TOKYO J. MATH.
VOL. 14, NO. 2, 1991

Commuting Involutions of Semisimple Groups

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Introduction.

In Oshima and Sekiguchi [OS1], a class of (non-Riemannian) symmetric spaces called those of "K_ε-type" or "$\mathfrak{k}_\varepsilon$-type" were introduced and analysis on such symmetric spaces were developed. It is an interesting problem to obtain a geometric characterization of symmetric spaces of $\mathfrak{k}_\varepsilon$-type. On the other hand, B.-Y. Chen and T. Nagano [CN] studied totally geodesic submanifolds of compact symmetric spaces by using the $(M_+(p), M_-(p))$-method. There is a similarity between the classification in [OS1, Appendix, Table 2] and [CN, p. 415, Tables I–III]. The motivation of our study is to clarify this similarity.

We are going to explain the results of this paper briefly. In this paper, a symmetric space means a coset space G/G^σ, where G is a connected semisimple Lie group, σ is an involution of G and $G^\sigma=\{g\in G\,;\,\sigma(g)=g\}$. And G and G^σ are not necessarily compact. Then, due to Berger [B], there is a Cartan involution θ of G commuting with σ. Moreover, $X=G/G^\sigma$ is regarded as a vector bundle over $M=G^\theta/G^{\theta,\sigma}$ $(G^{\theta,\sigma}=G^\theta\cap G^\sigma)$. For an involution τ of X, Y denotes the fixed point set of τ in X and N denotes the fixed point set of $\tau|M$ in M. Then Y is a symmetric space, N is a compact symmetric space and Y is a vector bundle over N. For the involution τ of X, we can choose an involution of G commuting with σ. We note here that studying N in Y is regarded as a generalization of the $(M_+(p), M_-(p))$-method of Chen-Nagano [CN]. There arises naturally a pair of involutions of G from the symmetric space X and its symmetric subspace Y. Conversing this argument, we shall study the relations among the symmetric spaces X, Y, M, N defined from a pair of involutions of G.

In this paper, we restrict our attention to studying a commuting pair of involutions of G in the following two cases: (1) the case where $X=G/G^\sigma$ is a complexification of $M=G^\theta/G^{\theta,\sigma}$ (then $X\simeq TM$, the tangent bundle to M as G^θ-space) and (2) the case where $X\simeq T^*M$, the cotangent bundle to M (then M is called a symmetric R-space which was studied in Nagano [N]). As a corollary of the case (1), we can give a geometric characterization of symmetric pairs of $\mathfrak{k}_\varepsilon$-type. As the main result of the study of the

Received November 21, 1990

case (2), we get an analogue of Borel embedding for a certain class of symmetric spaces (Theorem 4). A result similar to Theorem 4 was obtained by B. O. Makarevič [M].

§1. Let G be a connected real semisimple Lie group with finite center and let $\mathfrak{g}$ be its Lie algebra. In this paper, we denote an involution of G and the induced involution of $\mathfrak{g}$ by the same letter for the sake of simplicity.

Let θ, σ, τ be involutions of G commuting with each other. Denote $\mathfrak{g}^{\pm\alpha} = \{x \in \mathfrak{g}\,;\, \alpha(x) = \pm x\}$ for $\alpha \in \{\theta, \sigma, \tau\}$, denote $\mathfrak{g}^{\pm\alpha, \pm\beta} = \mathfrak{g}^{\pm\alpha} \cap \mathfrak{g}^{\pm\beta}$ for $\alpha, \beta \in \{\theta, \sigma, \tau\}$ and denote $\mathfrak{g}^{\pm\theta, \pm\sigma, \pm\tau} = \mathfrak{g}^{\pm\theta} \cap \mathfrak{g}^{\pm\sigma} \cap \mathfrak{g}^{\pm\tau}$. In the group case, denote $G^{\alpha} = \{x \in G\,;\, \alpha(x) = \pm x\}$ for $\alpha \in \{\theta, \sigma, \tau\}$, denote $G^{\alpha,\beta} = G^{\alpha} \cap G^{\beta}$ for $\alpha, \beta \in \{\theta, \sigma, \tau\}$ and denote $G^{\theta,\sigma,\tau} = G^{\theta} \cap G^{\sigma} \cap G^{\tau}$.

For an involution σ of $\mathfrak{g}$, $(\mathfrak{g}, \mathfrak{g}^{\sigma})$ is a symmetric pair. M. Berger [B] introduced the associated pair and the dual pair of $(\mathfrak{g}, \mathfrak{g}^{\sigma})$. In this paper, following the notation in [OS2, §1(1.2)–(1.4)], $(\mathfrak{g}, \mathfrak{g}^{\sigma})^{a}$ and $(\mathfrak{g}, \mathfrak{g}^{\sigma})^{d}$ are the associated pair and the dual pair of $(\mathfrak{g}, \mathfrak{g}^{\sigma})$, respectively. Moreover, we use the notation $(\mathfrak{g}, \mathfrak{g}^{\sigma})^{ad}$, $(\mathfrak{g}, \mathfrak{g}^{\sigma})^{da}$, $(\mathfrak{g}, \mathfrak{g}^{\sigma})^{ada}$ $(=(\mathfrak{g}, \mathfrak{g}^{\sigma})^{dad})$ as in [OS2].

