} H_k(M\times M,\ \boldsymbol{Z})\cong(\bigoplus_{p+q=k} H_p(M,\ \boldsymbol{Z})\otimes_{\boldsymbol{Z}} H_q(M,\ \boldsymbol{Z})) & \bigoplus_{\boldsymbol{Z}} & (\bigoplus_{p+q=k-1}\operatorname{Tor}(H_p(M,\ \boldsymbol{Z}),\ H_q(M,\ \boldsymbol{Z}))) \\ \downarrow [\mathscr{A}_1,\ \mathscr{A}_2]_k \quad \circlearrowleft \quad \downarrow \sum_{p+q=k}(f_p(\mathscr{A}_1)\otimes_{\boldsymbol{Z}} f_q(\mathscr{A}_2)) & & \downarrow \sum_{p+q=k-1}(f_p(\mathscr{A}_1),\ f_q(\mathscr{A}_2)) \\ H_k(M\times M,\ \boldsymbol{Z})\cong(\bigoplus_{p+q=k} H_p(M,\ \boldsymbol{Z})\otimes_{\boldsymbol{Z}} H_q(M,\ \boldsymbol{Z})) & \bigoplus_{\boldsymbol{Z}} & (\bigoplus_{p+q=k-1}\operatorname{Tor}(H_p(M,\ \boldsymbol{Z}),\ H_q(M,\ \boldsymbol{Z}))) \end{array}$$

Let F_k be the map defined in §1. From our construction of $F_k(\ ,\ )$ and $[\ ,\ ]_k$ we obtain

$$F_k(\mathscr{A}_1,\ \mathscr{A}_2)=[\mathscr{A}_1,\ \mathscr{A}_2]_k\circ\Delta_{*_k} \qquad \text{for each } k\in[0,\ 2d]\ . \tag{2.3.2}$$

To an endomorphism $\omega\in\operatorname{End}_{\boldsymbol{Z}} H_n(M,\ \boldsymbol{Z})$, write $\omega\otimes_{\boldsymbol{Z}}\text{id.}\in\operatorname{End}_{\boldsymbol{Q}} H_n(M,\boldsymbol{Q})$ for the induced map from ω by $H_n(M,\ \boldsymbol{Z})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}\cong H_n(M,\ \boldsymbol{Q})$. Let ${}^t(\omega\otimes_{\boldsymbol{Z}}\text{id.})$ $(\in\operatorname{End}_{\boldsymbol{Q}} H^n(M,\ \boldsymbol{Q}))$ denote its transposed endomorphism with respect to the Kronecker index for M. Let $\{f_n\}_{n=0}^{2d}$ be the homomorphisms in Theorem A.

LEMMA 2.3. *Let α and $\beta\in\mathrm{GSp}^+(g,\ \boldsymbol{Z})$. Write $\rho=\dim_{\boldsymbol{Q}}\bigoplus_{n=0}^{2d} H_n(M,\ \boldsymbol{Q})$, $[\Gamma\alpha\Gamma]_n=f_n(\Gamma\alpha\Gamma)$ and $[\Gamma\beta\Gamma]_n=f_n(\Gamma\beta\Gamma)$ for simplicity. Then we obtain*

$$\begin{aligned}\sum_{v=1}^{\rho}(-1)^{(\dim x'_v)}((([\Gamma\alpha\Gamma]_{(2d-(\dim x'_v))}\otimes_{\boldsymbol{Z}}\text{id.})\circ D_{(2d-(\dim x'_v))}\\ \circ{}^t([\Gamma\beta\Gamma]_{(\dim x'_v)}\otimes_{\boldsymbol{Z}}\text{id.}))(x'_v)\otimes_{\boldsymbol{Q}} a'_v)\\ =\sum_{v=1}^{\rho}(-1)^{(\dim a'_v)}([\Gamma\alpha\Gamma,\ \Gamma\beta\Gamma]_{2d}\otimes_{\boldsymbol{Z}}\text{id.})(a_v\otimes_{\boldsymbol{Q}} a'_v)\ .\end{aligned}$$

PROOF. We can write

$${}^t([\Gamma\alpha\Gamma]_{(\dim x_v)}\otimes_{\boldsymbol{Z}}\text{id.})(x_v)=\sum_{u=1}^{\rho}\alpha_{vu}x_u \quad \text{and}$$

$${}^t([\Gamma\beta\Gamma]_{(\dim x'_v)}\otimes_{\boldsymbol{Z}}\text{id.})(x'_v)=\sum_{u=1}^{\rho}\beta_{vu}x'_u$$

with some rational coefficients $\{\alpha_{vu}\}_{1\leq v\leq\rho,1\leq u\leq\rho}$ and $\{\beta_{vu}\}_{1\leq v\leq\rho,1\leq u\leq\rho}$. We can also write

$$([\Gamma\beta\Gamma]_{(\dim a'_v)}\otimes_{\boldsymbol{Z}}\text{id.})(a'_v)=\sum_{u=1}^{\rho}\lambda_{uv}a'_u \quad \text{and}$$

$$([\Gamma\alpha\Gamma]_{(\dim a_v)}\otimes_{\boldsymbol{Z}}\text{id.})(a_v)=\sum_{u=1}^{\rho}\lambda'_{uv}a_u$$

with some rational coefficients $\{\lambda_{uv}\}_{1\leq u\leq\rho,1\leq v\leq\rho}$ and $\{\lambda'_{uv}\}_{1\leq u\leq\rho,1\leq v\leq\rho}$. Notice that $\alpha_{vu}=\beta_{vu}=0$ if $\dim x_v\neq\dim x_u$ viz. $\dim x'_v\neq\dim x'_u$; and that $\lambda_{uv}=\lambda'_{uv}=0$ if $\dim a'_v\neq\dim a'_u$ viz. $\dim a_v\neq\dim a_u$. Let i and $j\in[1,\ \rho]\cap\boldsymbol{Z}$ with $\dim x'_i=$

$\dim x'_j$ viz. $\dim x_i = \dim x_j$. Then note that $\dim a'_j = \dim x'_j = \dim x'_i$ and $\dim a_j = \dim x_j = \dim x_i$. Let $\langle\ ,\ \rangle$ denote the Kronecker index for M. We have

$$\begin{aligned}\beta_{ij} &= \left\langle \sum_{u=1}^{\rho} \beta_{iu} x'_u,\ a'_j \right\rangle \\ &= \langle {}^t([\Gamma\beta\Gamma]_{(\dim x'_i)} \otimes_Z \mathrm{id.})(x'_i),\ a'_j \rangle = \langle x'_i,\ ([\Gamma\beta\Gamma]_{(\dim x'_i)} \otimes_Z \mathrm{id.})(a'_j) \rangle \\ &= \left\langle x'_i,\ \sum_{u=1}^{\rho} \lambda_{uj} a'_u \right\rangle = \lambda_{ij}\ ;\end{aligned}$$

$$\begin{aligned}\alpha_{ij} &= \left\langle \sum_{u=1}^{\rho} \alpha_{iu} x_u,\ a_j \right\rangle \\ &= \langle {}^t([\Gamma\alpha\Gamma]_{(\dim x_i)} \otimes_Z \mathrm{id.})(x_i),\ a_j \rangle = \langle x_i,\ ([\Gamma\alpha\Gamma]_{(\dim a_j)} \otimes_Z \mathrm{id.})(a_j) \rangle \\ &= \left\langle x_i,\ \sum_{u=1}^{\rho} \lambda'_{uj} a_u \right\rangle = \lambda'_{ij}\ .\end{aligned}$$

Hence we have:

$$([\Gamma\beta\Gamma]_{(\dim a'_j)} \otimes_Z \mathrm{id.})(a'_j) = \sum_{u=1}^{\rho} \beta_{uj} a'_u \quad \text{and}$$

$$([\Gamma\alpha\Gamma]_{(\dim a_i)} \otimes_Z \mathrm{id.})(a_i) = \sum_{u=1}^{\rho} \alpha_{ui} a_u\ .$$

