TOKYO J. MATH.
VOL. 15, NO. 1, 1992

# Global Solutions for the Heat Convection Equations in an Exterior Domain

Toshiaki HISHIDA and Yoshio YAMADA*

*Waseda University*

**Abstract.** A nonstationary problem of convection in the exterior domain to a heated sphere is studied. In the Boussinesq approximation, convection phenomena are governed by the system of the Navier-Stokes and heat equations. We find sufficient conditions on boundary and initial data to ensure the global existence of $L^p$-solutions for this system.

## 1. Introduction.

We consider the 3-dimensional laminar convection flow of a viscous incompressible fluid past a heated object. As a typical situation, we treat the case that the heat-source is a sphere of radius $R>0$. When its conducting surface is sufficiently warm compared with the ambient fluid maintained at a uniform temperature $T_\infty$ ($\geq 0$, constant), the buoyant forces against the acceleration due to gravity $\boldsymbol{g}(x)=g\nabla(1/|\boldsymbol{x}|)=-g\boldsymbol{x}/|\boldsymbol{x}|^3$ ($g>0$, gravitation constant) overcome the stabilizing effect of viscous forces and actually derive the convective motion.

Physically, early in this century, Bénard made various experiments on thermal instability of a horizontal layer of fluid heated from below. One can refer to Chandrasekhar [3; Chapter II], Drazin and Reid [5; Chapter 2] and Joseph [15; Chapter VIII] for comprehensive introduction to the Bénard-type convection problem.

By $\boldsymbol{v}(\boldsymbol{x}, t)=(v^1(\boldsymbol{x}, t), v^2(\boldsymbol{x}, t), v^3(\boldsymbol{x}, t))$, $\theta(\boldsymbol{x}, t)$ and $\pi(\boldsymbol{x}, t)$, we respectively denote the velocity field, temperature and pressure. As is well known, the equation of momentum is given by the Navier-Stokes equation:

$$\rho\{\boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\}=\rho\boldsymbol{g}-\nabla\pi+\mu\Delta\boldsymbol{v}\,,\qquad |\boldsymbol{x}|>R\,,\ t>0\,,\tag{1.1}$$

where $\rho$ denotes the density; the viscosity $\mu$ is assumed to be a positive constant. In the convection problem, the density variations are produced mainly by temperature and not by pressure. In what follows we assume that the equation of state is given by

Received April 18, 1991

* The second author was supported by Waseda University Grant for Special Research Projects (90A-89).

$$\rho=\rho(\theta)=\rho_\infty+\left[\frac{d\rho}{d\theta}\right]_{\theta=T_\infty}(\theta-T_\infty)=\rho_\infty-\chi\rho_\infty(\theta-T_\infty)\,, \tag{1.2}$$

namely, the density is linear with respect to temperature, where $\rho_\infty=\rho(T_\infty)$ is the ambient density and $\chi=-[(d/d\theta)\log\rho(\theta)]_{\theta=T_\infty}>0$ represents the volume expansion coefficient at the ambient temperature $T_\infty$. However, since $\chi$ is very small, we set $\rho=\rho_\infty$ not only in (1.1) but also in equations of continuity and energy; see (1.4) and (1.5) below. But there is one important exception. In the gravitational term $\rho\boldsymbol{g}$ which represents the buoyant force (per unit volume of fluid), we cannot neglect the density variations because $\boldsymbol{g}$ is quite large near the conducting surface. Such an approximation is called the Boussinesq approximation; see [3], [5] and [15]. Then (1.1) and (1.2) imply that

$$\boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=\{1-\chi(\theta-T_\infty)\}\boldsymbol{g}-\frac{\nabla\pi}{\rho_\infty}+\nu\Delta\boldsymbol{v}\,,\qquad |\boldsymbol{x}|>R\,,\ t>0\,, \tag{1.3}$$

where $\nu=\mu/\rho_\infty$ denotes the kinematic viscosity. The equations of continuity and energy are, respectively, given by

$$\nabla\cdot\boldsymbol{v}=0\,,\qquad |\boldsymbol{x}|>R\,,\ t\geq 0\,, \tag{1.4}$$

$$\theta_t+(\boldsymbol{v}\cdot\nabla)\theta=\kappa\Delta\theta\,,\qquad |\boldsymbol{x}|>R\,,\ t>0\,, \tag{1.5}$$

where the thermal conductivity $\kappa$ is assumed to be a positive constant. We supplement (1.3), (1.4) and (1.5) with boundary conditions

$$\begin{aligned}&\boldsymbol{v}=0\,,\quad \theta=T_w(t)>T_\infty\,, && |\boldsymbol{x}|=R\,,\ t>0\,,\\ &\boldsymbol{v}\to 0\,,\quad \theta\to T_\infty && \text{as } |\boldsymbol{x}|\to\infty\,,\ t>0\,,\end{aligned} \tag{1.6}$$

and initial ones

$$\boldsymbol{v}(\boldsymbol{x},0)=\boldsymbol{v}_0(\boldsymbol{x})\,,\qquad \theta(\boldsymbol{x},0)=\theta_0(\boldsymbol{x})\,,\qquad |\boldsymbol{x}|>R\,. \tag{1.7}$$