In the subsequent discussion, we frequently assume the existence of a Cartan involution commuting with given commuting involutions. This assumption is a consequence of the next lemma.

LEMMA 1. *If σ, τ are involutions of $\mathfrak{g}$ commuting with each other, there is a Cartan involution θ of G commuting with both σ, τ.*

After establishing this lemma, we are pointed out by T. Kobayashi that Lemma 1 is a special case of a more genral statement (cf. [H]).

§2. Let τ be an involution of G and let θ be a Cartan involution of G commuting with τ. Assume that G^{τ} is not compact. Then, due to [B], G/G^{τ} is regarded as a vector bundle over $G^{\theta}/G^{\theta,\tau}$. We explain this fact for later use. First introduce an equivalence relation "$\sim$" on $G^{\theta} \times \mathfrak{g}^{-\theta,-\tau}$ as follows:

$$(k, x) \sim (k', x') \quad \text{if and only if} \quad k(\exp x)G^{\tau} = k'(\exp x')G^{\tau}\,.$$

From the definition, we find that $(k, x) \sim (k', x')$ if and only if there is $m \in G^{\theta,\tau}$ such that $k' = km$, $x' = \mathrm{Ad}(m^{-1})x$. Then the relation "$\sim$" actually becomes an equivalence relation and $(G^{\theta} \times \mathfrak{g}^{-\theta,-\tau})/\sim \,\simeq G/G^{\tau}$. Now write $[k, x]$ for the equivalence class of (k, x). Defining $\pi([k, x]) = kG^{\theta,\tau}$, we find that π is a projection from G/G^{τ} to $G^{\theta}/G^{\theta,\tau}$ and in this way, G/G^{τ} is a vector bundle over $G^{\theta}/G^{\theta,\tau}$. From the definition, π is G^{θ}-equivariant.

Now let θ, σ, τ be involutions of G commuting with each other. Assume that θ is a Cartan involution of G. Then we have the following diagram of inclusions and projections:

$$\begin{tikzcd}[column sep=small]
G^{\theta\sigma}/G^{\theta\sigma,\tau} \arrow[rrr, hook, bend left] \arrow[dr] & & G^{\sigma}/G^{\sigma,\tau} \arrow[r, hook] \arrow[dl] & G/G^{\tau} \arrow[d] & G^{\sigma\tau}/G^{\sigma,\tau} \arrow[l, hook'] \arrow[dr] & & G^{\theta\sigma\tau}/G^{\theta\sigma,\tau} \arrow[lll, hook', bend right] \arrow[dl] \\
 & G^{\theta,\sigma}/G^{\theta,\sigma,\tau} \arrow[rr, hook] & & G^{\theta}/G^{\theta,\tau} & & G^{\theta,\sigma\tau}/G^{\theta,\sigma,\tau} \arrow[ll, hook'] & \\
G^{\theta\sigma}/G^{\theta\sigma,\theta\tau} \arrow[rrr, hook, bend right] \arrow[ur] & & G^{\sigma}/G^{\sigma,\theta\tau} \arrow[r, hook] \arrow[ul] & G/G^{\theta\tau} \arrow[u] & G^{\sigma\tau}/G^{\theta\sigma,\theta\tau} \arrow[l, hook'] \arrow[ur] & & G^{\theta\sigma\tau}/G^{\sigma,\theta\tau} \arrow[lll, hook', bend left] \arrow[ul]
\end{tikzcd}$$

DIAGRAM 1

Remark to the above diagram: G^{θ} is a maximal compact subgroup of G. The symmetric space G/G^{τ} is a vector bundle over $G^{\theta}/G^{\theta,\tau}$ as explained before. The symmetric spaces G/G^{τ} and $G/G^{\theta\tau}$ are mutually *associated* (cf. [B], [OS2, p. 437]). Both $G^{\theta\sigma}/G^{\theta\sigma,\tau}$ and $G^{\sigma}/G^{\sigma,\tau}$ are regarded as closed subspaces of G/G^{τ} and $G^{\theta,\sigma}/G^{\theta,\sigma,\tau}$ is regarded as a closed subspace of $G^{\theta}/G^{\theta,\tau}$. Moreover $G^{\theta\sigma}/G^{\theta\sigma,\tau}$ and $G^{\sigma}/G^{\sigma,\tau}$ are vector bundles over $G^{\theta,\sigma}/G^{\theta,\sigma,\tau}$.

In the sequel, we shall treat the case where G has a complex structure and both σ, τ are holomorphic involutions of G in §3 and the case where $G^{\theta}/G^{\theta,\tau}$ is a symmetric R-space in §4.

