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Automorphisms of Tensor Products of Irrational Rotation C^*-Algebras and the C^*-Algebra of Compact Operators

Kazunori KODAKA

Ryukyu University

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Abstract. Let θ be an irrational number and A_θ be the corresponding irrational rotation C^*-algebra. Let $\boldsymbol{K}$ be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space H. Let α be an automorphism of $A_\theta\otimes\boldsymbol{K}$ with $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes\boldsymbol{K})$. If the set of invertible elements in A_θ is dense in A_θ or α preserves the canonical dense $*$-subalgebra $F^\infty(A_\theta\otimes\boldsymbol{K})$ of $A_\theta\otimes\boldsymbol{K}$, then there are an automorphism β of A_θ and unitary elements w in the double centralizer $M(A_\theta\otimes\boldsymbol{K})$ of $A_\theta\otimes\boldsymbol{K}$ and W in $\boldsymbol{B}(H)$ such that $\alpha=\mathrm{Ad}(w)\circ(\beta\otimes\mathrm{Ad}(W))$.

§1. Preliminaries.

Let θ be an irrational number and A_θ be the corresponding irrational rotation C^*-algebra. Let u and v be unitary elements in A_θ with $uv=e^{2\pi i\theta}vu$ which generate A_θ. Let τ be the unique tracial state on A_θ. Let $\boldsymbol{K}$ (resp. $\boldsymbol{B}(H)$) be the C^*-algebra of all compact (resp. bounded) operators on a countably infinite dimensional Hilbert space H. Let $\{\varepsilon_j\}_{j\in\boldsymbol{Z}}$ be a completely orthonormal base of H and $\{e_{ij}\}_{i,j\in\boldsymbol{Z}}$ be matrix units of $\boldsymbol{K}$ with respect to $\{\varepsilon_j\}_{j\in\boldsymbol{Z}}$. Let Tr be the canonical trace on $\boldsymbol{K}$. For any C^*-algebra A we denote by $M(A)$ the double centralizer (the multiplier) of A and by A^+ the unitized C^*-algebra of A. We know $M(\boldsymbol{K})=\boldsymbol{B}(H)$. Let $\mathrm{Aut}(A)$ be the group of all automorphisms of A. Furthermore we denote by $\mathrm{tsr}(A)$ the topological stable rank of A.

§2. Automorphisms of $A_\theta\otimes\boldsymbol{K}$ with the trivial action on $K_0(A_\theta\otimes\boldsymbol{K})$.

By Riedel [6] or Anderson and Paschke [1], we can see that there is an irrational number θ such that the set of invertible elements in A_θ is dense in A_θ, i.e. $\mathrm{tsr}(A_\theta)=1$.

LEMMA 1. *Let* $\alpha\in\mathrm{Aut}(A_\theta\otimes\boldsymbol{K})$ *with* $\alpha_*=\mathrm{id}$ *on* $K_0(A_\theta\otimes\boldsymbol{K})$ *and let*

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$\mathrm{tsr}(A_\theta)=1$. *Then for any* $k\in \boldsymbol{Z}$ *there is a partial isometry* $w_k\in(A_\theta\otimes \boldsymbol{K})^+$ *such that*

$$w_k^*w_k=1\otimes e_{kk}\,,$$
$$w_kw_k^*=\alpha(1\otimes e_{kk})\,.$$

PROOF. Since $\mathrm{tsr}(A_\theta)=1$, $\mathrm{tsr}(A_\theta\otimes \boldsymbol{K})=1$ by Rieffel [8, Theorem 3.6]. Thus $(A_\theta\otimes \boldsymbol{K})^+$ has cancellation by Blackadar [2, Proposition 6.5.1]. Since $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes \boldsymbol{K})$, $[\alpha(1\otimes e_{kk})]=[1\otimes e_{kk}]$ in $K_0(A_\theta\otimes \boldsymbol{K})$ for any $k\in \boldsymbol{Z}$. Therefore for any $k\in \boldsymbol{Z}$ there is a partial isometry $w_k\in(A_\theta\otimes \boldsymbol{K})^+$ such that

$$w_k^*w_k=1\otimes e_{kk}\,,$$
$$w_kw_k^*=\alpha(1\otimes e_{kk})\,.$$

Q.E.D.

Let A be a C^*-algebra and B be its C^*-subalgebra. For any $\alpha\in\mathrm{Aut}(A)$ let $\alpha|_B$ be the monomorphism of B into A defined by $\alpha|_B(x)=\alpha(x)$ for any $x\in B$ and let $\tilde{\alpha}$ be the automorphism of $M(A)$ obtained by extending α. Moreover let

$$F(A_\theta\otimes \boldsymbol{K})=\{\textstyle\sum x_{ij}\otimes e_{ij}\mid x_{ij}\in A_\theta,\ x_{ij}=0 \text{ except for finitely many elements}\}\,.$$

LEMMA 2. *Let* α *be as in Lemma* 1. *We suppose that for any* $k\in \boldsymbol{Z}$ *there is a partial isometry* $w_k\in(A_\theta\otimes \boldsymbol{K})^+$ *such that*

$$w_k^*w_k=1\otimes e_{kk}\,,$$
$$w_kw_k^*=\alpha(1\otimes e_{kk})\,.$$

Then there is a unitary element $w\in M(A_\theta\otimes \boldsymbol{K})$ *such that* $\mathrm{Ad}(w^*)\circ\alpha|_{A_\theta\otimes e_{kk}}$ *is an automorphism of* $A_\theta\otimes e_{kk}$ *and*

$$(\mathrm{Ad}(w^*)\circ\alpha)(1\otimes e_{kk})=1\otimes e_{kk}$$

for any $k\in \boldsymbol{Z}$.