We expand the element

$$\mathfrak{X} \underset{\text{def.}}{=} ([\Gamma\alpha\Gamma]_{(2d-(\dim x'_*))} \otimes_Z \mathrm{id.}) \circ D_{(2d-(\dim x'_*))} \circ {}^t([\Gamma\beta\Gamma]_{(\dim x'_*)} \otimes_Z \mathrm{id.})(x'_*)$$

in terms of the basis $\{a_*\}_{*=1}^{\rho}$. Then

$$\begin{aligned}&(\text{its coefficient at } a_j) = \langle x_j,\ \mathfrak{X} \rangle \\ &\quad = \langle {}^t([\Gamma\alpha\Gamma]_{(2d-(\dim x'_*))} \otimes_Z \mathrm{id.})(x_j),\ D_{(2d-(\dim x'_*))} \circ {}^t([\Gamma\beta\Gamma]_{(\dim x'_*)} \otimes_Z \mathrm{id.})(x'_*) \rangle \\ &\quad = \left\langle \sum_{l=1}^{\rho} \alpha_{jl} x_l,\ D_{(2d-(\dim x'_*))}\left(\sum_{u=1}^{\rho} \beta_{*u} x'_u \right) \right\rangle = \left\langle \sum_{l=1}^{\rho} \alpha_{jl} x_l,\ \sum_{u=1}^{\rho} \beta_{*u} a_u \right\rangle \\ &\quad = \sum_{u=1}^{\rho} \alpha_{ju} \beta_{*u}\ .\end{aligned}$$

We have:

$$\begin{aligned}&\text{the left side of the equation in Lemma 2.3} \\ &= \sum_{*=1}^{\rho} (-1)^{(\dim x'_*)} \left(\sum_{j=1}^{\rho} \left(\sum_{u=1}^{\rho} \alpha_{ju} \beta_{*u} \right) a_j \right) \otimes_Q a'_* \\ &= \sum_{*=1}^{\rho} \sum_{j=1}^{\rho} \sum_{u=1}^{\rho} (-1)^{(\dim a'_*)} \alpha_{ju} \beta_{*u} a_j \otimes_Q a'_*\end{aligned}$$

since $\dim x'_* = \dim a'_*$, and

the right side of the equation in Lemma 2.3

$$=\sum_{*=1}^{\rho}(-1)^{(\dim a'_*)}(([\Gamma\alpha\Gamma]_{(\dim a_*)}\otimes_Z \mathrm{id.})(a_*)\otimes_Q ([\Gamma\beta\Gamma]_{(\dim a'_*)}\otimes_Z \mathrm{id.})(a'_*))$$

$$=\sum_{*=1}^{\rho}(-1)^{(\dim a'_*)}\Big(\Big(\sum_{l=1}^{\rho}\alpha_{l*}a_l\Big)\otimes_Q\Big(\sum_{k=1}^{\rho}\beta_{k*}a'_k\Big)\Big)$$

$$=\sum_{*=1}^{\rho}\sum_{l=1}^{\rho}\sum_{k=1}^{\rho}(-1)^{(\dim a'_*)+(\dim a'_k)+(\dim a'_k)}\alpha_{l*}\beta_{k*}a_l\otimes_Q a'_k$$

$$=\sum_{*=1}^{\rho}\sum_{l=1}^{\rho}\sum_{k=1}^{\rho}(-1)^{(\dim a'_k)}\alpha_{l*}\beta_{k*}a_l\otimes_Q a'_k$$

since $\beta_{k*}=0$ if $\dim a'_*\neq\dim a'_k$. Lemma 2.3 is proved. (We have given this proof by extending the method of Vick [21, p. 185, Lemma 6.10] to the case of our operators for the Hecke rings.)

§ 3. Coincidence Theorem, Fixed Point Theorem and proofs.

Let $\mathscr{A}_1=\sum_j m_j\Gamma\alpha_j\Gamma$ and $\mathscr{A}_2=\sum_k l_k\Gamma\beta_k\Gamma \in \mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. Recall $M=(\Gamma\ltimes \boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^{\sim}$. We define $\hat{L}_M(\mathscr{A}_1, \mathscr{A}_2)$ by

$$\hat{L}_M(\mathscr{A}_1, \mathscr{A}_2)=\sum_{n=0}^{2d}(-1)^n\mathrm{Tr}\{D_n^{-1}\circ(f_n(\mathscr{A}_1)\otimes_Z \mathrm{id.})\circ D_n\circ{}^t(f_{2d-n}(\mathscr{A}_2)\otimes_Z \mathrm{id.})\}\ .$$

Recall the definition of $L_M(\mathscr{A}_1, \mathscr{A}_2)$ given in the introduction. By the elementary matrix theory we have

$$L_M(\mathscr{A}_1, \mathscr{A}_2)=(-1)^{2d}\hat{L}_M(\mathscr{A}_1, \mathscr{A}_2)=\hat{L}_M(\mathscr{A}_1, \mathscr{A}_2)\ .$$

PROOF OF THEOREM 1.10. Write $\langle\ ,\ \rangle$ for the Kronecker index $\langle\ ,\ \rangle_{2d}$ on M. By the bilinearity of $L_M(\ ,\ )$ and $\langle\ {}_M(\ ,\ ), \bar{z}\rangle$ it is sufficient to show that $\langle\varepsilon_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma), \bar{z}\rangle=L_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)$ for any α and $\beta\in\mathrm{GSp}^+(g, \boldsymbol{Z})$. Write $[\Gamma\alpha\Gamma]_n=f_n(\Gamma\alpha\Gamma)$ and $[\Gamma\beta\Gamma]_n=f_n(\Gamma\beta\Gamma)$ for short. We have:

$$\hat{L}_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)=\sum_{*=1}^{\rho}(-1)^{(\dim x'_*)}\langle D^{-1}_{(2d-(\dim x'_*))}\circ([\Gamma\alpha\Gamma]_{(2d-(\dim x'_*))}\otimes_Z \mathrm{id.})$$
$$\circ D_{(2d-(\dim x'_*))}\circ{}^t([\Gamma\beta\Gamma]_{(\dim x'_*)}\otimes_Z \mathrm{id.})(x'_*), a'_*\rangle$$

$$=\sum_{*=1}^{\rho}(-1)^{(\dim x'_*)}\langle U^{\sim}, ([\Gamma\alpha\Gamma]_{(2d-(\dim x'_*))}\otimes_Z \mathrm{id.})\circ D_{(2d-(\dim x'_*))}$$
$$\circ{}^t([\Gamma\beta\Gamma]_{(\dim x'_*)}\otimes_Z \mathrm{id.})(x'_*)\otimes_Q a'_*\rangle \quad \text{by Lemma 2.1}\ ,$$

$$=\sum_{*=1}^{\rho}(-1)^{(\dim a'_*)}\langle U^{\sim}, ([\Gamma\alpha\Gamma, \Gamma\beta\Gamma]_{2d}\otimes_Z \mathrm{id.})(a_*\otimes_Q a'_*)\rangle$$

by Lemma 2.3 ,

(Here use $(-1)^{(\dim a'_*)}=(-1)^{(\dim a_*)(\dim a'_*)}$ since $(\dim a_*)+(\dim a'_*)=2d$.)

$$=\langle U^{\sim}, ([\Gamma\alpha\Gamma, \Gamma\beta\Gamma]_{2d}\otimes_Z \text{id.})\circ(\Delta_{*2d}\otimes_Z \text{id.})(\bar{z})\rangle \qquad \text{by Lemma 2.2},$$
$$=\langle U^{\sim}, (\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_{2d}\otimes_Z \text{id.})(\bar{z})\rangle \qquad \text{by (2.3.2)},$$

where $\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_{2d}\otimes_Z \text{id.} : H_{2d}(M, \boldsymbol{Z})\otimes_Z \boldsymbol{Q}\cong H_{2d}(M, \boldsymbol{Q})\rightarrow H_{2d}(M\times M, \boldsymbol{Z})\otimes_Z \boldsymbol{Q}\cong H_{2d}(M\times M, \boldsymbol{Q})$. Hence

$$\begin{aligned}\hat{L}_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma) &= \langle {}^t(\iota_{*2d}\otimes_Z \text{id.})(\bar{U}), (\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_{2d}\otimes_Z \text{id.})(\bar{z})\rangle\\ &= \langle {}^t((\iota_{*2d}\otimes_Z \text{id.})\circ(\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_{2d}\otimes_Z \text{id.}))(\bar{U}), \bar{z}\rangle\\ &= \langle {}^t((\iota_{*2d}\circ(F_{2d}(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)))\otimes_Z \text{id.})(\bar{U}), \bar{z}\rangle\\ &= \langle \bar{\varepsilon}_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma), \bar{z}\rangle\end{aligned}$$

by Definition 1.9. Theorem 1.10 is proved.

Now we give proofs of Theorems 2 and 3 in the introduction.