§**3.** In this section, we always assume that $\mathfrak{g}$ is a complex semisimple Lie algebra and G is a corresponding connected complex semisimple Lie group. The purpose of this section is to obtain a geometric characterization of a symmetric space of $\mathfrak{k}_{\varepsilon}$-type introduced in [OS1].

If θ is a Cartan involution of G, G^{θ} is a maximal compact subgroup of G and $\mathfrak{g}^{\theta}$ is a compact real form of $\mathfrak{g}$.

In the sequel of this section, σ, τ are $\boldsymbol{C}$-linear involutions of $\mathfrak{g}$ such that θ, σ, τ commute with each other. Then all the involutions $\theta\sigma$, $\theta\tau$, $\theta\sigma\tau$ are $\boldsymbol{R}$-linear involutions of $\mathfrak{g}$ but not $\boldsymbol{C}$-linear involutions. This implies that $\mathfrak{g}^{\theta\sigma}$, $\mathfrak{g}^{\theta\tau}$, $\mathfrak{g}^{\theta\sigma\tau}$ are real forms of $\mathfrak{g}$. On the other hand, it follows from the assumption that

$$\mathfrak{g}=\mathfrak{g}^{\theta}\oplus\mathfrak{g}^{-\theta}, \qquad \mathfrak{g}^{-\theta}=\sqrt{-1}\,\mathfrak{g}^{\theta}.$$

Moreover, it follows from the definition that $(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\tau})$ and $(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\theta\tau})$ are mutually associated.

PROPOSITION 2. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra, let* θ *be its Cartan involution and let* σ, τ *be* $\boldsymbol{C}$*-linear involutions of* $\mathfrak{g}$. *Assume that* θ, σ, τ *are mutually commutative. Then we have the following diagram.*

$$\begin{tikzcd}
(\mathfrak{g}^{\theta\tau}, \mathfrak{g}^{\theta\tau,\sigma}) \arrow[r, leftrightarrow, "\text{associated}"] \arrow[d, leftrightarrow, "\text{dual}"'] & (\mathfrak{g}^{\theta\tau}, \mathfrak{g}^{\theta\tau,\theta\sigma}) \arrow[r, leftrightarrow, "\text{dual}"] & (\mathfrak{g}^{\theta\sigma\tau}, \mathfrak{g}^{\theta\sigma,\tau}) \arrow[d, leftrightarrow, "\text{associated}"'] \\
(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\tau}) \arrow[r, leftrightarrow, "\text{associated}"'] & (\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\theta\tau}) \arrow[r, leftrightarrow, "\text{dual}"'] & (\mathfrak{g}^{\theta\sigma\tau}, \mathfrak{g}^{\theta\tau,\sigma})
\end{tikzcd}$$

DIAGRAM 2

PROOF. First show that $(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\tau})$ is a dual of $(\mathfrak{g}^{\theta\tau}, \mathfrak{g}^{\theta\tau,\sigma})$. Since

$$\mathfrak{g}^{\theta\tau}=\mathfrak{g}^{\theta\tau,\sigma,\tau}\oplus\mathfrak{g}^{\theta\tau,\sigma,-\tau}\oplus\mathfrak{g}^{\theta\tau,-\sigma,\tau}\oplus\mathfrak{g}^{\theta\tau,-\sigma,-\tau}$$

is a direct sum decomposition, we find from the definition of a dual symmetric pair that the dual of $(\mathfrak{g}^{\theta\tau}, \mathfrak{g}^{\theta\tau,\sigma})$ is

$$(\mathfrak{g}^{\theta\tau,\sigma,\tau}\oplus\sqrt{-1}\,\mathfrak{g}^{\theta\tau,\sigma,-\tau}\oplus\sqrt{-1}\,\mathfrak{g}^{\theta\tau,-\sigma,\tau}\oplus\mathfrak{g}^{\theta\tau,-\sigma,-\tau},\ \mathfrak{g}^{\theta\tau,\sigma,\tau}\oplus\sqrt{-1}\,\mathfrak{g}^{\theta\tau,-\sigma,\tau})$$
$$=(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\tau})\,.$$

Other cases are shown by a similar argument.

REMARK. (i) Among the six symmetric pairs appearing in Diagram 2, four symmetric pairs appear in Diagram 1. Changing the roles of τ and σ in Diagram 1, different four symmetric pairs of Diagram 2 appear in Diagram 1.