We now make the following change of variables and functions:

$$\boldsymbol{x}=R\boldsymbol{x}^*\,,\quad t=R^2t^*/\nu\,,\quad \boldsymbol{v}=\nu\boldsymbol{v}^*/R\,,$$
$$\theta-T_\infty=\nu^2\theta^*/\chi Rg \quad\text{and}\quad \pi-\rho_\infty g/|\boldsymbol{x}|=\rho_\infty\nu^2\pi^*/R^2\,.$$

By omitting the asterisks (for notational simplicity), (1.3)–(1.7) are reduced to the following nondimensional form:

$$\begin{cases}\boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=\theta\boldsymbol{e}-\nabla\pi+\Delta\boldsymbol{v}\,, & |\boldsymbol{x}|>1\,,\ t>0\,,\\ \nabla\cdot\boldsymbol{v}=0\,, & |\boldsymbol{x}|>1\,,\ t\geq 0\,,\\ \theta_t+(\boldsymbol{v}\cdot\nabla)\theta=\dfrac{1}{\mathrm{Pr}}\Delta\theta\,, & |\boldsymbol{x}|>1\,,\ t>0\,,\end{cases} \tag{P}$$

$$\begin{cases} v=0\,, \quad \theta=T_w^*(t)\,, & |x|=1\,,\ t>0\,, \\ v\to 0\,, \quad \theta\to 0 & \text{as } |x|\to\infty\,,\ t>0\,, \\ v(x,0)=v_0^*(x)\,, \quad \theta(x,0)=\theta_0^*(x)\,, & |x|>1\,, \end{cases}$$

where $e(x)=x/|x|^3$, $\mathrm{Pr}=\nu/\kappa$ (Prandtl number); the boundary and initial data are given by

$$T_w^*(t)=\chi Rg\{T_w(R^2t/\nu)-T_\infty\}/\nu^2\,, \qquad v_0^*(x)=Rv_0(Rx)/\nu\,,$$
$$\theta_0^*(x)=\chi Rg\{\theta_0(Rx)-T_\infty\}/\nu^2\,. \tag{1.8}$$

In the previous work [11] the first author has discussed the $L^p$-theory for a similar convection problem in a bounded domain. Many mathematicians have studied the convection phenomena from several points of view; see, e.g., Foias, Manley and Temam [6], Galdi and Padula [7], Joseph [14], Kirchgässner and Kielhöfer [17], Morimoto [21, 22], Ōeda [23, 24, 25], Rabinowitz [26] and the references therein. Among these, in [7] stability analysis for the exterior problem has been done although existence of solutions has not been discussed. In [12] the first author has recently shown the global existence of solutions for (P) with values in $L^2$ and derived some decay properties of such solutions when $T_w^*$ is independent of time and small.

It is the aim of this paper to construct global strong solutions for (P) with values in $L^p$ (contrary to [12]) and to study large time behavior of such solutions. We are interested in the class of initial data which decay too slowly as $|x|\to\infty$ to be square-summable. We also intend to investigate the influence of the presence of $T_w^*(t)$ on the global existence of solutions. Roughly speaking, our result shows that a unique strong $L^p$-solution exists globally in time when the initial data are small in a sense and $T_w^*(t)$ tends to zero as $t\to\infty$ (but this rate may be slow so that $T_w^*(t)$ is not summable over $(0,\infty)$). Our main tools are $L^p$-$L^q$ estimates for semigroups which are due to Borchers and Miyakawa [1], Giga and Sohr [10] and Iwashita [13].

The content of this paper is as follows. In section 2 we introduce the function spaces, operators and $L^p$-$L^q$ estimates for semigroups generated by them. In section 3 we state our main results: Theorem 1 (Global existence) and Theorem 2 (Decay property). In section 4 the crucial lemmas in deriving a priori estimates are presented. Theorems 1 and 2 are proved in sections 5 and 6, respectively.