Proof of Theorem 2. By the bilinearity of the Coincidence Index and the Coincidence Number it is sufficient to prove it in the case of $\mathscr{A}_1=\Gamma\alpha\Gamma$ and $\mathscr{A}_2=\Gamma\beta\Gamma$ for any α and $\beta\in \mathrm{GSp}^+(g, \boldsymbol{Z})$. Recall $I_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)=\langle U, \iota_{*2d}\circ\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_{2d}(z)\rangle_{2d}$. Tensoring these elements with $\boldsymbol{Q}$ over $\boldsymbol{Z}$ we have:

$$\begin{aligned}&I_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)\\ &\quad=\langle \bar{U}, (\iota_{*2d}\otimes_Z \text{id.})\circ(\{\Gamma\alpha\Gamma, \Gamma\beta\Gamma\}_{2d}\otimes_Z \text{id.})(\bar{z})\rangle_{2d}\\ &\quad=\langle \bar{U}, ((\iota_{*2d}\circ(F_{2d}(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)))\otimes_Z \text{id.})(\bar{z})\rangle_{2d}\\ &\quad=\langle {}^t((\iota_{*2d}\circ(F_{2d}(\Gamma\alpha\Gamma, \Gamma\beta\Gamma)))\otimes_Z \text{id.})(\bar{U}), \bar{z}\rangle_{2d}\\ &\quad=\langle \bar{\varepsilon}_M(\Gamma\alpha\Gamma, \Gamma\beta\Gamma), \bar{z}\rangle_{2d}\end{aligned}$$

by Definition 1.9. Now Theorem 2 is derived from Theorem 1.10.

Proof of Theorem 3. It is a consequence of Corollary 1.8 and Theorem 2.

§4. On l-adic cohomology groups, I.

Let X and Y be schemes which are separated and of finite type over an algebraically closed field $\mathscr{K}$ (e.g. $\bar{\boldsymbol{Q}}$, $\boldsymbol{C}$), let l be a prime number with $l\neq$ Characteristic of $\mathscr{K}$, let $u: X\rightarrow Y$ be a finite morphism, and let $H_c^n(X, \boldsymbol{Q}_l)$ and $H_c^n(Y, \boldsymbol{Q}_l)$ be the l-adic cohomology groups of X and Y with compact support respectively where $n\,(\geqq 0)\in \boldsymbol{Z}$ (cf. Artin et al. [1], Carter [3], Deligne [5], Hartshorne [6] and Srinivasan [20]). Then it is well known that u induces functorially a canonical map

$u^* \in \mathrm{Hom}_{Q_l}(H_c^n(Y, \boldsymbol{Q}_l) \to H_c^n(X, \boldsymbol{Q}_l))$ for each $n\ (\geqq 0) \in \boldsymbol{Z}$. In this section we prove Theorem 4 in the introduction. Recall $M = (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})^{\sim}$ where $\Gamma = \Gamma_g(N)$. Since M is a non-singular complex projective variety, one has $H^n(M, \boldsymbol{C}) \cong H^n(M, \boldsymbol{Q}_l) \otimes_{Q_l} \boldsymbol{C}$ by the comparison theorem of the l-adic cohomology theory. One has the non-degenerate bilinear forms of the Kronecker index $H^n(M, \boldsymbol{C}) \times H_n(M, \boldsymbol{C}) \to \boldsymbol{C}\ (n \in [0, 2d])$. The anti-ring homomorphisms $\{f^{(n)}\}_{n \geqq 0}$ in Theorem 4 are compatible with the homomorphisms $\{f_n\}_{n \geqq 0}$ in Theorem A and these bilinear forms of the Kronecker index.

We need:

LEMMA 4.1 (Artin et al. [1]; also cf. Carter [3; p. 203] and Srinivasan [20, p. 53]). *Suppose that $\mathscr{G}$ is a finite group of automorphisms of X such that a strict quotient $\mathscr{G} \backslash X$ exists. Then*

$$H_c^n(\mathscr{G} \backslash X, \boldsymbol{Q}_l) \cong H_c^n(X, \boldsymbol{Q}_l)^{\mathscr{G}} \qquad \textit{for each } n\ (\geqq 0) \in \boldsymbol{Z}\,.$$

For the definition of "strict quotient" see e.g. Carter [3, p. 13].

Under the condition of Proposition D in §1 we see that $(G_2 \backslash \mathfrak{D})^{\sim}$ is a strict quotient $\mathscr{G} \backslash (G_1 \backslash \mathfrak{D})^{\sim}$ where $\mathscr{G} = G_2/G_1$.

PROOF OF THEOREM 4. Recall $\Gamma = \Gamma_g(N)$ and $\mathfrak{D} = \mathfrak{H}_g \times \boldsymbol{C}^{gw}$. Let $\alpha \in \mathrm{GSp}^+(g, \boldsymbol{Z})$. Write $c = r(\alpha)$ and $\Gamma \alpha \Gamma = \cup_{i=1}^{\mu} \Gamma \alpha_i$ (disjoint). See the commutative diagram (0.1) in the introduction. We use the same notations given in (0.1). Notice that those π_i, $\pi^{(i)}$, π, $(\alpha_i, 0)^{\sim}$ and $[\pi]$ are all finite morphisms. Let us fix $n \in \boldsymbol{Z}$ with $0 \leqq n \leqq 2d = 2\langle g \rangle + 2gw$ arbitrarily. Consider the element:

$$\Upsilon \underset{\mathrm{def.}}{=} \sum_{i=1}^{\mu} \pi_i^* \circ (\alpha_i, 0)^{\sim *} \circ [\pi]^* \in \mathrm{Hom}_{Q_l}(H^n(M, \boldsymbol{Q}_l), H^n((\Gamma_g(c^2N) \ltimes (c\boldsymbol{Z})^{2gw} \backslash \mathfrak{D})^{\sim}, \boldsymbol{Q}_l))\,.$$

Choose $\gamma_i \in \Gamma$ for each $i \in [1, \mu]$ arbitrarily. Then we obtain:

$$(\gamma_i \alpha_i, 0)^{\sim *} \circ [\pi]^* = ([\pi] \circ (\gamma_i, 0)^{\sim} \circ (\alpha_i, 0)^{\sim})^* = ([\pi] \circ (\alpha_i, 0)^{\sim})^* = (\alpha_i, 0)^{\sim *} \circ [\pi]^*$$

since we have $\mathrm{id.} \circ [\pi] = [\pi] \circ (\gamma_i, 0)^{\sim}$ (cf. the diagram in front of (2.3) in Hatada [8]). Hence the above Υ does not depend on the choice of representatives $\{\alpha_i\}_{i=1}^{\mu}$ for $\Gamma \backslash \Gamma \alpha \Gamma$.

Now we show:

$$(4.2) \qquad (\gamma, \boldsymbol{m})^{\sim *} \circ \Upsilon = \Upsilon \qquad \text{for all} \quad (\gamma, \boldsymbol{m}) \in \Gamma \ltimes \boldsymbol{Z}^{2gw}\,.$$

Let $(\gamma, \boldsymbol{m}) \in \Gamma \ltimes \boldsymbol{Z}^{2gw}$. We have a permutation $\begin{pmatrix} \Gamma\alpha_1 & \Gamma\alpha_2 & \cdots & \Gamma\alpha_\mu \\ \Gamma\alpha_1\gamma & \Gamma\alpha_2\gamma & \cdots & \Gamma\alpha_\mu\gamma \end{pmatrix}$ of $\Gamma \backslash \Gamma \alpha \Gamma$, which gives the permutation $\langle \gamma \rangle$ of the numbers $\{1, 2, \cdots, \mu\}$

with $\Gamma\alpha_i\gamma=\Gamma\alpha_{\langle\gamma\rangle(i)}$ $(1\leqq i\leqq\mu)$. Write $\alpha_i\cdot\gamma=\gamma_i'\cdot\alpha_{\langle\gamma\rangle(i)}$ with some $\gamma_i'\in\Gamma$ for each $i\in[1,\mu]$. Then we obtain:

$$(\alpha_i,\mathbf{0})\cdot(\gamma,\boldsymbol{m})=(\gamma_i',\boldsymbol{x}_i)\cdot(\alpha_{\langle\gamma\rangle(i)},\mathbf{0})\qquad\text{with some }\boldsymbol{x}_i\in\boldsymbol{Z}^{2gw}$$

in $\mathrm{GSp}^+(g,\boldsymbol{Q})\ltimes\boldsymbol{Q}^{2gw}$ for each $i\in[1,\mu]$. Hence we have

$$(\alpha_i,\mathbf{0})^\sim\circ(\gamma,\boldsymbol{m})^\sim=(\gamma_i',\boldsymbol{x}_i)^\sim\circ(\alpha_{\langle\gamma\rangle(i)},\mathbf{0})^\sim\ .$$