PROOF. There is a partial isometry w_k in $(A_\theta\otimes \boldsymbol{K})^+$ for any $k\in \boldsymbol{Z}$ such that

$$w_k^*w_k=1\otimes e_{kk}\,,$$
$$w_kw_k^*=\alpha(1\otimes e_{kk})$$

by the assumptions. Let (π_τ, H_τ) be the cyclic representation of A_θ associated with τ. Since A_θ is simple, π_τ is faithful. Thus we can identify $A_\theta\otimes \boldsymbol{K}$ with $\pi_\tau(A_\theta)\otimes \boldsymbol{K}$. For any $\sum_{j\in \boldsymbol{Z}}\xi_j\otimes\varepsilon_j\in H_\tau\otimes H$ and $n\geqq m\geqq 1$,

$$\begin{aligned}\Big\|\Big(\sum_{|k|=m}^{n} w_k\Big)\Big(\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\Big\|^2 &= \Big(\Big(\sum_{|k|,|l|=m}^{n} w_l^* w_k\Big)\Big(\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\Big|\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\\ &= \Big(\Big(\sum_{|k|=m}^{n} w_k^*\alpha(1\otimes e_{kk})w_k\Big)\Big(\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\Big|\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\\ &= \Big(\Big(\sum_{|k|=m}^{n} 1\otimes e_{kk}\Big)\Big(\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\Big|\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\\ &= \Big(\sum_{|k|=m}^{n}\xi_k\otimes\varepsilon_k\Big|\sum_{j\in Z}\xi_j\otimes\varepsilon_j\Big)\\ &= \sum_{|k|=m}^{n}\sum_{j\in Z}(\xi_k|\xi_j)(\varepsilon_k|\varepsilon_j)\\ &= \sum_{|k|=m}^{n}\|\xi_k\|^2 .\end{aligned}$$

Hence since $\sum_{|k|=m}^{n}\|\xi_k\|^2\to 0$ as $m, n\to\infty$, $\{\sum_{|k|\leq n} w_k\}$ is a Cauchy sequence with respect to the strong topology. Thus we can define $w=\sum_{k\in Z} w_k$. Then $w\in \boldsymbol{B}(H_\tau\otimes H)$. For any $x=\sum x_{ij}\otimes e_{ij}\in F(A_\theta\otimes \boldsymbol{K})$

$$\begin{aligned}\Big\|\sum_{|k|=m}^{n} w_k x\Big\|^2 &= \Big\|\Big(\sum_{|j|=m}^{n} x^* w_j^*\Big)\Big(\sum_{|k|=m}^{n} w_k x\Big)\Big\|\\ &= \Big\|x^*\Big(\sum_{|k|=m}^{n} 1\otimes e_{kk}\Big)x\Big\|\\ &\leqq \|x\|\,\Big\|\Big(\sum_{|k|=m}^{n} 1\otimes e_{kk}\Big)x\Big\|\\ &= \|x\|\,\Big\|\Big(\sum_{|k|=m}^{n} 1\otimes e_{kk}\Big)\Big(\sum_{i,j} x_{ij}\otimes e_{ij}\Big)\Big\|\\ &= \|x\|\,\Big\|\sum_j\sum_{|k|=m}^{n} x_{kj}\otimes e_{kj}\Big\|\to 0\end{aligned}$$

as $m, n\to\infty$. Hence $\{\sum_{|k|\leq m} w_k x\}$ is a Cauchy sequence with respect to the norm topology. Thus $wx\in A_\theta\otimes\boldsymbol{K}$ for any $x\in F(A_\theta\otimes\boldsymbol{K})$. Since $F(A_\theta\otimes\boldsymbol{K})$ is dense in $A_\theta\otimes\boldsymbol{K}$ with respect to the norm topology, for any $x\in A_\theta\otimes\boldsymbol{K}$ there is a sequence $\{x_n\}$ in $F(A_\theta\otimes\boldsymbol{K})$ such that $\|x_n-x\|\to 0$ as $n\to\infty$. Then $\|wx-wx_n\|\leqq\|w\|\,\|x-x_n\|\to 0$ as $n\to\infty$. Therefore $wx\in A_\theta\otimes\boldsymbol{K}$ for any $x\in A_\theta\otimes\boldsymbol{K}$. Similarly we obtain that $xw\in A_\theta\otimes\boldsymbol{K}$ for any $x\in A_\theta\otimes\boldsymbol{K}$. Hence $w\in M(A_\theta\otimes\boldsymbol{K})$. Furthermore we see that

$$\begin{aligned}w^*w &= \sum_{j,k\in Z} w_j^* w_k = \sum_{k\in Z} 1\otimes e_{kk} = 1\otimes 1 ,\\ ww^* &= \sum_{j,k\in Z} w_j w_k^* = \sum_{k\in Z}\alpha(1\otimes e_{kk}) = \tilde{\alpha}(1\otimes 1) = 1\otimes 1\end{aligned}$$