(ii) In general, any symmetric pair is expressed in the form $(\mathfrak{g}^{\theta\tau}, \mathfrak{g}^{\theta\tau,\sigma})$ for some complex semisimple Lie algebra $\mathfrak{g}$ and its involutions θ, τ, σ with the conditions of this section. We will explain this more precisely. Any symmetric pair has the form $(\mathfrak{g}_0, \mathfrak{g}_0^\sigma)$, where $\mathfrak{g}_0$ is a real semisimple Lie algebra and σ is its involution. Take a Cartan involution θ of $\mathfrak{g}_0$ commuting with σ and denote by $\mathfrak{g}$ the compexification of $\mathfrak{g}_0$. Extend σ to $\mathfrak{g}$ as a C-linear involution. Then $\mathfrak{g}_u=\mathfrak{g}_0^\theta\oplus\sqrt{-1}\,\mathfrak{g}_0^{-\theta}$ is a compact real form of $\mathfrak{g}$ and θ coincides with the restriction to $\mathfrak{g}_0$ of the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_u$. Noting this, denote by θ the conjugation of $\mathfrak{g}$. Then clearly σ and θ commute with each other. Since the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ commutes with both θ, σ, there is a C-linear involution τ of $\mathfrak{g}$ such that $\theta\tau$ is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. It is obvious that τ commutes with both θ, σ. Then $(\mathfrak{g}_0, \mathfrak{g}_0^\sigma)=(\mathfrak{g}^{\theta\tau}, \mathfrak{g}^{\theta\tau,\sigma})$.

In the sequel, we shall explain the $(M_+(p), M_-(p))$-method due to Chen-Nagano [CN] which is one of the methods of geometric constructions of pairs of commuting involutions of $\mathfrak{g}$. Let τ be a holomorphic involution of G and let θ be a Cartan involution of G commuting with τ. Then $M=G^\theta/G^{\theta,\tau}$ is a compact symmetric space. Take any point p of M. If $p=g_0G^{\theta,\tau}$ for some $g_0\in G^\theta$, the automorphism s_p of M defined by $s_p(gG^{\theta,\tau})=g_0\tau(g_0^{-1})\tau(g)G^{\theta,\tau}$ $(\forall g\in G^\theta)$ is an involutive isometry of M with respect to G^θ-invariant Riemannian metric. This implies that the automorphism τ_p of G^θ defined by $\tau_p(g)=g_0\tau(g_0^{-1})\tau(g)\tau(g_0)g_0^{-1}$ $(\forall g\in G^\theta)$ is an involution of G^θ and $\tau_p(g)s_p(q)=s_p(gq)$ for any $g \in G^{\theta}$, $q \in M$.

Put $o = eG^{\theta,\tau} \in M$ and consider a closed geodesic (=circle) $\mathscr{C}$ passing through o. Then there is $x_0 \in \mathfrak{g}^{\theta,-\tau}$ with the conditions (i) $\mathscr{C}(t) = \exp(tx_0)G^{\theta,\tau}$ $(0 \leq t \leq 1)$, (ii) $\mathscr{C}(t) \neq \mathscr{C}(t')$ if $0 \leq t < t' < 1$, (iii) $\mathscr{C}(0) = \mathscr{C}(1) = o$. For the geodesic $\mathscr{C}$, $p = \mathscr{C}(\frac{1}{2})$ is the antipodal point to o. Put $g_0 = \exp(\frac{1}{2}x_0)$. Since, by definition, $\tau(g_0) = \exp(\frac{1}{2}\tau(x_0)) = \exp(-\frac{1}{2}x_0) = g_0^{-1}$, we find that $g_0\tau(g_0)^{-1} = g_0^2 \in G^{\theta,\tau}$. This implies that $g_0^2 = \tau(g_0^2) = g_0^{-2}$. Therefore $g_0^4 = e$.

Let $M_+(p) = G^{\theta,\tau}p$ be the $G^{\theta,\tau}$-orbit of p in M. Since p is the antipodal point to o of the geodesic $\mathscr{C}$, it follows that $s_p s_o = s_o s_p$. Since both $\tau_p\tau$ and $\tau\tau_p$ are continuous maps of G^{θ} and since $\tau\tau_p(e) = \tau_p\tau(e)$, we find that $\tau_p\tau(g) = \tau\tau_p(g)$ for any $g \in G^{\theta}$.

THEOREM 3. *Let G, θ, τ, M be as above. Moreover τ_p is the involution of G corresponding to the antipodal point $p = g_0G^{\theta,\tau}$ to o with respect to a circle passing through o. By definition, $\sigma = \tau_p = \mathrm{Ad}(g_0^2) \circ \tau$ is a holomorphic involution of G. Then θ, σ, τ commute with each other and the symmetric pair $(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\tau})$ is of $\mathfrak{k}_\varepsilon$-type in the sense of* [OS2, (1.2)]. *In particular, $(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\tau})$ is self-dual and $(\mathfrak{g}^{\theta\sigma\tau}, \mathfrak{g}^{\theta\sigma,\tau})$ is self-associated.*