Now let $i\in[1,\mu]$ be fixed arbitrarily and write $h=\langle\gamma\rangle^{-1}(i)$. Then

$$(\alpha_h,\mathbf{0})\cdot(\gamma,\boldsymbol{m})=(\gamma_h',\boldsymbol{x}_h)\cdot(\alpha_i,\mathbf{0})\ .$$

Look at the following commutative diagrams.

$$\begin{array}{ccccc}
(\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{(\gamma,\boldsymbol{m})^\sim} & (\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{(\alpha_h,\mathbf{0})^\sim} & ((\alpha_h,\mathbf{0})(\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw})(\alpha_h,\mathbf{0})^{-1}\backslash\mathfrak{D})^\sim \\
 & & \downarrow{\scriptstyle\pi_h} & & \downarrow{\scriptstyle p} \\
 & & ((\alpha_h,\mathbf{0})^{-1}(\Gamma_g(cN)\ltimes(c\boldsymbol{Z})^{2gw})(\alpha_h,\mathbf{0})\backslash\mathfrak{D})^\sim & \xrightarrow{(\alpha_h,\mathbf{0})^\sim} & (\Gamma_g(cN)\ltimes(c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim \\
 & & & & \downarrow{\scriptstyle[\pi]} \\
 & & & & M=(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^\sim
\end{array}$$

$$\begin{array}{ccccc}
(\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{(\alpha_i,\mathbf{0})^\sim} & ((\alpha_i,\mathbf{0})(\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw})(\alpha_i,\mathbf{0})^{-1}\backslash\mathfrak{D})^\sim & \xrightarrow{(\gamma_h',\boldsymbol{x}_h)^\sim} & (\Gamma'\backslash\mathfrak{D})^\sim \\
\downarrow{\scriptstyle\pi_i} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
((\alpha_i,\mathbf{0})^{-1}(\Gamma_g(cN)\ltimes(c\boldsymbol{Z})^{2gw})(\alpha_i,\mathbf{0})\backslash\mathfrak{D})^\sim & \xrightarrow{(\alpha_i,\mathbf{0})^\sim} & (\Gamma_g(cN)\ltimes(c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{(\gamma_h',\boldsymbol{x}_h)^\sim} & (\Gamma_g(cN)\ltimes(c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim \\
 & & \downarrow{\scriptstyle[\pi]} & & \downarrow{\scriptstyle[\pi]} \\
 & & M=(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{\text{id.}} & M=(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^\sim
\end{array}$$

Here we have written $\Gamma'=(\gamma_h',\boldsymbol{x}_h)\cdot(\alpha_i,\mathbf{0})(\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw})(\alpha_i,\mathbf{0})^{-1}\cdot(\gamma_h',\boldsymbol{x}_h)^{-1}=(\alpha_h,\mathbf{0})(\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw})(\alpha_h,\mathbf{0})^{-1}$. All the horizontal lines denote the isomorphisms given by Proposition C. All the vertical lines denote the canonical morphisms given by Proposition B. Use the relation of $(\alpha_h,\mathbf{0})^\sim\circ(\gamma,\boldsymbol{m})^\sim=(\gamma_h',\boldsymbol{x}_h)^\sim\circ(\alpha_i,\mathbf{0})^\sim$ in the above diagrams. We obtain:

$$\begin{aligned}
&(\gamma,\boldsymbol{m})^{\sim*}\circ\pi_h^*\circ(\alpha_h,\mathbf{0})^{\sim*}\circ[\pi]^*\\
&\quad=([\pi]\circ(\alpha_h,\mathbf{0})^\sim\circ\pi_h\circ(\gamma,\boldsymbol{m})^\sim)^*=([\pi]\circ p\circ(\alpha_h,\mathbf{0})^\sim\circ(\gamma,\boldsymbol{m})^\sim)^*\\
&\quad=([\pi]\circ p\circ(\gamma_h',\boldsymbol{x}_h)^\sim\circ(\alpha_i,\mathbf{0})^\sim)^*=([\pi]\circ(\alpha_i,\mathbf{0})^\sim\circ\pi_i)^*\\
&\quad=\pi_i^*\circ(\alpha_i,\mathbf{0})^{\sim*}\circ[\pi]^*\ .
\end{aligned}$$

Since $h=\langle\gamma\rangle^{-1}(i)$ and $\langle\gamma\rangle$ is a permutation, (4.2) is proved. Then by Lemma 4.1,

$$\varUpsilon\underset{\text{def.}}{=}\sum_{i=1}^{\mu}\pi_i^*\circ(\alpha_i,\mathbf{0})^{\sim*}\circ[\pi]^*$$

is regarded as an element of $\mathrm{End}_{Q_l}(H^n(M, \boldsymbol{Q}_l))$. Namely there exists a unique element, for which we write $f^{(n)}(\Gamma\alpha\Gamma)$, of $\mathrm{End}_{Q_l}(H^n(M, \boldsymbol{Q}_l))$ such that $\Upsilon = \pi^* \circ f^{(n)}(\Gamma\alpha\Gamma)$ where $\pi = \pi^{(i)} \circ \pi_i$.

(4.3) DEFINITION OF $f^{(n)}$. We have got $f^{(n)}(\Gamma\alpha\Gamma)$ for each double coset $\Gamma\alpha\Gamma$ with $\alpha \in \mathrm{GSp}^+(g, \boldsymbol{Z})$ in the above. Extend this $f^{(n)}$ to the whole of $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \otimes_Z \boldsymbol{Q}_l$ uniquely as a $\boldsymbol{Q}_l$-linear map.

Next we show:

(4.4)
$$f^{(n)}(\Gamma\beta\Gamma \cdot \Gamma\alpha\Gamma) = f^{(n)}(\Gamma\alpha\Gamma) \circ f^{(n)}(\Gamma\beta\Gamma) \quad \text{for all } \alpha \text{ and } \beta \in \mathrm{GSp}^+(g, \boldsymbol{Z}) .$$

Here $\cdot$ denotes the multiplication in the Hecke ring, and $\circ$ denotes the composition of two endomorphisms. Write $\Gamma\alpha\Gamma = \bigcup_{i=1}^{\mu} \Gamma\alpha_i$ (disjoint) and $\Gamma\beta\Gamma = \bigcup_{x=1}^{\mu'} \Gamma\beta_x$ (disjoint). Write $c = r(\alpha)$ and $b = r(\beta)$. Look at the following commutative diagrams.

(4.5)
$$\begin{array}{cccc}
 & (\Gamma_g(b^2c^2N) \ltimes (bc\boldsymbol{Z})^{2gw} \backslash \mathfrak{D})^{\sim} & & \\
 & \downarrow{\scriptstyle r_{1i}} & & \\
 & ((\alpha_i, 0)^{-1}(\Gamma_g(b^2cN) \ltimes (bc\boldsymbol{Z})^{2gw})(\alpha_i, 0) \backslash \mathfrak{D})^{\sim} & \xrightarrow{(\alpha_i, 0)^{\sim}} & (\Gamma_g(b^2cN) \ltimes (bc\boldsymbol{Z})^{2gw} \backslash \mathfrak{D})^{\sim} \\
 & \downarrow{\scriptstyle r_{2ix}} & & \downarrow{\scriptstyle s_{1x}} \\
r_0 \Bigg\downarrow & ((\beta_x \cdot \alpha_i, 0)^{-1}(\Gamma_g(bcN) \ltimes (bc\boldsymbol{Z})^{2gw})(\beta_x \cdot \alpha_i, 0) \backslash \mathfrak{D})^{\sim} & \xrightarrow{(\alpha_i, 0)^{\sim}} & ((\beta_x, 0)^{-1}(\Gamma_g(bcN) \ltimes (bc\boldsymbol{Z})^{2gw})(\beta_x, 0) \backslash \mathfrak{D})^{\sim} \\
 & \downarrow{\scriptstyle r_{3ix}} & & \downarrow{\scriptstyle s_{2x}} \\
 & ((\alpha_i, 0)^{-1}(\Gamma_g(cN) \ltimes (c\boldsymbol{Z})^{2gw})(\alpha_i, 0) \backslash \mathfrak{D})^{\sim} & \xrightarrow{(\alpha_i, 0)^{\sim}} & (\Gamma_g(cN) \ltimes (c\boldsymbol{Z})^{2gw} \backslash \mathfrak{D})^{\sim} \\
 & \downarrow{\scriptstyle r_{4i}} & & \downarrow{\scriptstyle s_3} \\
 & M = (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})^{\sim} & & M = (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})^{\sim}
\end{array}$$

Here we set $r_0 = r_{4i} \circ r_{3ix} \circ r_{2ix} \circ r_{1i}$.