since $\tilde{\alpha}$ is strictly continuous by Busby [3, Proposition 3.8]. Thus w is a unitary element. And for any $k\in \boldsymbol{Z}$

$$\begin{aligned} w(1\otimes e_{kk})w^* &= \sum_{j\in \boldsymbol{Z}} w_j(1\otimes e_{kk})w^* \\ &= \sum_{j\in \boldsymbol{Z}} w_j w_j^* w_j(1\otimes e_{kk})w^* \\ &= w_k(1\otimes e_{kk})\sum_{j\in \boldsymbol{Z}} w_j^* w_j w_j^* \\ &= w_k(1\otimes e_{kk})w_k^* \\ &= \alpha(1\otimes e_{kk}) . \end{aligned}$$

Hence

$$(\mathrm{Ad}(w^*)\circ\alpha)(1\otimes e_{kk}) = 1\otimes e_{kk}$$

for any $k\in \boldsymbol{Z}$. Furthermore for any $x\in A_\theta$ and $k\in \boldsymbol{Z}$

$$\begin{aligned} (\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{kk}) &= (\mathrm{Ad}(w^*)\circ\alpha)(1\otimes e_{kk})(\mathrm{Ad}(w^*)\circ\tilde{\alpha})(x\otimes 1) \\ &\quad \times (\mathrm{Ad}(w^*)\circ\alpha)(1\otimes e_{kk}) \\ &= (1\otimes e_{kk})(\mathrm{Ad}(w^*)\circ\tilde{\alpha})(x\otimes 1)(1\otimes e_{kk}) . \end{aligned}$$

Since

$$(1\otimes e_{kk})(A_\theta\otimes \boldsymbol{B}(H))(1\otimes e_{kk}) = A_\theta\otimes e_{kk}$$

for any $k\in \boldsymbol{Z}$, we obtain that

$$(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{kk}) \in A_\theta\otimes e_{kk}$$

for any $x\in A_\theta$ and $k\in \boldsymbol{Z}$. Hence $\mathrm{Ad}(w^*)\circ\alpha$ is a monomorphism of $A_\theta\otimes e_{kk}$ into $A_\theta\otimes e_{kk}$. However if we repeat the above discussion, we can see that $\alpha^{-1}\circ\mathrm{Ad}(w)$ is a monomorphism of $A_\theta\otimes e_{kk}$ into $A_\theta\otimes e_{kk}$ and clearly

$$(\mathrm{Ad}(w^*)\circ\alpha)(\alpha^{-1}\circ\mathrm{Ad}(w)) = (\alpha^{-1}\circ\mathrm{Ad}(w))(\mathrm{Ad}(w^*)\circ\alpha) = \mathrm{id} .$$

Thus $(\mathrm{Ad}(w^*)\circ\alpha)|_{A_\theta\otimes e_{kk}}$ is an automorphism of $A_\theta\otimes e_{kk}$ for any $k\in \boldsymbol{Z}$.

Q.E.D.

Now let $\alpha\in \mathrm{Aut}(A_\theta\otimes \boldsymbol{K})$. We suppose that α satisfies the following conditions:

1) $\alpha(1\otimes e_{kk}) = 1\otimes e_{kk}$,

2) $\alpha|_{A_\theta\otimes e_{kk}} \in \mathrm{Aut}(A_\theta\otimes e_{kk})$ for any $k\in \boldsymbol{Z}$.

Then for any $k\in \boldsymbol{Z}$ there is an automorphism β of A_θ such that for any $x\in A_\theta$

$$\alpha(x\otimes e_{kk}) = \beta_k(x)\otimes e_{kk} .$$

Let V be the unitary element in $\boldsymbol{B}(H)$ defined by

$$V\varepsilon_k = \varepsilon_{k+1}$$

for any $k \in \boldsymbol{Z}$. Then $Ve_{kk}V^* = e_{k+1\,k+1}$ for any $k \in \boldsymbol{Z}$.

LEMMA 3. *Let $\alpha \in \mathrm{Aut}(A_\theta \otimes \boldsymbol{K})$ satisfy the above conditions and β_k, $k \in \boldsymbol{Z}$ be as above. Then there are a unitary element $w \in M(A_\theta \otimes \boldsymbol{K})$ and an automorphism β of A_θ such that for any $x \in A_\theta$ and $k \in \boldsymbol{Z}$*

$$(\mathrm{Ad}(w^*) \circ \alpha)(x \otimes e_{kk}) = \beta(x) \otimes e_{kk} .$$

PROOF. Let $\tilde{\alpha}$ be the automorphism of $M(A_\theta \otimes \boldsymbol{K})$ obtained by extending α. Let V be as above. Since $\alpha(1 \otimes e_{kk}) = 1 \otimes e_{kk}$ for any $k \in \boldsymbol{Z}$,

$$\tilde{\alpha}(1 \otimes V)(1 \otimes e_{kk}) = (1 \otimes e_{k+1\,k+1})\tilde{\alpha}(1 \otimes V)$$

for any $k \in \boldsymbol{Z}$. And for any $x \in A_\theta$

$$\tilde{\alpha}(x \otimes 1)\tilde{\alpha}(1 \otimes V) = \tilde{\alpha}(1 \otimes V)\tilde{\alpha}(x \otimes 1) .$$