PROOF. We keep the notation before the theorem. Put $g(t) = \exp(tx_0)$. Then $\mathscr{C}(t) = g(t)G^{\theta,\tau}$, $g_0 = g(1/2)$, $g_0^4 = e$. Since x_0 is semisimple, there is a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{\theta,-\tau}$ containing x_0. Then $\tau\sigma(x) = x$ for any $x \in \mathfrak{a}$. This implies that $\mathfrak{a}$ is contained in $\mathfrak{g}^{\theta,-\tau,-\sigma}$. On the other hand, we have $\mathrm{Ad}(g_0)\mathfrak{g}^{\theta\tau} = \mathfrak{g}^{\theta\sigma}$, $\mathrm{Ad}(g_0)\tau = \sigma\,\mathrm{Ad}(g_0)$, $\mathrm{Ad}(g_0)\sigma = \tau\,\mathrm{Ad}(g_0)$. Therefore, noting that $\sqrt{-1}\,\mathfrak{g}^{\theta} = \mathfrak{g}^{-\theta}$, we find that $\sqrt{-1}\,\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}^{\theta\sigma,-\sigma}$ and that $\sqrt{-1}\,\mathfrak{a}$ is contained in $\mathfrak{g}^{\theta\sigma,-\tau,-\sigma}$. Since $\mathfrak{g}^{\theta\sigma} = \mathfrak{g}^{\theta\sigma,\sigma} \oplus \mathfrak{g}^{\theta\sigma,-\sigma}$ is a Cartan decomposition of $\mathfrak{g}^{\theta\sigma}$, the totality of the roots of $\mathfrak{g}^{\theta\sigma}$ with respect to $\sqrt{-1}\,\mathfrak{a}$ becomes the restricted root system of $\mathfrak{g}^{\theta\sigma}$ which we denote by Σ. Moreover, for any root $\alpha \in \Sigma$, denote by $\mathfrak{g}^{\theta\sigma}(\sqrt{-1}\,\mathfrak{a}, \alpha)$ the root space of $\mathfrak{g}^{\theta\sigma}$ belonging to α. Then, for each root $\alpha \in \Sigma$, there is a real number $\varepsilon(\alpha)$ such that $\tau(x) = \varepsilon(\alpha)\sigma(x)$ for any $x \in \mathfrak{g}^{\theta\sigma}(\sqrt{-1}\,\mathfrak{a}, \alpha)$. From the definition, $\varepsilon(\alpha)^2 = 1$. On the other hand, it is clear from the definition that for any $x \in \mathfrak{g}^{\theta\sigma}$ such that $[x, \mathfrak{a}] = 0$, $\tau(x) = \sigma(x)$ holds. Therefore, from [OS2, (1.9.3), (1.12)], we find that $(\mathfrak{g}^{\theta\sigma}, \mathfrak{g}^{\theta\sigma,\tau})$ is a symmetric pair of $\mathfrak{k}_\varepsilon$-type. The rest of the statements follow from the properties of symmetric pairs of $\mathfrak{k}_\varepsilon$-type (cf. [OS2]).

Now we explain the converse of the theorem.

Let σ be a C-linear involution of $\mathfrak{g}$ and let θ be a Cartan involution of $\mathfrak{g}$ commuting with σ. Then clearly, $\mathfrak{g}^{\theta\sigma}$ is a real form of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{g}^{\theta\sigma,-\sigma}$ and let Σ be the root system of $(\mathfrak{g}^{\theta\sigma}, \mathfrak{a})$. For a signature $\varepsilon\colon \Sigma \to \{1, -1\}$ (cf. [OS2]), define an involution τ of $\mathfrak{g}$ by

$$\tau(x) = \begin{cases} \sigma(x) & \text{if } x \in \mathfrak{g}^{\theta\sigma},\ [x, \mathfrak{a}] = 0, \\ \varepsilon(\alpha)\sigma(x) & \text{if } x \in \mathfrak{g}^{\theta\sigma}(\mathfrak{a}, \alpha),\ \alpha \in \Sigma, \end{cases}$$

where $\mathfrak{g}^{\theta\sigma}(\mathfrak{a}, \alpha)$ is the root space of $\mathfrak{g}^{\theta\sigma}$ belonging to α. Then τ is uniquely extended to a C-linear involution of $\mathfrak{g}$. For the involution τ, we find from [OS1, Lemma 1.3] that there is $x_0 \in \mathfrak{a}$ such that $\sigma\tau(x)=\mathrm{Ad}(\exp(ix_0))(x)$ for any $x \in \mathfrak{g}$. If $G=\mathrm{Int}\,\mathfrak{g}$, then θ, σ, τ are lifted to involutions of G. Put $M=G^\theta/G^{\theta,\tau}$. Then $\mathscr{C}(t)=\exp(tix_0)$ $(t \in \boldsymbol{R})$ is a circle passing through $o=eG^{\theta,\tau}$ and if p is the antipodal point to o, it follows that $\sigma|G^\theta=\tau_p$.

From the discussions above, a geometric characterization of symmetric spaces of $\mathfrak{k}_\varepsilon$-type is obtained at least in Lie algebra case. In particular, it clarifies the relation between [OS1, Appendix, Table 2] and [CN, p. 415, Tables I–III].