(4.6)
$$\begin{array}{ccc}
(\Gamma_g(b^2cN) \ltimes (bc\boldsymbol{Z})^{2gw} \backslash \mathfrak{D})^{\sim} & & \\
\downarrow{\scriptstyle s_{1x}} & & \\
((\beta_x, 0)^{-1}(\Gamma_g(bcN) \ltimes (bc\boldsymbol{Z})^{2gw})(\beta_x, 0) \backslash \mathfrak{D})^{\sim} & \xrightarrow{(\beta_x, 0)^{\sim}} & (\Gamma_g(bcN) \ltimes (bc\boldsymbol{Z})^{2gw} \backslash \mathfrak{D})^{\sim} \\
\downarrow{\scriptstyle s_{4x}} & & \downarrow{\scriptstyle u_1} \\
((\beta_x, 0)^{-1}(\Gamma_g(bN) \ltimes (b\boldsymbol{Z})^{2gw})(\beta_x, 0) \backslash \mathfrak{D})^{\sim} & \xrightarrow{(\beta_x, 0)^{\sim}} & (\Gamma_g(bN) \ltimes (b\boldsymbol{Z})^{2gw} \backslash \mathfrak{D})^{\sim} \\
\downarrow{\scriptstyle s_{5x}} & & \downarrow{\scriptstyle u_2} \\
M = (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})^{\sim} & & M = (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})^{\sim}
\end{array}$$

In the commutative diagrams (4.5) and (4.6), all the vertical lines denote the canonical morphisms given by Proposition B; all the horizontal lines denote the isomorphisms given by Proposition C. They are all finite morphisms. Notice that $s_3\circ s_{2x}=s_{5x}\circ s_{4x}$.

Recall the definition of the multiplication $\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma$ in $\mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ given in Shimura [18, p. 52, (3.1.1)]. Namely $\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma=\sum m(\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma; W)W$ where the sum is extended over all $W=\Gamma\theta\Gamma\subset\Gamma\beta\Gamma\alpha\Gamma$, and $m(\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma; W)\underset{\text{def.}}{=}$ the number of (x, i) such that $\Gamma\beta_x\alpha_i=\Gamma\theta$ (for a fixed θ). This $m(\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma; W)$ depends only on $\Gamma\beta\Gamma$, $\Gamma\alpha\Gamma$ and W, and not on the choice of representatives $\{\alpha_i\}_{i=1}^{\mu}$, $\{\beta_x\}_{x=1}^{\mu'}$ and θ.

Using the notations used in the above commutative diagrams (4.5) and (4.6) we obtain:

$$
\begin{aligned}
&r_0^*\circ f^{(n)}(\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma)\\
&=\sum_{i=1}^{\mu}\sum_{x=1}^{\mu'} r_{1i}^*\circ r_{2ix}^*\circ(\alpha_i, 0)^{\sim *}\circ(\beta_x, 0)^{\sim *}\circ u_1^*\circ u_2^* && \text{by (4.3)}\\
&=\sum_{i=1}^{\mu}\sum_{x=1}^{\mu'} r_{1i}^*\circ(\alpha_i, 0)^{\sim *}\circ s_{1x}^*\circ(\beta_x, 0)^{\sim *}\circ u_1^*\circ u_2^*\\
&=\sum_{i=1}^{\mu} r_{1i}^*\circ(\alpha_i, 0)^{\sim *}\circ s_{1x}^*\circ s_{4x}^*\circ s_{5x}^*\circ f^{(n)}(\Gamma\beta\Gamma) && \text{by Lemma 4.1 and (4.2)}\\
&=\sum_{i=1}^{\mu} r_{1i}^*\circ(\alpha_i, 0)^{\sim *}\circ s_{1x}^*\circ s_{2x}^*\circ s_3^*\circ f^{(n)}(\Gamma\beta\Gamma) && \text{since } s_3\circ s_{2x}=s_{5x}\circ s_{4x}\\
&=\sum_{i=1}^{\mu} r_{1i}^*\circ r_{2ix}^*\circ s_{ix}^*\circ(\alpha_i, 0)^{\sim *}\circ s_3^*\circ f^{(n)}(\Gamma\beta\Gamma)\\
&=r_0^*\circ f^{(n)}(\Gamma\alpha\Gamma)\circ f^{(n)}(\Gamma\beta\Gamma) && \text{by Lemma 4.1 and (4.2)}\,.
\end{aligned}
$$

Thus we obtain (4.4) using Lemma 4.1. Theorem 4 is proved.

Now we show

PROPOSITION 4.7 (Lemma 1 in Hatada [9]). *There is a ring homomorphism* $h_0\colon \mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\to\mathrm{End}_{\boldsymbol{Z}}\, Z_0(M, \boldsymbol{Z})$. *Under the notations in* (0.1) *in the introduction, it is given by*

$$(h_0(\Gamma\alpha\Gamma))(Q)=\sum_{i=1}^{\mu}[\pi]\circ(\alpha_i, 0)^{\sim}\circ\pi_i(Q')$$

where Q *is a closed point of* M; Q' *is a closed point of* (*domain of* $\pi^{(i)}\circ\pi_i$) *such that* $Q=\pi^{(i)}\circ\pi_i(Q')$; $\alpha\in\mathrm{GSp}^+(g, \boldsymbol{Z})$ *and* $\Gamma\alpha\Gamma=\bigcup_{i=1}^{\mu}\Gamma\alpha_i$ (*disjoint*).

PROOF. Our proof is similar to the above one of (4.4). We give it for the convenience of the reader. Let Q be a closed point of M. Let α and $\beta\in\mathrm{GSp}^+(g, \boldsymbol{Z})$ with $r(\alpha)=c$ and $r(\beta)=b$. Use the commutative diagrams (4.5) and (4.6). Let Q'' be a closed point of $(\Gamma_g(b^2c^2N)\ltimes(bcZ)^{2gw}\backslash\mathfrak{D})^{\sim}$. Then

$$\begin{aligned}
&(h_0(\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma))(Q)\\
&=\sum_{i=1}^{\mu}\sum_{x=1}^{\mu'}u_2\circ u_1\circ(\beta_x,0)^{\sim}\circ(\alpha_i,0)^{\sim}\circ r_{2ix}\circ r_{1i}(Q'')\\
&=\sum_{i=1}^{\mu}\sum_{x=1}^{\mu'}u_2\circ u_1\circ(\beta_x,0)^{\sim}\circ s_{1x}\circ(\alpha_i,0)^{\sim}\circ r_{1i}(Q'')\\
&=\sum_{i=1}^{\mu}\sum_{x=1}^{\mu'}u_2\circ(\beta_x,0)^{\sim}\circ s_{4x}\circ s_{1x}\circ(\alpha_i,0)^{\sim}\circ r_{1i}(Q'')\\
&=\sum_{i=1}^{\mu}h_0(\Gamma\beta\Gamma)\circ s_{5x}\circ s_{4x}\circ s_{1x}\circ(\alpha_i,0)^{\sim}\circ r_{1i}(Q'')\\
&=\sum_{i=1}^{\mu}h_0(\Gamma\beta\Gamma)\circ s_3\circ s_{2x}\circ s_{1x}\circ(\alpha_i,0)^{\sim}\circ r_{1i}(Q'')\\
&=\sum_{i=1}^{\mu}h_0(\Gamma\beta\Gamma)\circ s_3\circ(\alpha_i,0)^{\sim}\circ r_{3ix}\circ r_{2ix}\circ r_{1i}(Q'')\\
&=(h_0(\Gamma\beta\Gamma)\circ h_0(\Gamma\alpha\Gamma))(Q)\,.
\end{aligned}$$

Q.E.D.

§ 5. On l-adic cohomology groups, II.

Let X and Y be schemes as are defined in § 4. Let $u: X\to Y$ be a morphism. Let $\mathcal{S}$ (resp. $\mathcal{S}'$) be an abelian sheaf on Y (resp. X) with respect to the étale topology. We denote by $u^*\mathcal{S}$ (resp. $R^m u_*\mathcal{S}'$) the inverse image sheaf of $\mathcal{S}$ on X (resp. the m-th direct image sheaf of $\mathcal{S}'$ on Y). Then one has an induced canonical homomorphism of abelian groups $u^*: H^n(Y,\mathcal{S})\to H^n(X,u^*\mathcal{S})$ for each $n\geq 0$ (cf. Artin et al. [1], Deligne [5], Srinivasan [20]).