Furthermore

$$x \otimes 1 = \sum_{k \in \boldsymbol{Z}} x \otimes e_{kk}$$

where the summation is taken with respect to the strict topology. Hence we have

$$\tilde{\alpha}(x \otimes 1) = \sum_{k \in \boldsymbol{Z}} \beta_k(x) \otimes e_{kk}$$

since $\tilde{\alpha}$ is strictly continuous. Thus

$$\begin{aligned}
\Big(\sum_{k \in \boldsymbol{Z}} \beta_k(x) \otimes e_{kk}\Big)\tilde{\alpha}(1 \otimes V) &= \tilde{\alpha}(1 \otimes V)\Big(\sum_{k \in \boldsymbol{Z}} \beta_k(x) \otimes e_{kk}\Big) \\
&= \sum_{k \in \boldsymbol{Z}} \tilde{\alpha}(1 \otimes V)(\beta_k(x) \otimes e_{kk}) \\
&= \sum_{k \in \boldsymbol{Z}} \tilde{\alpha}(1 \otimes V)(1 \otimes e_{kk})(\beta_k(x) \otimes e_{kk}) \\
&= \sum_{k \in \boldsymbol{Z}} (1 \otimes e_{k+1\,k+1})\tilde{\alpha}(1 \otimes V)(\beta_k(x) \otimes e_{kk})
\end{aligned}$$

since

$$\tilde{\alpha}(1 \otimes V)(1 \otimes e_{kk}) = (1 \otimes e_{k+1\,k+1})\tilde{\alpha}(1 \otimes V)$$

for any $k \in \boldsymbol{Z}$. Since $\tilde{\alpha}(1 \otimes V)$ is a unitary element, we obtain that

$$\sum_{k \in \boldsymbol{Z}} \beta_k(x) \otimes e_{kk} = \sum_{k \in \boldsymbol{Z}} (1 \otimes e_{k+1\,k+1})\tilde{\alpha}(1 \otimes V)(\beta_k(x) \otimes e_{kk})\tilde{\alpha}(1 \otimes V)^* .$$

Hence for any $k \in \boldsymbol{Z}$

$$\begin{aligned}\beta_k(x)\otimes e_{kk} &= (1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(\beta_{k-1}(x)\otimes e_{k-1\,k-1})\tilde{\alpha}(1\otimes V)^*(1\otimes e_{kk}) \\ &= (1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(\beta_{k-1}(x)\otimes e_{kk})(1\otimes V)\tilde{\alpha}(1\otimes V)^*(1\otimes e_{kk}) \\ &= (1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk})(\beta_{k-1}(x)\otimes e_{kk}) \\ &\quad \times(1\otimes e_{kk})(1\otimes V)\tilde{\alpha}(1\otimes V)^*(1\otimes e_{kk})^* .\end{aligned}$$

Since $(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk})$ is in $\boldsymbol{B}(H_\tau)\otimes e_{kk}\cap M(A_\theta\otimes\boldsymbol{K})$, we see that

$$(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk})\in A_\theta\otimes e_{kk} ,$$

which is a unitary element in $A_\theta\otimes e_{kk}$. Hence for any $k\in\boldsymbol{Z}$ there is a unitary element $y_k\in A_\theta$ such that

$$\beta_k(x)=y_k\beta_{k-1}(x)y_k^*$$

for any $x\in A_\theta$. Let $\beta=\beta_0$ and

$$w_k=\begin{cases} y_k y_{k-1}\cdots y_1 & \text{if } k>0 \\ 1 & \text{if } k=0 \\ y_{k+1}^* y_{k+2}^*\cdots y_0^* & \text{if } k<0 .\end{cases}$$

Then $w_k\in A_\theta$ and $\beta_k=\mathrm{Ad}(w_k)\circ\beta$ for any $k\in\boldsymbol{Z}$. Let $w=\sum_{k\in\boldsymbol{Z}} w_k\otimes e_{kk}$ where the summation is taken with respect to the strong topology. For any $x=\sum x_{ij}\otimes e_{ij}\in F(A_\theta\otimes\boldsymbol{K})$ and $n\geqq m\geqq 1$,

$$\begin{aligned}\left\|\left(\sum_{|k|=m}^{n} w_k\otimes e_{kk}\right)x\right\|^2 &= \left\|x^*\left(\sum_{|j|=m}^{n} w_j^*\otimes e_{jj}\right)\left(\sum_{|k|=m}^{n} w_k\otimes e_{kk}\right)x\right\| \\ &\leqq \|x\|\left\|\left(\sum_{|k|=m}^{n} 1\otimes e_{kk}\right)\left(\sum_{i,j} x_{ij}\otimes e_{ij}\right)\right\| \\ &= \|x\|\left\|\sum_j \sum_{|k|=m}^{n} x_{kj}\otimes e_{kj}\right\| \\ &\to 0 \quad \text{as } m, n\to\infty .\end{aligned}$$