## 2. Preliminaries.

We set $\Omega=\{x\in \boldsymbol{R}^3\,;\,|x|>1\}$ and let $W^{l,p}(\Omega)$ ($l=0,1,2\,;\ 1\le p\le\infty$) be the usual Sobolev space over the domain $\Omega$ such that $W^{0,p}(\Omega)=L^p(\Omega)$. We denote by $\|\cdot\|_p$ the norm of $L^p(\Omega)$ and that of $L^p(\Omega)^3$. Define the operator $B$ in $L^p(\Omega)$ by $B=-(1/\mathrm{Pr})\Delta$ with domain $D_p(B)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ for $1<p<\infty$. Let $C_{0,\sigma}^\infty(\Omega)$ be the set of all

$C^\infty$-real vector fields $\boldsymbol{v}$ with compact support in $\Omega$ such that $\nabla\cdot\boldsymbol{v}=0$. By $L^p_\sigma(\Omega)$ we denote the completion of $C^\infty_{0,\sigma}(\Omega)$ in $L^p(\Omega)^3$. Then the Helmholtz decomposition $L^p(\Omega)^3=L^p_\sigma(\Omega)\oplus G^p(\Omega)$ holds for $1<p<\infty$ with $G^p(\Omega)=\{\nabla\pi\in L^p(\Omega)^3\,;\,\pi\in L^p_{\text{loc}}(\bar{\Omega})\}$; see Miyakawa [20; Theorem 1.6]. Let $P$ be the bounded projection from $L^p(\Omega)^3$ onto $L^p_\sigma(\Omega)$ associated with this decomposition. We introduce the Stokes operator $A$ in $L^p_\sigma(\Omega)$ by $A=-P\Delta$ with domain $D_p(A)=D_p(B)^3\cap L^p_\sigma(\Omega)$ for $1<p<\infty$. Due to Giga [8; Theorem 1′ in p. 327], the Stokes operator generates an analytic semigroup $\{e^{-tA}\,;\,t\geq 0\}$ of class $(C_0)$ on all $L^p_\sigma(\Omega)$ for $1<p<\infty$ (see also Deuring [4]). Moreover, Borchers and Sohr [2] have proved the boundedness property on all $L^p_\sigma(\Omega)$ of this semigroup; that is, $\|e^{-tA}\|_{p\to p}\leq C_p$ uniformly for all $t\geq 0$, where $\|\cdot\|_{p\to q}$ denotes the operator norm from $L^p_\sigma(\Omega)$ to $L^q_\sigma(\Omega)$. The following $L^p$-$L^q$ estimates for this semigroup have been investigated by Borchers and Miyakawa [1; Corollary 4.6], Giga and Sohr [10; (1.7) in p. 105] and Iwashita [13; Theorems 1.2 and 1.3]:

$$\|e^{-tA}\|_{p\to q}\leq Ct^{-\alpha}\,,\qquad t>0\,, \tag{2.1}$$

$$\|\nabla e^{-tA}\|_{p\to q}\leq Ct^{-\alpha-1/2}\,,\qquad t>0\,, \tag{2.2}$$

with $\alpha=3(1/p-1/q)/2$ and a positive constant $C=C(p,q)$ independent of $t$. Estimate (2.1) holds for $1<p\leq q<\infty$, while (2.2) for $1<p\leq q\leq 3$; see [13]. According to [10], (2.2) also holds for $1<p<3/2$, $0\leq\alpha\leq 1$ or $1<p\leq 2$, $0\leq\alpha\leq 1/2$. On the other hand, it should be noted that both (2.1) and (2.2) are valid for $1\leq p\leq q\leq\infty$ if we replace $e^{-tA}$ by $e^{-tB}$, which denotes the semigroup generated by $-B$.

## 3. Results.

To start with, we introduce an auxiliary function $\phi(x,t)$ defined by

$$\begin{aligned}
&\phi_t=\frac{1}{\mathrm{Pr}}\Delta\phi\,, && |x|>1\,,\ t>0\,,\\
&\phi=T^*_w(t)\,, && |x|=1\,,\ t>0\,,\\
&\phi\to 0 && \text{as } |x|\to\infty \text{ and } \phi(\cdot,0)=0\,.
\end{aligned} \tag{3.1}$$

Setting $\tilde{\theta}=\theta-\phi$, we can formulate (P) to the abstract Cauchy problem of the form:

$$\begin{aligned}
&dv/dt+Av=F(v,\tilde{\theta})+P\phi e\,, && t>0\,;\ v(0)=v_0^*\,,\\
&d\tilde{\theta}/dt+B\tilde{\theta}=G(v,\tilde{\theta})\,, && t>0\,;\ \tilde{\theta}(0)=\theta_0^*\,,
\end{aligned} \tag{CP}$$

where

$$F(v,\tilde{\theta})=-P(v\cdot\nabla)v+P\tilde{\theta}e\,,$$

$$G(v,\tilde{\theta})=-(v\cdot\nabla)\tilde{\theta}-(v\cdot\nabla)\phi\,.$$

We study (CP) via the following system of integral equations:

$$v(t)=v^0(t)+\int_0^t e^{-(t-s)A}F(v,\tilde{\theta})(s)ds\,,\qquad t\geq 0\,,$$
$$\tilde{\theta}(t)=\theta^0(t)+\int_0^t e^{-(t-s)B}G(v,\tilde{\theta})(s)ds\,,\qquad t\geq 0\,,\tag{IP}$$

where

$$v^0(t)=e^{-tA}v_0^*+\int_0^t e^{-(t-s)A}(P\phi e)(s)ds\,,$$
$$\theta^0(t)=e^{-tB}\theta_0^*\,.$$

We now put the following hypotheses on the data given by (1.8):

(H1) $\{v_0^*,\theta_0^*\}\in L_\sigma^3(\Omega)\times L^m(\Omega)$ for some $1<m<3$.

(H2) $T_w^*\in C^{1+\tau}(\boldsymbol{R}_+\,;\boldsymbol{R}_+)$ for some $\tau>0$; $t^\delta T_w^*(t)$ is bounded from above in $t>0$ for some $\delta$ satisfying $\max\{1/2,3/2m-1/2\}<\delta<\min\{1,3/2m\}$.

From now on, we set $[T_w^*]=\sup_{0<t<\infty}t^\delta T_w^*(t)$. It follows from $T_w^*\in C^{1+\tau}$ that (3.1) has a unique solution of class $C^1(\boldsymbol{R}_+\,;L^p(\Omega))$, which is explicitly given by (4.6) below. Therefore, by the same manner as in [11; Section 5.1], it is seen that any solution of (IP) becomes a strong solution to (CP) in a suitable $L^p$-topology. We thus state results on global existence and decay of solutions to (IP).

THEOREM 1. *Suppose that* (H1) *and* (H2) *hold. Then there exists a positive constant $\varepsilon$ depending only on $m$ and $\delta$ such that if*

$$\|v_0^*\|_3+\|\theta_0^*\|_m+[T_w^*]<\varepsilon\,,$$

*then* (IP) *has a unique solution $\{v,\tilde{\theta}\}$ on* $[0,\infty)$ *with the following properties ($\mathscr{B}$ denotes the class of bounded continuous functions)*:

$$\{t^{(1-3/p)/2}v\,,\ t^{3(1/m-1/q)/2}\tilde{\theta}\}\in\mathscr{B}([0,\infty)\,;L_\sigma^p(\Omega)\times L^q(\Omega))$$
$$\text{for } 3\leq p<\infty \text{ and } \begin{cases} m\leq q<m_0 & \text{if } 1<m\leq 3/2\,,\\ m\leq q\leq\infty & \text{if } 3/2<m<3\,,\end{cases}\tag{3.2}$$

$$\{t^{1/2}\nabla v,\ t^{3(1/m-1/q)/2+1/2}\nabla\tilde{\theta}\}\in\mathscr{B}([0,\infty)\,;L^3(\Omega)^9\times L^q(\Omega)^3)$$
$$\text{for } m\leq q<m_1\,,\tag{3.3}$$

*where all the functions vanish at $t=0$ except when $\{p,q\}=\{3,m\}$ in* (3.2), *for which $\{v(0),\tilde{\theta}(0)\}=\{v_0^*,\theta_0^*\}$. Also, $m_i$ $(i=0,1)$ are defined by $1/m_i=1/m-(2-i)/3$ ($m_0=\infty$ if $m=3/2$).*

COROLLARY 1. *Let $\tilde{\theta}$ be the solution in Theorem* 1. *Let $q$ satisfy $3\leq q<\infty$ (resp.*

$3 \le q < m_0$) *in case* $3/2 \le m < 3$ (*resp.* $1 < m < 3/2$). *Then the temperature* $\theta = \tilde{\theta} + \phi$ *satisfies*

$$\|\theta(t)\|_q \le C t^{-\alpha}, \qquad t \ge 1,$$

*with*

$$\alpha = \begin{cases} \frac{3}{2}\left(\frac{1}{m} - \frac{1}{q}\right) & \text{if } 3 \le q < \frac{3m}{3-2m\delta}, \\ \delta - \eta & \text{if } q \ge \frac{3m}{3-2m\delta}, \end{cases}$$