Let m and n denote non-negative integers. In this section all our compactifications of the analytic quotient spaces are the toroidal compactifications with respect to the $\Sigma^{(1)}$ and $\Sigma^{(0)}$ given in the introduction. Then we have a proper canonical morphism $E: M=(\Gamma\ltimes Z^{2gw}\backslash\mathfrak{H}_g\times C^{gw})^{\sim}\to {}_0M=(\Gamma\backslash\mathfrak{H}_g)^{\sim}$. More generally let $\Phi\ltimes L\in\mathcal{F}$ where $\mathcal{F}$ is the set given in the introduction. Then we have a proper canonical morphism $\mathcal{E}: (\Phi\ltimes L\backslash\mathfrak{H}_g\times C^{gw})^{\sim}\to(\Phi\backslash\mathfrak{H}_g)^{\sim}$. Let l be a prime number. Write $\mathcal{L}_k=Z/(l^k)$ which is a constant sheaf on $(\Phi\ltimes L\backslash\mathfrak{H}_g\times C^{gw})^{\sim}$ with respect to the étale topology. One has

$$(\operatorname*{proj-lim}_{k\to+\infty}H^n((\Phi\backslash\mathfrak{H}_g)^{\sim}_{\text{ét}},R^m\mathcal{E}_*\mathcal{L}_k))\otimes_{Z_l}Q_l=H^n((\Phi\backslash\mathfrak{H}_g)^{\sim},R^m\mathcal{E}_*Q_l)\,.$$

Recall $E: M\to{}_0M$. We show

THEOREM 5.1. *The Hecke ring* $\mathrm{HR}(\Gamma,\mathrm{GSp}^+(g,Z))\otimes_Z Q_l$ *over* Q_l *acts* *naturally on* $H^n({}_0M, R^mE_*\boldsymbol{Q}_l)$, *i.e. there exists an anti-ring homomorphism* $\mathfrak{f}^{(n,m)}: \mathrm{HR}(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\otimes_{\boldsymbol{Z}} \boldsymbol{Q}_l \to \mathrm{End}_{\boldsymbol{Q}_l}(H^n({}_0M, R^mE_*\boldsymbol{Q}_l))$ *for each pair, m and n, of non-negative integers.*

PROOF. Let $\alpha\in\mathrm{GSp}^+(g, \boldsymbol{Z})$. Write $c=r(\alpha)$ and $\Gamma\alpha\Gamma=\cup_{i=1}^{\mu}\Gamma\alpha_i$ (disjoint). See the diagram (0.1) in the introduction. We denote by $[\pi]$, $(\alpha_i, 0)^\sim$, π_i, $\pi^{(i)}$ and π the morphisms given in (0.1) for an arbitrarily fixed integer $w\geqq 0$. They are finite morphisms. In the case of $w=0$ write ${}_0[\pi]$, $\alpha_i^\sim$, ${}_0\pi_i$, ${}_0\pi^{(i)}$ and ${}_0\pi$ for the morphisms $[\pi]$, $(\alpha_i, 0)^\sim$, π_i, $\pi^{(i)}$ and π respectively. For short write $\psi(\alpha_i)$ for ${}_0[\pi]\circ\alpha_i^\sim\circ{}_0\pi_i$ now. Consider the commutative diagram:

(5.1.1)
$$\begin{array}{ccc}
(\Gamma_g(c^2N)\ltimes(c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{[\pi]\circ(\alpha_i,\,0)^\sim\circ\pi_i} & M \\
\Big\downarrow E_1 & & \Big\downarrow E \\
(\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim & \xrightarrow{\quad\psi(\alpha_i)\quad} & {}_0M
\end{array}$$

Here E_1 denotes the canonical proper morphism. First we show

(5.1.2) Let m be fixed arbitrarily. We have a natural $\boldsymbol{Q}_l$-linear map

$$\varphi_{i,n}\circ\psi(\alpha_i)_n^* \;:\; H^n({}_0M, R^mE_*\boldsymbol{Q}_l)\longrightarrow H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^mE_{1*}\boldsymbol{Q}_l)$$

for each $n\geqq 0$.

PROOF OF (5.1.2). From (5.1.1) we have a natural homomorphism of abelian sheaves: $\psi(\alpha_i)^*R^mE_*\mathscr{L}_k\to R^mE_{1*}([\pi]\circ(\alpha_i, 0)^\sim\circ\pi_i)^*\mathscr{L}_k$. Then we have a $\boldsymbol{Q}_l$-linear map $\varphi_{i,n}: H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, \psi(\alpha_i)^*R^mE_*\boldsymbol{Q}_l)\to H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^mE_{1*}\boldsymbol{Q}_l)$ taking projective limits as $k\to+\infty$ and tensoring them with $\boldsymbol{Q}_l$ over $\boldsymbol{Z}_l$. On the other hand we have the canonical $\boldsymbol{Q}_l$-linenr map $\psi(\alpha_i)_n^*: H^n({}_0M, R^mE_*\boldsymbol{Q}_l)\to H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, \psi(\alpha_i)^*R^mE_*\boldsymbol{Q}_l)$. (5.1.2) is proved.

(5.1.3) Consider $\sum_{i=1}^{\mu}\varphi_{i,n}\circ\psi(\alpha_i)_n^*$ which is an element of $\mathrm{Hom}_{\boldsymbol{Q}_l}(H^n({}_0M, R^mE_*\boldsymbol{Q}_l), H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^mE_{1*}\boldsymbol{Q}_l))$. This does not depend on the choice of representatives $\{\alpha_i\}_{i=1}^{\mu}$ for $\Gamma\backslash\Gamma\alpha\Gamma$. Hence we write $\psi(\Gamma\alpha\Gamma)_n^*=\sum_{i=1}^{\mu}\varphi_{i,n}\circ\psi(\alpha_i)_n^*$.

PROOF OF (5.1.3). Let $\gamma_i\in\Gamma$. Replace α_i by $\gamma_i\alpha_i$ in (0.1). Then ${}_0[\pi]\circ(\gamma_i\alpha_i)^\sim\circ{}_0\pi_i={}_0[\pi]\circ\gamma_i^\sim\circ\alpha_i^\sim\circ{}_0\pi_i=\mathrm{id.}\circ{}_0[\pi]\circ\alpha_i^\sim\circ{}_0\pi_i={}_0[\pi]\circ\alpha_i^\sim\circ{}_0\pi_i$. (See the following commutative diagram.) (5.1.3) is proved.

$$\begin{array}{ccc}
(\Gamma_g(cN)\backslash\mathfrak{H}_g)^\sim & \xrightarrow{\quad\gamma^\sim\quad} & (\Gamma_g(cN)\backslash\mathfrak{H}_g)^\sim \\
\Big\downarrow {}_0[\pi] & & \Big\downarrow {}_0[\pi] \\
{}_0M & \xrightarrow{\quad\mathrm{id.}\quad} & {}_0M
\end{array} \qquad (\gamma\in\Gamma)$$

Let $\mathscr{L}_k = \boldsymbol{Z}/(l^k)$ be the étale sheaf on M. Let E_2: $(\Gamma_g(c^2N)\ltimes \boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^\sim \rightarrow (\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim$ denote the canonical proper morphism. Apply the Base Change Theorem (cf. Artin et al. [1, Expose XII]) to the following commutative diagram.

$$\begin{array}{ccccc} & & \overset{\pi}{\frown\!\!\!\longrightarrow} & & \\ (\Gamma_g(c^2N)\ltimes (c\boldsymbol{Z})^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{\text{can.}} & (\Gamma_g(c^2N)\ltimes \boldsymbol{Z}^{2gw}\backslash\mathfrak{D})^\sim & \xrightarrow{\pi^\wedge} & M \\ \downarrow{\scriptstyle E_1} & & \downarrow{\scriptstyle E_2} & & \downarrow{\scriptstyle E} \\ (\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim & \xrightarrow{\text{id.}} & (\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim & \xrightarrow{{}_0\pi} & {}_0M \end{array}$$