Hence $\{\sum_{|k|\leq n} w_k x\}$ is a Cauchy sequence with respect to the norm topology. Thus $wx\in A_\theta\otimes\boldsymbol{K}$ for any $x\in F(A_\theta\otimes\boldsymbol{K})$. For any $x\in A_\theta\otimes\boldsymbol{K}$ there is a sequence $\{x_n\}$ in $F(A_\theta\otimes\boldsymbol{K})$ such that $\|x_n-x\|\to 0$ as $n\to\infty$. Then $\|wx-wx_n\|\leqq\|x-x_n\|\to 0$ as $n\to\infty$. Hence $wx\in A_\theta\otimes\boldsymbol{K}$ for any $x\in A_\theta\otimes\boldsymbol{K}$. Similarly $xw\in A_\theta\otimes\boldsymbol{K}$ for any $x\in A_\theta\otimes\boldsymbol{K}$. Therefore $w\in M(A_\theta\otimes\boldsymbol{K})$. Furthermore for any $x\in A_\theta$ and $k\in\boldsymbol{Z}$

$$\begin{aligned}(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{kk}) &= w^*\alpha(x\otimes e_{kk})w \\ &= w^*(\beta_k(x)\otimes e_{kk})w \\ &= w^*(w_k\beta(x)w_k^*\otimes e_{kk})w\end{aligned}$$

$$= w^*(w_k\beta(x)w_k^*\otimes e_{kk})(\sum_{j\in \boldsymbol{Z}} w_j\otimes e_{jj})$$
$$= w^*(w_k\beta(x)\otimes e_{kk})$$
$$= (\sum_{j\in \boldsymbol{Z}} w_j^*\otimes e_{jj})(w_k\beta(x)\otimes e_{kk})$$
$$= \beta(x)\otimes e_{kk}\,. \qquad \text{Q.E.D.}$$

LEMMA 4. *Let* $\alpha\in \mathrm{Aut}(A_\theta\otimes \boldsymbol{K})$. *We suppose that there is an automorphism* β *of* A_θ *such that*

$$\alpha(x\otimes e_{kk}) = \beta(x)\otimes e_{kk}$$

for any $x\in A_\theta$ *and* $k\in \boldsymbol{Z}$. *Then there is a unitary element* $W\in \boldsymbol{B}(H)$ *such that*

$$\alpha = \beta\otimes \mathrm{Ad}(W)\,.$$

PROOF. Let $\tilde{\alpha}$ and V be as in Lemma 3. For any $k\in \boldsymbol{Z}$

$$\tilde{\alpha}(1\otimes V)\tilde{\alpha}(1\otimes e_{kk}) = (1\otimes e_{k+1\,k+1})\tilde{\alpha}(1\otimes V)$$

and for any $x\in A_\theta$

$$\tilde{\alpha}(1\otimes V)\tilde{\alpha}(x\otimes 1) = \tilde{\alpha}(x\otimes 1)\tilde{\alpha}(1\otimes V)\,.$$

Thus

$$\tilde{\alpha}(1\otimes V)\tilde{\alpha}(\sum_{k\in \boldsymbol{Z}} x\otimes e_{kk}) = \tilde{\alpha}(\sum_{k\in \boldsymbol{Z}} x\otimes e_{kk})\tilde{\alpha}(1\otimes V)\,.$$

Since $\sum_{k\in \boldsymbol{Z}} x\otimes e_{kk}$ is taken with respect to the strict topology and $\tilde{\alpha}$ is strictly continuous, we have that

$$\sum_{k\in \boldsymbol{Z}} \tilde{\alpha}(1\otimes V)\tilde{\alpha}(x\otimes e_{kk}) = \sum_{k\in \boldsymbol{Z}} \tilde{\alpha}(x\otimes e_{kk})\tilde{\alpha}(1\otimes V)\,.$$

Hence

$$\sum_{k\in \boldsymbol{Z}} \tilde{\alpha}(1\otimes V)(\beta(x)\otimes e_{kk}) = \sum_{k\in \boldsymbol{Z}} (\beta(x)\otimes e_{kk})\tilde{\alpha}(1\otimes V)\,.$$

Thus

$$\tilde{\alpha}(1\otimes V)(\beta(x)\otimes e_{k-1\,k-1}) = (\beta(x)\otimes e_{kk})\tilde{\alpha}(1\otimes V)$$

for any $k\in \boldsymbol{Z}$ since

$$\tilde{\alpha}(1\otimes V)(1\otimes e_{kk}) = (1\otimes e_{k+1\,k+1})\tilde{\alpha}(1\otimes V)\,.$$

Therefore for any $k\in \boldsymbol{Z}$

$$\tilde{\alpha}(1\otimes V)(1\otimes V)^*(\beta(x)\otimes e_{kk})(1\otimes V)=(\beta(x)\otimes e_{kk})\tilde{\alpha}(1\otimes V) .$$

Hence

$$(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk})(\beta(x)\otimes e_{kk}) \\ =(\beta(x)\otimes e_{kk})(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk}) .$$

Clearly

$$(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk}) \in M(A_\theta\otimes \boldsymbol{K})\cap(\boldsymbol{B}(H_\tau)\otimes e_{kk}) \\ =A_\theta\otimes e_{kk} .$$

And since x is an arbitrary element in A_θ, we see that

$$(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk}) \in (A_\theta\otimes e_{kk})'$$

for any $k\in \boldsymbol{Z}$. Hence for any $k\in \boldsymbol{Z}$ there is a $\lambda_k\in \boldsymbol{C}$ such that

$$(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)(1\otimes V)^*(1\otimes e_{kk})=\lambda_k(1\otimes e_{kk}) .$$