§4. In this section, we treat the case where $G^\theta/G^{\theta,\tau}$ is a symmetric $\boldsymbol{R}$-space. To explain our situation more precisely, let G be a connected real semisimple Lie group and let θ, τ be commuting involutions of G. Assume that θ is a Cartan involution. In the sequel, we always assume that there is a parabolic subgroup P of G such that $P=G^\tau N$ is its Levi decomposition for the unique unipotent radical N. This assumption implies that $G/P \simeq G^\theta/G^{\theta,\tau}$. On the other hand, it follows from [N, Th. 4.4] that G/G^τ is isomorphic to the cotangent bundle to $G^\theta/G^{\theta,\tau}$ as G-space. Noting this, we denote by π_P the natural projection of G/G^τ to G/P. In section 2, we already explained that G/G^τ is a vector bundle over $G^\theta/G^{\theta,\tau}$. There π is meant to be the projection of G/G^τ onto $G^\theta/G^{\theta,\tau}$. We stress here the difference between π and π_P. The latter is G-equivariant whereas the former is only G^θ-equivariant.

Now let us review the results of [N] briefly. Let $\mathfrak{g}$ be the Lie algebra of G. Then there is a unique element z of $\mathfrak{g}^{-\theta,\tau}$ (up to signature) such that $\mathfrak{g}_0=\mathfrak{g}^\tau$, $\mathfrak{g}_{-1}\oplus\mathfrak{g}_1=\mathfrak{g}^{-\tau}$, where $\mathfrak{g}_d=\{x\in\mathfrak{g}\,;\,[z,x]=dz\}$ $(\forall d\in\boldsymbol{R})$. The element z has the following properties:

$$[z,[z,x]]=x \qquad (\forall x\in\mathfrak{g}^{-\tau}),$$
$$[z,\mathfrak{g}_{\pm1}]=\mathfrak{g}_{\pm1},$$
$$\mathfrak{g}_{\pm1}=\{x\pm[z,x]\,;\,x\in\mathfrak{g}^{\theta,-\tau}\}.$$

THEOREM 4. *Let G, θ, τ be as above and moreover assume that G/G^τ is irreducible. If σ_0 is an involution of G^θ commuting with τ, the following hold:*

(i) *There is a unique involution σ of G such that $\sigma|G^\theta=\sigma_0$, $\sigma(z)=z$ and that σ commutes with both θ, τ.*

(ii) *$G^\sigma\cap P$ is a parabolic subgroup of G^σ and $G^\sigma/G^{\sigma,\tau}$ is isomorphic to the cotangent bundle over $G^{\theta,\sigma}/G^{\theta,\sigma,\tau}\simeq G^\sigma/G^\sigma\cap P$ as G^σ-space. Similarly, $G^{\sigma\tau}\cap P$ is a parabolic subgroup of $G^{\sigma\tau}$ and $G^{\sigma\tau}/G^{\sigma,\tau}$ is isomorphic to the cotangent bundle over $G^{\theta,\sigma\tau}/G^{\theta,\sigma,\tau}\simeq G^{\sigma\tau}/G^{\sigma\tau}\cap P$ as $G^{\sigma\tau}$-space.*

(iii) *Let $\iota: G^{\theta\sigma}/G^{\theta\sigma,\tau}\to G/G^\tau$ and $\iota': G^{\theta\sigma\tau}/G^{\theta\sigma,\tau}\to G/G^\tau$ be natural inclusions. Then $\pi_P\circ\iota$ (resp. $\pi_P\circ\iota'$) is a bijective $G^{\theta\sigma}$-equivariant (resp. $G^{\theta\sigma\tau}$-equivariant) map and both $\mathrm{Im}(\pi_P\circ\iota)$ and $\mathrm{Im}(\pi_P\circ\iota')$ are open subsets of $G^\theta/G^{\theta,\tau}$.*

(iv) *Regard $G^{\theta,\sigma}/G^{\theta,\sigma,\tau}$ and $G^{\theta,\sigma\tau}/G^{\theta,\sigma,\tau}$ as closed subsets of $G^\theta/G^{\theta,\tau}$. Then $G^{\theta\sigma}/G^{\theta\sigma,\tau}$ (resp. $G^{\theta\sigma\tau}/G^{\theta\sigma,\tau}$) is isomorphic to the cotangent bundle over $G^{\theta,\sigma}/G^{\theta,\sigma,\tau}$ (resp. $G^{\theta,\sigma\tau}/G^{\theta,\sigma,\tau}$) as $G^{\theta,\sigma}$-space (resp. $G^{\theta,\sigma\tau}$-space).*

PROOF. (i) First note that it is sufficient to show the case where the action of G on G/G^τ is effective. In fact, suppose that (i) is true for the effective action case. In general, let G_1 be the quotient group of G which acts on G/G^τ effectively. Then (i) is true for G_1. So there is an involution σ' of G_1 with the conditions mentioned in (i). Let σ be the induced involution of $\mathfrak{g}$ from σ'. Noting the Cartan decomposition $G=G^\theta\exp(\mathfrak{g}^{-\theta})$, define an involution σ of G by $\sigma(k\cdot\exp x)=\sigma(k)\exp\sigma(x)$ for any $k\in G^\theta$, $x\in\mathfrak{g}^{-\theta}$. Then σ is the required involution on G.