*where* $\eta > 0$ *is an arbitrary small number.*

THEOREM 2. *There exists a constant* $\varepsilon_* \in (0, \varepsilon]$ *such that if*

$$\|v_0^*\|_3 + \|\theta_0^*\|_m + [T_w^*] < \varepsilon_*,$$

*then the solution* $\{v, \tilde{\theta}\}$ *in Theorem* 1 *enjoys the following decay properties*:

$$\lim_{t \to \infty} \|v(t)\|_3 = 0, \tag{3.4}$$

$$\lim_{t \to \infty} \|\tilde{\theta}(t)\|_m = 0. \tag{3.5}$$

COROLLARY 2. *The solution* $\{v, \tilde{\theta}\}$ *in Theorem* 2 *satisfies*

$$\lim_{t \to \infty} t^{(1-3/p)/2} \|v(t)\|_p = 0, \tag{3.6}$$

$$\lim_{t \to \infty} t^{3(1/m-1/q)/2} \|\tilde{\theta}(t)\|_q = 0, \tag{3.7}$$

*for the same* $\{p, q\}$ *as in* (3.2).

COROLLARY 3. *Let* $\tilde{\theta}$ *be the solution in Theorem* 2. *Then the temperature* $\theta = \tilde{\theta} + \phi$ *satisfies*

$$\lim_{t \to \infty} t^{\alpha} \|\theta(t)\|_q = 0,$$

*for the same* $q$ *and* $\alpha$ *as in Corollary* 1.

REMARK 1. For initial data in $L^3_\sigma \times L^1$, only the local solution can be constructed (even if $T_w^*$ is independent of time). According to [11; Theorem 1], the local solvability with data in $L^3_\sigma \times L^1$ is critical for the interior problem in $\mathbf{R}^3$. However, employing $L^p$-$L^q$ estimates, one can give an affirmative answer to it. In general, the $L^p$-$L^q$ estimates work better than the estimates in fractional powers spaces which are used in [11] when we intend to derive the solvability for data in the marginal class.

REMARK 2. It is expected that the steady conductive state (motionless state) is stable when $T_w^*$ is independent of time and small. However, Theorem 1 gives no answer to it. For this purpose, we need to investigate the linearized operator around such state. This has been done in [7] and [12] within the framework of $L^2$-theory.

## 4. Lemmas.

In this section we show two lemmas which are crucial in deriving a priori estimates for (IP). From now on, the same letter $C$ denotes various positive constants which may change from line to line. The first lemma is

LEMMA 4.1. *Let* $1<p<\infty$, $1\leq q<\infty$, $1\leq r\leq\infty$ *and* $1/r\leq 1/p+1/q\leq 1$. *Then*

$$\|e^{-tB}(\boldsymbol{v}\cdot\nabla)f\|_r\leq Ct^{-\beta-1/2}\|\boldsymbol{v}\|_p\|f\|_q\,,\qquad t>0\,, \tag{4.1}$$

$$\|\nabla e^{-tB}(\boldsymbol{v}\cdot\nabla)f\|_r\leq Ct^{-\beta-1}\|\boldsymbol{v}\|_p\|f\|_q\,,\qquad t>0\,, \tag{4.2}$$

*hold with* $\beta=3(1/p+1/q-1/r)/2$ *and a constant* $C=C(p,q,r)$ *independent of* $\boldsymbol{v}\in L^p_\sigma(\Omega)$, $f\in L^q(\Omega)$ *and* $t>0$.

PROOF. It suffices to show (4.1) and (4.2) for $\boldsymbol{v}\in C^\infty_{0,\sigma}(\Omega)$ and $f\in C^\infty_0(\Omega)$. Let $U(\boldsymbol{x},\boldsymbol{y},t)$ be the fundamental solution associated with $e^{-tB}$; then, since $\nabla\cdot\boldsymbol{v}=0$, we obtain

$$\begin{aligned}&[e^{-tB}(\boldsymbol{v}\cdot\nabla)f](\boldsymbol{x})\,,\qquad |\boldsymbol{x}|>1\,,\ t>0\,,\\ &=\int_\Omega U(\boldsymbol{x},\boldsymbol{y},t)[(\boldsymbol{v}\cdot\nabla_y)f](\boldsymbol{y})d\boldsymbol{y}\\ &=\int_\Omega U(\boldsymbol{x},\boldsymbol{y},t)[\nabla_y\cdot(\boldsymbol{v}f)](\boldsymbol{y})d\boldsymbol{y}\\ &=-\int_\Omega \nabla_y U(\boldsymbol{x},\boldsymbol{y},t)\cdot(\boldsymbol{v}f)(\boldsymbol{y})d\boldsymbol{y}\,.\end{aligned} \tag{4.3}$$