We obtain ${}_0\pi^* R^m E_* \mathscr{L}_k \cong R^m E_{2*} \pi^{\wedge *} \mathscr{L}_k$. By our choice of the toroidal compactifications, $R^m E_{2*}\mathscr{L}_k \cong R^m E_{1*}\mathscr{L}_k$. Taking projective limits and tensoring them with $\boldsymbol{Q}_l$ over $\boldsymbol{Z}_l$ we have $H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, {}_0\pi^* R^m E_* \boldsymbol{Q}_l) \cong H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^m E_{2*}\boldsymbol{Q}_l) \cong H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^m E_{1*}\boldsymbol{Q}_l)$ from $H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, {}_0\pi^* R^m E_* \mathscr{L}_k) \cong H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^m E_{2*}\pi^{\wedge *}\mathscr{L}_k)$. We use the results on the trace in Artin et al. [1, Expose XVII, §6] and Srinivasan [20, p. 53]. Recall that ${}_0M$ is a strict quotient of $(\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim$ by the group $\mathscr{G} = \Gamma_g(N)/\Gamma_g(c^2N)$. Write $\Xi_k = {}_0\pi^* R^m E_* \mathscr{L}_k$, which is an étale sheaf on $(\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim$. Then ${}_0\pi_* \Xi_k$ is an étale sheaf on ${}_0M$. By the action of $\mathscr{G}$ on $(\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim$, $\mathscr{G}$ operates on the sheaves Ξ_k and ${}_0\pi_*\Xi_k$. From the method of the proof of Srinivasan [20, p. 53, (5.10)] one obtains:

$$(5.1.4) \qquad \begin{aligned} H^n({}_0M, R^m E_* \boldsymbol{Q}_l) &\cong H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, {}_0\pi^* R^m E_* \boldsymbol{Q}_l)^{\mathscr{G}} \\ &\cong H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^m E_{1*} \boldsymbol{Q}_l)^{\mathscr{G}} \end{aligned}$$

for each pair, m and n, of non-negative integers.

(5.1.4) is a generalization of Lemma 4.1. Now we show:

$$(5.1.5) \qquad \psi(\Gamma\alpha\Gamma)_n^*(\chi) \in H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^m E_{1*}\boldsymbol{Q}_l)^{\mathscr{G}}$$

for all $\chi \in H^n({}_0M, R^m E_* \boldsymbol{Q}_l)$.

Proof of (5.1.5). Let $\gamma \in \Gamma$. Let $\gamma^{\sim *}$ denote the endomorphism of $H^n((\Gamma_g(c^2N)\backslash\mathfrak{H}_g)^\sim, R^m E_{1*}\boldsymbol{Q}_l)$. We use the notations in our proof of (4.2) in §4. Recall $\Gamma\alpha\Gamma = \cup_{i=1}^{\mu} \Gamma\alpha_i$ and that $\langle\gamma\rangle$ denotes the permutation of $\{1, 2, \cdots, \mu\}$. Write $h = \langle\gamma\rangle^{-1}(i)$ for an integer-valued variable $i \in [1, \mu]$. Namely $\Gamma\alpha_h\gamma = \Gamma\alpha_i$. We have

$$\begin{aligned} {}_0[\pi]\circ\alpha_h^\sim\circ{}_0\pi_h\circ\gamma^\sim &= {}_0[\pi]\circ p\circ\alpha_h^\sim\circ\gamma^\sim \\ &= {}_0[\pi]\circ\gamma_h'^\sim\circ\alpha_{\langle\gamma\rangle(h)}^\sim\circ{}_0\pi_{\langle\gamma\rangle(h)} \qquad \text{where } \gamma_h' \in \Gamma \\ &= {}_0[\pi]\circ\alpha_i^\sim\circ{}_0\pi_i \qquad \text{since } {}_0[\pi]\circ\gamma_h'^\sim = {}_0[\pi]\,. \end{aligned}$$

We have also $\gamma^{\sim *}\circ\varphi_{h,n}\circ({}_0[\pi]\circ\alpha_h^{\sim}\circ{}_0\pi_h)^* = \varphi_{\langle\gamma\rangle(h),n}\circ({}_0[\pi]\circ\alpha^{\sim}\circ{}_0\pi_h\circ\gamma^{\sim})^*$. Hence

$$\begin{aligned}\gamma^{\sim *}\circ\psi(\Gamma\alpha\Gamma)_n^*(\chi) &= \sum_{h=1}^{\mu}\varphi_{\langle\gamma\rangle(h),n}\circ({}_0[\pi]\circ\alpha^{\sim}\circ{}_0\pi_h\circ\gamma^{\sim})^*(\chi)\\ &= \sum_{i=1}^{\mu}\varphi_{i,n}\circ({}_0[\pi]\circ\alpha_i^{\sim}\circ{}_0\pi_i)^*(\chi)\\ &= \psi(\Gamma\alpha\Gamma)_n^*(\chi)\ .\end{aligned}$$

(5.1.5) is proved.

From (5.1.4) and (5.1.5) we can define $\mathfrak{f}^{(n,m)}(\Gamma\alpha\Gamma)$ uniquely by

$$\mathfrak{f}^{(n,m)}(\Gamma\alpha\Gamma) = \Theta\in\mathrm{End}_{\boldsymbol{Q}_l} H^n({}_0M,\ R^mE_*\boldsymbol{Q}_l)\quad\text{with}\quad {}_0\pi^*\circ\Theta = \psi(\Gamma\alpha\Gamma)_n^*\ .$$

As a $\boldsymbol{Q}_l$-linear map we can extend the domain of $\mathfrak{f}^{(n,m)}$ to the whole of $\mathrm{HR}(\Gamma,\ \mathrm{GSp}^+(g,\boldsymbol{Z}))\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l$ uniquely. We can prove the following (5.1.6) by the same way as in the proof of (4.4).

$$\text{(5.1.6)}\qquad \mathfrak{f}^{(n,m)}(\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma) = \mathfrak{f}^{(n,m)}(\Gamma\alpha\Gamma)\circ\mathfrak{f}^{(n,m)}(\Gamma\beta\Gamma)$$

for all α and $\beta\in\mathrm{GSp}^+(g,\boldsymbol{Z})$.

Theorem 5.1 is proved.

In the same way we obtain Theorem 5 in the introduction. We may leave the proof to the reader.

REMARK 5.2. Theorems 4 and 5 in the introduction also hold true for the natural Siegel modular schemes over $\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{F}}_p$ with $p\nmid N$ instead of those schemes over $\boldsymbol{C}$. Here $\Gamma=\Gamma_g(N)\subset\mathrm{Sp}(g,\boldsymbol{Z})$ and $l\neq p$.

REMARK 5.3. Let α and $\beta\in\mathrm{GSp}^+(g,\boldsymbol{Z})$. Under the conditions in this section we obtain the following equivalence.

$$\mathscr{X}_{{}_0M}(\Gamma\alpha\Gamma)\cap\mathscr{X}_{{}_0M}(\Gamma\beta\Gamma)\neq\varnothing \iff \mathscr{X}_M(\Gamma\alpha\Gamma)\cap\mathscr{X}_M(\Gamma\beta\Gamma)\neq\varnothing\ .$$

REMARK 5.4. Assume $g=1$ and $N=5$. Then $\Gamma=\Gamma(5)$, and the genus of the curve ${}_0M=(\Gamma\backslash\mathfrak{H}_1)^{\sim}$ is 0. Let α be any element of $\mathrm{GSp}^+(1,\boldsymbol{Z})$. Write $\alpha'=r(\alpha)\alpha^{-1}$ now. Then

$$\begin{aligned}L_{{}_0M}(\Gamma\alpha\Gamma,\ \Gamma 1_2\Gamma) &= \mathrm{Tr}(f_0(\Gamma\alpha\Gamma)\otimes_{\boldsymbol{Z}}\mathrm{id.}) + \mathrm{Tr}(f_0(\Gamma\alpha'\Gamma)\otimes_{\boldsymbol{Z}}\mathrm{id.})\\ &\qquad\text{by Hatada [8, Theorem 4]},\\ &= \sharp(\Gamma\backslash\Gamma\alpha\Gamma)+\sharp(\Gamma\backslash\Gamma\alpha'\Gamma)\geqq 2>0\ .\end{aligned}$$

From Theorem 3 we get $\mathscr{X}_{{}_0M}(\Gamma\alpha\Gamma)\cap\mathscr{X}_{{}_0M}(\Gamma 1_2\Gamma)\neq\varnothing$. For $M=(\Gamma\ltimes\boldsymbol{Z}^{2w}\backslash\mathfrak{H}_1\times\boldsymbol{C}^w)^{\sim}$ with any integer $w\geqq 0$, we have $\mathscr{X}_M(\Gamma\alpha\Gamma)\cap\mathscr{X}_M(\Gamma 1_2\Gamma)\neq\varnothing$.