For this reason, we assume that G acts on G/G^τ effectively for a moment. Define σ by

$$\sigma(x)=\begin{cases}\sigma_0(x) & (x\in\mathfrak{g}^\theta)\\ [z,\sigma_0([z,x])] & (x\in\mathfrak{g}^{-\theta,-\tau})\,.\end{cases}$$

Then we find that σ is a linear transformation of the vector space $V=\mathfrak{g}^{\theta,\tau}\oplus\mathfrak{g}^{\theta,-\tau}\oplus\mathfrak{g}^{-\theta,-\tau}$. We find from the property of z mentioned above that σ^2 is the identity transformation of V. Now, using σ, define an automorphism φ' of $G^\theta\times\mathfrak{g}^{-\theta,-\tau}$ by

$$\varphi'(k,x)=(\sigma_0(k),\sigma(x))\qquad(\forall(k,x)\in G^\theta\times\mathfrak{g}^{-\theta,-\tau})\,.$$

Since φ' preserves the equivalence relation $\sim$ on $G^\theta\times\mathfrak{g}^{-\theta,-\tau}$ (see §2), it follows that φ' induces an automorphism φ of $(G^\theta\times\mathfrak{g}^{-\theta,-\tau})/\sim\;\simeq G/G^\tau$. It is clear from the definition that φ^2 is the identity automorphism of G/G^τ. Put $\sigma(g)=\varphi g\varphi^{-1}$ for any $g\in G$. Then $\sigma(g)$ is an automorphism of G/G^τ. Therefore both G and $\sigma(G)$ are subgroups of the automorphism group of G/G^τ. This combined with Kobayashi-Nomizu [KN] gives $\sigma(G)=G$. This implies that σ is an involution of G. From the definition, we find that $\sigma|G^\theta=\sigma_0$ and that σ commutes with θ, τ. From the choice of z, it follows that $\sigma(z)\in\mathfrak{g}^{-\theta,\tau}$, moreover, $\sigma(z)$ is contained in the center of $\mathfrak{g}^\tau$. Since G/G^τ is irreducible, the center $\mathfrak{c}$ of $\mathfrak{g}^\tau$ is one or two dimensional. In the case where $\dim\mathfrak{c}=2$, $\mathfrak{g}^\tau$ is complex semisimple and $\mathfrak{c}=\boldsymbol{C}z$. But eigenvalues of $\mathrm{ad}(\sigma(z))$ are 1, -1. Hence $\sigma(z)\in\boldsymbol{R}z$. On the other hand, if $\dim\mathfrak{c}=1$, then clearly, $\mathfrak{c}=\boldsymbol{R}z$. We have thus found that $\sigma(z)=cz$ for some $c\in\boldsymbol{R}$. Since $\sigma^2=1$, it follows that $c^2=1$. In the case where $c=-1$, we have $\theta\sigma(z)=z$. Therefore, in this case, we take $\theta\sigma$ instead of σ. Then we can take such a σ that $\sigma(z)=z$ holds. We have thus proved (i).

(ii) Since $z\in\mathfrak{g}^\sigma$, $\mathfrak{g}^\sigma\cap\mathfrak{g}_0=\mathfrak{g}^{\sigma,\tau}$, $\mathfrak{g}^{\sigma,-\tau}=\mathfrak{g}^\sigma\cap\mathfrak{g}_{-1}\oplus\mathfrak{g}^\sigma\cap\mathfrak{g}_1$, it follows that $G^\sigma\cap P$ is a parabolic subgroup of G^σ. The rest follows from properties of symmetric R-spaces.

(iii) First show that $\pi_P\circ\iota$ is injective. For this purpose, take $g\in G^{\theta\sigma}$ and assume that $\pi_P(\iota(gG^{\theta\sigma,\tau}))=eP$. Since $P=G^\tau N$, there are $h\in G^\tau$ and $n\in N$ such that $g=hn$. Let $\bar{N}$ be the opposite of N. Since $\theta\sigma(g)=g$, it follows that $hn=\theta\sigma(hn)=\theta\sigma(h)\theta\sigma(n)$. Then $\theta\sigma(h)\in G^\tau$ and $\theta\sigma(n)\in\bar{N}$. These imply that $h=\theta\sigma(h)$, $n=e$. Hence $g=h\in G^\tau\cap G^{\theta\sigma}$ and the injectivity of $\pi_P\circ\iota$ follows.