Thus we have that

$$\tilde{\alpha}(1\otimes V)(1\otimes e_{k-1\,k-1})=\lambda_k(1\otimes V)(1\otimes e_{k-1\,k-1})$$

for any $k\in \boldsymbol{Z}$ since

$$(1\otimes e_{kk})\tilde{\alpha}(1\otimes V)=\tilde{\alpha}(1\otimes V)(1\otimes e_{k-1\,k-1}) .$$

Then

$$\alpha(1\otimes e_{10})=\alpha(1\otimes Ve_{00}) \\ =\tilde{\alpha}(1\otimes V)(1\otimes e_{00}) \\ =\lambda_1(1\otimes V)(1\otimes e_{00})\in 1\otimes \boldsymbol{K} .$$

Similarly for any $m, n\in \boldsymbol{Z}$

$$\alpha(1\otimes e_{mn})=\alpha(1\otimes V^m e_{00}V^{-n})\in 1\otimes \boldsymbol{K} .$$

Since $\{1\otimes e_{mn}\mid m, n\in \boldsymbol{Z}\}$ generate $1\otimes \boldsymbol{K}$, $\alpha|_{1\otimes K}$ is an automorphism of $1\otimes \boldsymbol{K}$. Therefore there is a unitary element $W\in \boldsymbol{B}(H)$ such that

$$\alpha|_{1\otimes K}=\mathrm{Ad}(1\otimes W) .$$

For any $x\in A_\theta$ and $X\in \boldsymbol{K}$

$$\alpha(x\otimes X)=\alpha(\sum_{k\in \boldsymbol{Z}}(x\otimes e_{kk})(1\otimes X))$$

where the summation is taken with respect to the strict topology in

$M(A_\theta \otimes \boldsymbol{K})$. The automorphism $\tilde{\alpha}$ of $M(A_\theta \otimes \boldsymbol{K})$ is strictly continuous. Hence

$$\begin{aligned}\alpha(x\otimes X) &= \sum_{k\in \boldsymbol{Z}} \tilde{\alpha}(x\otimes e_{kk})\tilde{\alpha}(1\otimes X)\\ &= \sum_{k\in \boldsymbol{Z}} \alpha(x\otimes e_{kk})\alpha(1\otimes X)\\ &= (\sum_{k\in \boldsymbol{Z}} \beta(x)\otimes e_{kk})(1\otimes W)(1\otimes X)(1\otimes W)^*\\ &= (\beta\otimes \mathrm{Ad}(W))(x\otimes X)\ .\end{aligned}$$

Thus $\alpha(x)=(\beta\otimes \mathrm{Ad}(W))(x)$ for any $x\in A_\theta\otimes \boldsymbol{K}$. Q.E.D.

THEOREM 5. *Let $\alpha\in \mathrm{Aut}(A_\theta\otimes \boldsymbol{K})$ with $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes \boldsymbol{K})$ and let* $\mathrm{tsr}(A_\theta)=1$. *Then there are an automorphism β of A_θ and unitary elements $w\in M(A_\theta\otimes \boldsymbol{K})$, $W\in \boldsymbol{B}(H)$ such that*

$$\alpha=\mathrm{Ad}(w)\circ(\beta\otimes \mathrm{Ad}(W))\ .$$

PROOF. This is immediate by Lemmas 1, 2, 3 and 4. Q.E.D.

§ 3. Automorphisms of $A_\theta\otimes \boldsymbol{K}$ with the trivial action on $K_0(A_\theta\otimes \boldsymbol{K})$ and preserving $F^\infty(A_\theta\otimes \boldsymbol{K})$.

Let A_θ^∞ be the dense $*$-subalgebra of smooth elements of A_θ with respect to the canonical action of the two dimensional torus. Let

$$F^\infty(A_\theta\otimes \boldsymbol{K})=\{\sum_{i,j\in \boldsymbol{Z}} x_{ij}\otimes e_{ij}\in F(A_\theta\otimes \boldsymbol{K})\ |\ x_{ij}\in A_\theta^\infty\}\ .$$

LEMMA 6. *Let $\alpha\in \mathrm{Aut}(A_\theta\otimes \boldsymbol{K})$ with $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes \boldsymbol{K})$. We suppose that $\alpha(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K})$. Then there is a unitary element $w\in M(A_\theta\otimes \boldsymbol{K})$ such that*

1) $(\mathrm{Ad}(w^*)\circ\alpha)(1\otimes e_{kk})=1\otimes e_{kk}$,
2) $\mathrm{Ad}(w^*)\circ\alpha|_{A_\theta\otimes e_{kk}}$ *is an automorphism of $A_\theta\otimes e_{kk}$ for any $k\in \boldsymbol{Z}$,*
2) $(\mathrm{Ad}(w^*)\circ\alpha)(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K})$.