Here we note that

$$|\nabla_y U(\boldsymbol{x},\boldsymbol{y},t)|\leq W(\boldsymbol{x}-\boldsymbol{y},t)\,, \tag{4.4}$$

where

$$W(\boldsymbol{x},t)=Ct^{-2}\exp\left\{-\frac{|\boldsymbol{x}|^2}{Ct}\right\};$$

see, e.g., Ladyženskaja, Solonnikov and Ural'ceva [19; Theorem 16.3 in Chapter IV]. Since $W(\boldsymbol{x},t)$ satisfies

$$\|W(\cdot,t)\|_{L^{r_0}(\boldsymbol{R}^3)}\leq Ct^{-3(1-1/r_0)/2-1/2}$$

for $r_0 \geq 1$, it follows from (4.3) and Hausdorff-Young's inequality that

$$\|e^{-tB}(v\cdot\nabla)f\|_r \leq \left\| \int_\Omega W(\cdot - y, t)\,|(vf)(y)|\,dy \right\|_r$$
$$\leq Ct^{-3(1-1/r_0)/2-1/2}\|v\|_p\|f\|_q$$

with $1+1/r = 1/r_0 + 1/p + 1/q$; so,

$$\|e^{-tB}(v\cdot\nabla)f\|_r \leq Ct^{-3(1/p+1/q-1/r)/2-1/2}\|v\|_p\|f\|_q\,.$$

Estimate (4.2) is derived in the same way as above, by making use of

$$|\nabla_x\nabla_y U(x, y, t)| \leq Ct^{-5/2}\exp\left\{-\frac{|x-y|^2}{Ct}\right\}.$$

Thus we have proved Lemma 4.1. Q.E.D.

We next derive the decay estimate for the auxiliary function $\phi(x, t)$ defined by (3.1):

LEMMA 4.2. *For every* $0<\eta<1$ *and* $3 \leq p < \infty$,

$$\|\phi(t)\|_p \leq C[T_w^*]t^{-\delta+\eta/2p}\,, \qquad t>0\,, \tag{4.5}$$

*holds with a constant* $C = C(\delta, \eta, p)$ *independent of* $t>0$.

PROOF. With use of the fundamental solution $U(x, y, t)$ associated with $e^{-tB}$, it is possible to write down the function $\phi(x, t)$ by the following exact formula:

$$\phi(x, t) = -\frac{1}{\Pr}\int_0^t\int_{\partial\Omega}\frac{\partial}{\partial n_y}U(x, y, t-s)dS_yT_w^*(s)ds\,, \tag{4.6}$$

where $\partial\Omega$ is the unit spherical surface, $dS$ is the usual surface element on $\partial\Omega$ and $\partial/\partial n$ is the outward normal derivative to $\partial\Omega$. By virtue of

$$\left|\frac{\partial}{\partial n_y}U(x, y, t)\right| = |\nabla_y U(x, y, t)\cdot n_y|$$
$$\leq Ct^{-2}\exp\left\{-\frac{|x-y|^2}{Ct}\right\} \qquad \text{(see (4.4))}$$

and

$$\int_{\partial\Omega}\exp\left\{-\frac{|x-y|^2}{Ct}\right\}dS_y = \frac{Ct}{|x|}\left[\exp\left\{-\frac{(|x|-1)^2}{Ct}\right\} - \exp\left\{-\frac{(|x|+1^2)}{Ct}\right\}\right]$$
$$\leq \frac{Ct}{|x|}\exp\left\{-\frac{(|x|-1)^2}{Ct}\right\},$$

we get

$$\left\| \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}_y} U(\cdot, y, t-s) dS_y \right\|_p^p$$

$$\leq C^p (t-s)^{-p} \int_{\Omega} \frac{1}{|\boldsymbol{x}|^p} \exp\left\{ -\frac{p(|\boldsymbol{x}|-1)^2}{C(t-s)} \right\} d\boldsymbol{x}$$

$$= 4\pi C^p (t-s)^{-p} \sqrt{C(t-s)/p} \int_0^\infty \{\sqrt{C(t-s)/p}\ \tau + 1\}^{-p+2} e^{-\tau^2} d\tau . \tag{4.7}$$

Let $0<\eta<1$ and $3\leq p<\infty$ be arbitrarily fixed. Since

$$\{\sqrt{C(t-s)/p}\ \tau+1\}^{-p+2} \leq \{\sqrt{C(t-s)/p}\ \tau\}^{\eta-1} ,$$

the right-hand side of (4.7) is bounded from above by

$$C(\eta, p)(t-s)^{-p+\eta/2} \int_0^\infty \tau^{\eta-1} e^{-\tau^2} d\tau = C(\eta, p)\Gamma(\eta/2)(t-s)^{-p+\eta/2} ,$$

where $\Gamma(\cdot)$ denotes the gamma function. Hence by (4.6),

$$\|\phi(t)\|_p \leq \frac{1}{\mathrm{Pr}} \int_0^t \left\| \int_{\partial\Omega} \frac{\partial}{\partial \boldsymbol{n}_y} U(\cdot, y, t-s) dS_y \right\|_p T_w^*(s) ds$$

$$\leq C(\eta, p)[T_w^*] \int_0^t (t-s)^{-1+\eta/2p} s^{-\delta} ds$$

$$= C(\eta, p) B(\eta/2p, 1-\delta)[T_w^*] t^{-\delta+\eta/2p} ,$$

where $B(\cdot, \cdot)$ denotes the beta function. This completes the proof of Lemma 4.2.