In the following Remarks 5.5 and 5.6 assume $g=1$ and $N=6$, and write $\Gamma=\Gamma(6)$, ${}_0M=(\Gamma\backslash\mathfrak{H}_1)^{\sim}$ and $M=(\Gamma\ltimes \boldsymbol{Z}^{2w}\backslash\mathfrak{H}_1\times \boldsymbol{C}^{w})^{\sim}$ for any integer $w\geqq 0$. Then note that the genus of the curve ${}_0M$ is 1. Hence the space $S_2(\Gamma)$ of cusp forms of weight 2 with respect to Γ is one dimensional over $\boldsymbol{C}$. It is spanned by

$$C(z)=\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)^{1/6} \quad \text{where} \quad q=\exp(2\pi\sqrt{-1}z)$$

over $\boldsymbol{C}$ (see e.g. Shimura [18, p. 50].)

REMARK 5.5. Set $\beta=\begin{pmatrix}5&3\\8&5\end{pmatrix}\in \mathrm{GSp}^+(1, \boldsymbol{Z})$. Then $\Gamma\beta\Gamma=\Gamma\beta$. We see that β transforms any cusp P of ${}_0M$ to another cusp $\neq P$ of ${}_0M$ and that $\gamma\beta$ is not an elliptic element for any $\gamma\in\Gamma$. ($\mathrm{Tr}(\gamma\beta)\equiv 4 \pmod 6 \not\equiv 0, \pm 1 \pmod 6$.) Hence $\mathscr{X}_{{}_0M}(\Gamma\beta\Gamma)\cap\mathscr{X}_{{}_0M}(\Gamma 1_2\Gamma)=\varnothing$ and $\mathscr{X}_M(\Gamma\beta\Gamma)\cap\mathscr{X}_M(\Gamma 1_2\Gamma)=\varnothing$. From Theorem 3 we get

$$L_M(\Gamma\beta\Gamma,\ \Gamma 1_2\Gamma)=0\ .$$

Since $\#(\Gamma\backslash\Gamma\beta\Gamma)=\#(\Gamma\backslash\Gamma\beta^{\iota}\Gamma)=1$ where $\beta^{\iota}=r(\beta)\beta^{-1}=\beta^{-1}$, we have

$$L_{{}_0M}(\Gamma\beta\Gamma,\ \Gamma 1_2\Gamma)=1-\mathrm{Tr}(f_1(\Gamma\beta\Gamma)\otimes_{Z}\mathrm{id.})+1=0$$

using Hatada [8, Theorem 4]. By the Hodge decomposition, $H^1({}_0M, \boldsymbol{C})\cong S_2(\Gamma)\oplus\overline{S_2(\Gamma)}$. Note that $C(z)^6$ is a cusp form on $\mathrm{Sp}(1, \boldsymbol{Z})$ and that $\{\lambda\in\boldsymbol{C} \mid \mathrm{Re}\,\lambda=1,\ |\lambda|=1\}=\{1\}$. Hence we get the functional equation

$$C((5z+3)/(8z+5))(8z+5)^{-2}=C(z)$$

which we can prove by the other method using the transformation formula of the Dedekind η function.

REMARK 5.6. Recall $\Gamma=\Gamma(6)$. Let $p\geqq 5$ be a prime number. Write $T'(p)\underset{\mathrm{def.}}{=}\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GSp}^+(1, \boldsymbol{Z})\,\middle|\, ad-bc=p.\ a-1\equiv b\equiv c\equiv 0 \pmod 6.\right\}$ and $\alpha=\begin{pmatrix}1&0\\0&p\end{pmatrix}$. Then we have $T'(p)=\Gamma\alpha\Gamma=\bigcup_{j=1}^{p+1}\Gamma\beta_j$ (disjoint) $=\bigcup_{j=1}^{p+1}\beta_j'\Gamma$ (disjoint) with some $\{\beta_j\}_{j=1}^{p+1}$ and $\{\beta_j'\}_{j=1}^{p+1}$ in $T'(p)$. Let λ_p be the eigenvalue of the Hecke operator $[T'(p)]_2$, corresponding to $T'(p)$, acting on $S_2(\Gamma)$. Namely $(C|[T'(p)]_2)(z)=\lambda_p C(z)$. It is well known that $|\lambda_p|\leqq 2\sqrt{p}$. Let ι be the involution of $\mathrm{GSp}^+(1, \boldsymbol{Z})$ given by $x\mapsto r(x)x^{-1}$. Note $\Gamma^{\iota}=\Gamma$. We have $T'(p)^{\iota}=\bigcup_{j=1}^{p+1}\Gamma\beta_j'^{\iota}$ (disjoint). Compute $L_{{}_0M}(\Gamma\alpha\Gamma,\ \Gamma 1_2\Gamma)$ using Hatada [8, Theorem 4] and the isomorphism $H^1({}_0M, \boldsymbol{C})\cong S_2(\Gamma)\oplus\overline{S_2(\Gamma)}$.

$$L_{0M}(\Gamma\alpha\Gamma,\ \Gamma 1_2\Gamma)=(p+1)-\mathrm{Tr}(f_1(\Gamma\alpha\Gamma)\otimes_Z \mathrm{id.})+(p+1)$$
$$=2(p+1-\mathrm{Re}\ \lambda_p)\neq 0$$

since $|p+1-\mathrm{Re}\ \lambda_p|\geqq p+1-|\lambda_p|\geqq p+1-2\sqrt{p}=(\sqrt{p}-1)^2>0$. Then from Theorem 3 we get

$$\mathscr{X}_{0M}(\Gamma\alpha\Gamma,\ \Gamma 1_2\Gamma)\neq\varnothing\ .$$

Hence we have $\mathscr{X}_M(\Gamma\alpha\Gamma,\ \Gamma 1_2\Gamma)\neq\varnothing$.

References

[1] M. ARTIN, A. GROTHENDIECK and J. L. VERDIER, *Théorie des Topos et Cohomologie Étale des Schémas* (SGA 4), Lecture Notes in Math., **269**, **270**, **305** (1972-1973), Springer.

[2] A. ASH, D. MUMFORD, M. RAPOPORT and Y. TAI, *Smooth Compactification of Locally Symmetric Varieties*, Math. Sci. Press, 1975.

[3] R. W. CARTER, *Finite Groups of Lie Type*, John Wiley, 1985.

[4] C.-L. CHAI, *Compactification of Siegel Moduli Schemes*, London Math. Soc. Lecture Note Series, **107** (1985), Cambridge Univ. Press.

[5] P. DELIGNE, *Cohomologie Étale* (SGA 4½), Lecture Notes in Math., **569** (1977), Springer.

[6] R. HARTSHORNE, *Algebraic Geometry*, Springer, 1977.

[7] K. HATADA, Siegel cusp forms as holomorphic differential forms on certain compact varieties, Math. Ann., **262** (1983), 503-509.

[8] K. HATADA, Homology groups, differential forms and Hecke rings on Siegel modular varieties, *Topics in Mathematical Analysis*, (dedicated to the memory of A. L. Cauchy) (ed. T. M. Rassias), World Scientific Publ., 1989, pp. 371-409.

[9] K. HATADA, Correspondences for Hecke rings and l-adic cohomology groups on smooth compactifications of Siegel modular varieties, Proc. Japan Acad. Ser. A, **65** (1989), 62-65.

[10] J.-I. IGUSA, A desingularization problem in the theory of Siegel modular functions, Math. Ann., **168** (1967), 228-260.

[11] S. LEFSCHETZ, Intersections and transformations of complexes and manifolds, Trans. Amer. Math. Soc., **28** (1926), 1-49.

[12) S. ŁOJASIEWICZ, Triangulation of semianalytic sets, Ann. Scuola Norm. Sup. Pisa Ser. III, **18** (1964), 449-474.

[13] H. MAASS, *Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Math., **216** (1971), Springer.

[14] Y. NAMIKAWA, A new compactification of the Siegel space and degeneration of abelian varieties, I, Math. Ann., **221** (1976), 57-141; II, ibid. (1976), 201-241.

[15] Y. NAMIKAWA, *Toroidal Compactification of Siegel Spaces*, Lecture Notes in Math., **812** (1980), Springer.

[16] I. SATAKE, On the compactification of the Siegel space, J. Indian Math. Soc., **20** (1956), 259-281.

[17] G. SHIMURA, On modular correspondences for $\mathrm{Sp}(N, \boldsymbol{Z})$ and their congruence relations, Proc. Nat. Acad. Sci. U.S.A., **49** (1963), 824-828.

[18] G. SHIMURA, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten; Princeton University Press, 1971.

[19] T. SHIODA, On elliptic modular surfaces, J. Math. Soc. Japan, **24** (1972), 20-59.
[20] B. SRINIVASAN, *Representations of Finite Chevalley Groups*, Lecture Notes in Math., **764** (1979), Springer.
[21] J. W. VICK, *Homology Theory*, Academic Press, 1973.

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