PROOF. Since $\alpha(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K})$, for any $k\in \boldsymbol{Z}$ there is an $n\in \boldsymbol{N}$ such that $1\otimes e_{kk}$ and $\alpha(1\otimes e_{kk})$ are in $M_n(A_\theta^\infty)$ where $M_n(A_\theta^\infty)$ is the $n\times n$-matrix algebra over A_θ^∞. By Rieffel [8] A_θ^∞ has cancellation. Hence for any $k\in \boldsymbol{Z}$ there is a partial isometry $w_k\in M_n(A_\theta^\infty)\subset F^\infty(A_\theta\otimes \boldsymbol{K})$ such that

$$\begin{aligned}w_k^* w_k &= 1\otimes e_{kk}\ ,\\ w_k w_k^* &= \alpha(1\otimes e_{kk})\ .\end{aligned}$$

And in the same way as in Lemma 2 let $w=\sum_{k\in Z} w_k$. Then we can easily obtain 1) and 2). And for any $x=\sum_{i,j} x_{ij}\otimes e_{ij}\in F^\infty(A_\theta\otimes \boldsymbol{K})$

$$\begin{aligned} wxw^* &= (\sum_{k\in Z} w_k)(\sum_{i,j} x_{ij}\otimes e_{ij})w^* \\ &= (\sum_{i,j} w_i x_{ij}\otimes e_{ij})(\sum_{k\in Z} w_k^*) \\ &= \sum_{i,j} w_i x_{ij} w_j^* \otimes e_{ij} \in F^\infty(A_\theta\otimes \boldsymbol{K}) . \end{aligned}$$

Hence we obtain 3). Q.E.D.

Let $\alpha\in \mathrm{Aut}(A_\theta\otimes \boldsymbol{K})$. We suppose that α satisfies the following conditions:

1) $\alpha(1\otimes e_{kk})=1\otimes e_{kk}$,
2) $\alpha|_{A_\theta\otimes e_{kk}}$ is an automorphism of $A_\theta\otimes e_{kk}$ for any $k\in \boldsymbol{Z}$,
3) $\alpha(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K})$.

Let $A_\theta^\infty\otimes e_{kk}=\{x\otimes e_{kk} \mid x\in A_\theta^\infty\}$. Then $\alpha(A_\theta^\infty\otimes e_{kk})=A_\theta^\infty\otimes e_{kk}$ for any $k\in \boldsymbol{Z}$ by 2), 3). Hence since $A_\theta\otimes e_{kk}$ is isomorphic to A_θ, for any $k\in \boldsymbol{Z}$ there is an automorphism β_k of A_θ such that for any $x\in A_\theta$

$$\beta_k(A_\theta^\infty)=A_\theta^\infty ,$$
$$\alpha(x\otimes e_{kk})=\beta_k(x)\otimes e_{kk} .$$

LEMMA 7. *Let $\alpha\in \mathrm{Aut}(A_\theta\otimes \boldsymbol{K})$ satisfy the above conditions. Let β_k, $k\in \boldsymbol{Z}$ be as above. Then there are a unitary element $w\in M(A_\theta\otimes \boldsymbol{K})$ and an automorphism β of A_θ with $\beta(A_\theta^\infty)=A_\theta^\infty$ such that*

$$(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{kk})=\beta(x)\otimes e_{kk}$$

for any $x\in A_\theta$ and $k\in \boldsymbol{Z}$ and that

$$(\mathrm{Ad}(w^*)\circ\alpha)(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K}) .$$

PROOF. Let V be as in Lemma 3. In the same way as in Lemma 3 we obtain that for any $k\in \boldsymbol{Z}$ there is a unitary element $y_k\in A_\theta$ such that

$$\beta_k(x)=y_k\beta_{k-1}(x)y_k^*$$

for any $x\in A_\theta$. Since $\beta_k(A_\theta^\infty)=A_\theta^\infty$ for any $k\in \boldsymbol{Z}$, $y_kA_\theta^\infty y_k^*=A_\theta^\infty$ for any $k\in \boldsymbol{Z}$. Let β, w_k ($k\in \boldsymbol{Z}$) and w be as in Lemma 3. By the definition of w_k, $w_kA_\theta^\infty w_k^*=A_\theta^\infty$ for any $k\in \boldsymbol{Z}$. Therefore

$$\mathrm{Ad}(w)(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K}) .$$

Hence we obtain the conclusion in the same way as in Lemma 3.

Q.E.D.

Let

$$F(\boldsymbol{K})=\{\sum c_{ij}e_{ij} \mid c_{ij}\in \boldsymbol{C},\ c_{ij}=0 \text{ except for finitely many elements}\}$$

and let R_τ be the linear map of $F(A_\theta\otimes \boldsymbol{K})$ to $F(\boldsymbol{K})$ defined by

$$R_\tau(\sum x_{ij}\otimes e_{ij})=\sum \tau(x_{ij})e_{ij}$$

for any $\sum x_{ij}\otimes e_{ij}\in F(A_\theta\otimes \boldsymbol{K})$. By Tomiyama [11] R_τ can be extended to a bounded linear map of $A_\theta\otimes \boldsymbol{K}$ onto $\boldsymbol{K}$. We also denote it by R_τ.