Q.E.D.

## 5. Proof of Theorem 1.

To solve (IP), we carry out the iteration procedure (which is almost similar to [11; Section 4]; see also Giga and Miyakawa [9]). Consider the successive sequences $\{\boldsymbol{v}^n, \theta^n\}$ defined by

$$\boldsymbol{v}^{n+1}(t) = \boldsymbol{v}^0(t) + \int_0^t e^{-(t-s)A} F(\boldsymbol{v}^n, \theta^n)(s) ds ,$$

$$\theta^{n+1}(t) = \theta^0(t) + \int_0^t e^{-(t-s)B} G(\boldsymbol{v}^n, \theta^n)(s) ds , \qquad n = 0, 1, 2, \cdots .$$

We will divide the proof of Theorem 1 into five steps (i)–(v).

(i) For notational convenience, we set $\sigma = 3/\mathrm{m}$; then, $1<\sigma<3$ and $\max\{1/2,$

$(\sigma-1)/2\}<\delta<\min\{1, \sigma/2\}$. Let $0<\eta<1$ be a fixed number so small that

$$\eta<3\min\left\{1-\sigma+2\delta, \frac{2\delta-1}{\sigma-1}\right\}. \tag{5.1}$$

With use of such $\eta$, we define $\hat{\delta}$ by

$$(1-\eta/3)(\sigma-2\hat{\delta})=\sigma-2\delta\,. \tag{5.2}$$

Then, because of (5.1), $\hat{\delta}$ also satisfies

$$\max\{1/2, (\sigma-1)/2\}<\hat{\delta}<\min\{1, \sigma/2\}\,. \tag{5.3}$$

Since $0<\sigma-2\hat{\delta}<1$, we can use Lemma 4.2 with $\eta$ fixed above and $p=3/(\sigma-2\hat{\delta})$. Then, by virtue of (5.2), we get

$$\|\phi(t)\|_{3/(\sigma-2\hat{\delta})}\leq C[T_w^*]t^{-\delta}, \qquad t>0\,, \tag{5.4}$$

with a constant $C=C(\sigma, \delta, \eta)$.

(ii) Combining estimates (2.1), (2.2) and (4.1) ($f=\tilde{\theta}, \phi$) with Hölder's inequality, one can formally get the following estimates ($\gamma$ is a positive parameter determined later):

$$\begin{aligned}\|v^{n+1}(t)\|_{3/\hat{\delta}}\leq\, & C t^{(\hat{\delta}-1)/2}\|v_0^*\|_3\\ &+C\|e\|_{3/(3-\sigma)}\int_0^t(t-s)^{3(\hat{\delta}-1)/2}\|\phi(s)\|_{3/(\sigma-2\hat{\delta})}ds\\ &+C\int_0^t(t-s)^{-1/2}\|v^n(s)\|_{3/\hat{\delta}}\|\nabla v^n(s)\|_3 ds\\ &+C\|e\|_{3/(3-\sigma)}\int_0^t(t-s)^{(\sigma+\hat{\delta}-\gamma-3)/2}\|\theta^n(s)\|_{3/\gamma}ds\,,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\|\nabla v^{n+1}(t)\|_3\leq\, & C t^{-1/2}\|v_0^*\|_3\\ &+C\|e\|_{3/(3-\sigma)}\int_0^t(t-s)^{\hat{\delta}-3/2}\|\phi(s)\|_{3/(\sigma-2\hat{\delta})}ds\\ &+C\int_0^t(t-s)^{-(\hat{\delta}+1)/2}\|v^n(s)\|_{3/\hat{\delta}}\|\nabla v^n(s)\|_3 ds\\ &+C\|e\|_{3/(3-\sigma)}\int_0^t(t-s)^{(\sigma-\gamma-3)/2}\|\theta^n(s)\|_{3/\gamma}ds\,,\end{aligned} \tag{5.6}$$

$$\begin{aligned}\|\theta^{n+1}(t)\|_{3/\gamma}\leq\, & C t^{(\gamma-\sigma)/2}\|\theta_0^*\|_m\\ &+C\int_0^t(t-s)^{-(\hat{\delta}+1)/2}\|v^n(s)\|_{3/\hat{\delta}}\|\theta^n(s)\|_{3/\gamma}ds\end{aligned}$$

$$+C\int_0^t (t-s)^{(\hat{\delta}-\sigma+\gamma-1)/2}\|v^n(s)\|_{3/\delta}\|\phi(s)\|_{3/(\sigma-2\delta)}ds\,. \tag{5.7}$$