Next we will show that $\mathrm{Im}(\pi_P\circ\iota)$ is open. Put $o=eG^\tau$. This is regarded as an element of $G^{\theta\sigma}/G^{\theta\sigma,\tau}$. Moreover we need the identifications

$$T_o(G^{\theta\sigma}/G^{\theta\sigma,\tau})\simeq\mathfrak{g}^{\theta\sigma}/\mathfrak{g}^{\theta\sigma,\tau}\,,$$

$$T_o(G/G^\tau) \simeq \mathfrak{g}/\mathfrak{g}^\tau ,$$
$$T_{\pi_P(o)}(G/P) \simeq \mathfrak{g}/\mathfrak{g}^\tau \oplus \mathfrak{g}_1^{-\tau} .$$

(Here we assumed that $\mathrm{Lie}(N) = \mathfrak{g}^\tau \oplus \mathfrak{h}^{-\tau}$.) Then

$$\begin{aligned}&(\pi_P \circ \iota)_*(x \bmod \mathfrak{g}^{\theta\sigma,\tau})\\ &= (\pi_P)_*(x \bmod \mathfrak{g}^\tau)\\ &= \frac{1}{2}(x - [z, x]) \bmod \mathfrak{g}^\tau \oplus \mathfrak{g}_1^{-\tau} .\end{aligned}$$

On the other hand, from the property of z, the maps

$$\mathfrak{g}^{\theta,\sigma,-\tau} \longrightarrow \mathfrak{g}_{-1}^{-\tau,\sigma} , \qquad x \mapsto \frac{1}{2}(x - [z, x]) ,$$

$$\mathfrak{g}^{-\theta,-\sigma,-\tau} \longrightarrow \mathfrak{g}_{-1}^{-\tau,-\sigma} , \qquad x \mapsto \frac{1}{2}(x - [z, x])$$

are linear isomorphisms. Noting that

$$\mathfrak{g}^{\theta\sigma}/\mathfrak{g}^{\theta\sigma,\tau} \simeq \mathfrak{g}^{\theta,\sigma,-\tau} \oplus \mathfrak{g}^{-\theta,-\sigma,-\tau}$$
$$\mathfrak{g}/\mathfrak{g}^\tau \oplus \mathfrak{g}_1^{-\tau} \simeq \mathfrak{g}_{-1}^{-\tau,\sigma} \oplus \mathfrak{g}_{-1}^{-\tau,-\sigma} ,$$

we find that if $x \in \mathfrak{g}^{\theta,\sigma,-\tau}$, $y \in \mathfrak{g}^{-\theta,-\sigma,-\tau}$, then

$$\begin{aligned}&(\pi_P \circ \iota)_*(x + y \bmod \mathfrak{g}^{\theta\sigma,\tau})\\ &= \frac{1}{2}\{(x - [z, x]) + (y - [z, y])\} \bmod \mathfrak{g}^\tau \oplus \mathfrak{g}_1^{-\tau} .\end{aligned}$$

Then we find that $(\pi_P \circ \iota)_* T_o(G^{\theta\sigma}/G^{\theta\sigma,\tau}) = T_{\pi_P(o)}(G/P)$. Since $\pi_P \circ \iota$ is $G^{\theta\sigma}$-equivariant, it follows that the image of $\pi_P \circ \iota$ is an open subset of $G/P \simeq G^\theta/G^{\theta,\tau}$.

(iv) If M is a C^∞-manifold, and L is its closed submanifold, then

$$0 \longrightarrow T_L^*M \longrightarrow T^*M|L \longrightarrow T^*L \longrightarrow 0$$

is an exact sequence. Consider the case where $M = G^\theta/G^{\theta,\tau}$ and $L = G^{\theta,\sigma}/G^{\theta,\sigma,\tau}$. Then it follows from (ii) that

$$T^*M \simeq G/G^\tau \simeq (G^\theta \times \mathfrak{g}^{-\theta,-\tau})/\sim ,$$
$$T^*L \simeq G^\sigma/G^{\sigma,\tau} \simeq (G^{\theta,\sigma} \times \mathfrak{g}^{-\theta,-\sigma,-\tau})/\sim .$$

On the other hand, since $\mathfrak{g}^{-\theta,-\tau} = \mathfrak{g}^{-\theta,-\sigma,-\tau} \oplus \mathfrak{g}^{-\theta,\sigma,-\tau}$, we have

$$T_L^*M \simeq (G^{\theta,\sigma} \times \mathfrak{g}^{-\theta,\sigma,-\tau})/\sim \; \simeq G^{\theta\sigma}/G^{\theta\sigma,\tau} .$$

Similar argument goes well for the case where $L=G^{\theta\sigma\tau}/G^{\theta\sigma,\tau}$.

REMARK. (i) Theorem 4, (ii) is regarded as an extension of Theorem 7.2 in [CN] to the case of symmetric R-spaces.

(ii) Theorem 4, (iii) is regarded as a generalization of "Borel embedding" of a Hermitian symmetric space of the non-compact type to its compact dual. A result similar to Theorem 4 was obtained by Makarevič [M].

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