LEMMA 8. *With the above notations*

$$F^\infty(A_\theta\otimes \boldsymbol{K})\cap 1\otimes \boldsymbol{K}=1\otimes F(\boldsymbol{K}) .$$

PROOF. It is clear that

$$F^\infty(A_\theta\otimes \boldsymbol{K})\cap 1\otimes \boldsymbol{K}\supset 1\otimes F(\boldsymbol{K}) .$$

We suppose that $x\in F^\infty(A_\theta\otimes \boldsymbol{K})\cap 1\otimes \boldsymbol{K}$. Then we can write

$$x=\sum_{i,j} x_{ij}\otimes e_{ij}=1\otimes X$$

where $\sum x_{ij}\otimes e_{ij}\in F^\infty(A_\theta\otimes \boldsymbol{K})$ and $X\in \boldsymbol{K}$. Hence

$$X=R_\tau(1\otimes X)=\sum_{i,j} \tau(x_{ij})e_{ij} .$$

Thus $X\in F(\boldsymbol{K})$. Therefore $x\in 1\otimes F(\boldsymbol{K})$. Q.E.D.

THEOREM 9. *Let $\alpha\in \mathrm{Aut}(A_\theta\otimes \boldsymbol{K})$ with $\alpha_*=\mathrm{id}$ on $K_0(A_\theta\otimes \boldsymbol{K})$. We suppose that $\alpha(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K})$. Then there are an automorphism β of A_θ with $\beta(A_\theta^\infty)=A_\theta^\infty$ and unitary elements $w\in M(A_\theta\otimes \boldsymbol{K})$, $W\in \boldsymbol{B}(H)$ with $WF(\boldsymbol{K})W^*=F(\boldsymbol{K})$ such that*

$$\alpha=\mathrm{Ad}(w)\circ(\beta\otimes \mathrm{Ad}(W)) .$$

PROOF. By Lemmas 6 and 7 we can see that there are an automorphism β of A_θ with $\beta(A_\theta^\infty)=A_\theta^\infty$ and a unitary element $w\in M(A_\theta\otimes \boldsymbol{K})$ such that

$$(\mathrm{Ad}(w^*)\circ\alpha)(x\otimes e_{kk})=\beta(x)\otimes e_{kk}$$

for any $x\in A_\theta$ and $k\in \boldsymbol{Z}$ and that

$$(\mathrm{Ad}(w^*)\circ\alpha)(F^\infty(A_\theta\otimes \boldsymbol{K}))=F^\infty(A_\theta\otimes \boldsymbol{K}) .$$

Hence $\mathrm{Ad}(w^*)\circ\alpha$ satisfies the assumptions of Lemma 4. Thus there is a unitary element $W\in \boldsymbol{B}(H)$ such that

$$\mathrm{Ad}(w^*)\circ\alpha=\beta\otimes\mathrm{Ad}(W)\ .$$

Therefore

$$\alpha=\mathrm{Ad}(w)\circ(\beta\otimes\mathrm{Ad}(W))\ .$$

Furthermore for any $X\in F(\boldsymbol{K})$, we have that $1\otimes X\in F^{\infty}(A_\theta\otimes\boldsymbol{K})$. Since $(\mathrm{Ad}(w^*)\circ\alpha)(F^{\infty}(A_\theta\otimes\boldsymbol{K}))=F^{\infty}(A_\theta\otimes\boldsymbol{K})$,

$$\begin{aligned}(\mathrm{id}\otimes\mathrm{Ad}(W))(1\otimes X)&=(\beta\otimes\mathrm{Ad}(W))(1\otimes X)\\ &=(\mathrm{Ad}(w^*)\circ\alpha)(1\otimes X)\in F^{\infty}(A_\theta\otimes\boldsymbol{K})\ .\end{aligned}$$

Hence $1\otimes WXW^*\in F^{\infty}(A_\theta\otimes\boldsymbol{K})\cap 1\otimes\boldsymbol{K}$. Thus $1\otimes WXW^*\in 1\otimes F(\boldsymbol{K})$ by Lemma 8. Therefore $WXW^*\in F(\boldsymbol{K})$.

References

[1] J. ANDERSON and W. PASCHKE, The rotation algebra, Preprint Series, M.S.R.I. Berkeley, February (1985).

[2] B. BLACKADAR, *K-theory for Operator Algebras*, M.S.R.I. Publications, Springer-Verlag, 1986.

[3] R. C. BUSBY, Double centralizers and extensions of C^*-algebras, Trans. Amer. Math. Soc., **132** (1968), 79-99.

[4] G. A. ELLIOTT, The diffeomorphism group of the irrational rotation C^*-algebra, C. R. Math. Rep. Acad. Sci. Canada, **8** (1986), 329-334.

[5] G. K. PEDERSEN, *C^*-Algebras and their Automorphism Groups*, Academic Press, 1979.

[6] N. RIEDEL, On the topological stable rank of irrational rotation algebras, J. Operator Theory, **13** (1985), 143-150.

[7] M. A. RIEFFEL, C^*-algebras associated with irrational rotations, Pacific J. Math., **93** (1981), 415-429.

[8] ———, Dimension and stable rank in the K-theory of C^*-algebras, Proc. London Math. Soc., **46** (1983), 301-333.

[9] ———, The cancellation theorem for projective modules over irrational rotation C^*-algebras, Proc. London Math. Soc., **47** (1983), 285-302.

[10] M. TAKESAKI, *Theory of Operator Algebras I*, Springer-Verlag, 1979.

[11] J. TOMIYAMA, Applications of Fubini type theorem to the tensor products of C^*-algebras, Tohoku Math. J., **19** (1967), 213-226.

Present Address:
DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, RYUKYU UNIVERSITY
NISHIHARA-CHO, OKINAWA 903-01, JAPAN