Note that $e(x)=x/|x|^3\in L^{3/(3-\sigma)}(\Omega)^3$ because of $0<3-\sigma<2$. We choose and fix $\gamma$ such that

$$\{\sigma-\hat{\delta}-1\}^+<\gamma<\sigma-1\,. \tag{5.8}$$

Then it follows from a delicate verification that by (5.3) and (5.8), we can actually employ (2.1), (2.2) and (4.1); moreover, the integrability of each term is also ensured.

(iii) It is important to establish

$$t^{(1-\delta)/2}\|v^n(t)\|_{3/\delta}\le J_n\,, \tag{5.9}$$

$$t^{1/2}\|\nabla v^n(t)\|_3\le \hat{J}_n\,, \tag{5.10}$$

$$t^{(\sigma-\gamma)/2}\|\theta^n(t)\|_{3/\gamma}\le K_n\,, \tag{5.11}$$

for all $n=0, 1, 2, \cdots$ and $0<t<\infty$ with some positive constants $J_n$, $\hat{J}_n$ and $K_n$ which are independent of $t$. Indeed, it follows from (5.4)–(5.7) that (5.9)–(5.11) hold true with $J_n$, $\hat{J}_n$ and $K_n$ determined by

$$\begin{aligned} J_{n+1}=&J_0+CB(1/2, \hat{\delta}/2)J_n\hat{J}_n\\ &+CB((\sigma+\hat{\delta}-\gamma-1)/2, (\gamma-\sigma+2)/2)K_n\,, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \hat{J}_{n+1}=&\hat{J}_0+CB((1-\hat{\delta})/2, \hat{\delta}/2)J_n\hat{J}_n\\ &+CB((\sigma-\gamma-1)/2, (\gamma-\sigma+2)/2)K_n\,, \end{aligned} \tag{5.13}$$

$$\begin{aligned} K_{n+1}=&K_0+CB((1-\hat{\delta})/2, (\hat{\delta}-\sigma+\gamma+1)/2)J_nK_n\\ &+C[T_w^*]B((\hat{\delta}-\sigma+\gamma+1)/2, (1-\hat{\delta})/2)J_n\,, \end{aligned} \tag{5.14}$$

with

$$J_0=C\|v_0^*\|_3+C[T_w^*]B((3\hat{\delta}-1)/2, 1-\hat{\delta})\,,$$

$$\hat{J}_0=C\|v_0^*\|_3+C[T_w^*]B(\hat{\delta}-1/2, 1-\hat{\delta})\,,$$

$$K_0=C\|\theta_0^*\|_m\,.$$

By (5.12)–(5.14), there exists a positive constant $\varepsilon_0$ such that if

$$E(v_0^*, \theta_0^*, T_w^*):=\|v_0^*\|_3+\|\theta_0^*\|_m+[T_w^*]<\varepsilon_0\,,$$

then

$$\max\{J_n, \hat{J}_n, K_n\}\le CE(v_0^*, \theta_0^*, T_w^*)\,,$$

with a constant $C$ independent of $n$. We next consider the differences of the successive sequences; then one can inductively deduce

$$\left.\begin{aligned} &t^{(1-\hat{\delta})/2}\|v^{n+1}(t)-v^n(t)\|_{3/\hat{\delta}} \\ &t^{1/2}\|\nabla v^{n+1}(t)-\nabla v^n(t)\|_3 \end{aligned}\right\}\le CE(v_0^*, \theta_0^*, T_w^*)L_n\,, \tag{5.15}$$

$$t^{(\sigma-\gamma)/2}\|\theta^{n+1}(t)-\theta^n(t)\|_{3/\gamma}\le CE(v_0^*, \theta_0^*, T_w^*)M_n\,, \tag{5.16}$$

for all $n$ and $0<t<\infty$, where $L_n$ and $M_n$ are determined by

$$\begin{bmatrix} L_{n+1} \\ M_{n+1} \end{bmatrix} = C\begin{bmatrix} E(v_0^*, \theta_0^*, T_w^*) & 1 \\ E(v_0^*, \theta_0^*, T_w^*) & E(v_0^*, \theta_0^*, T_w^*) \end{bmatrix}\begin{bmatrix} L_n \\ M_n \end{bmatrix},$$

with $L_0=M_0=1$. Thus, if $E(v_0^*, \theta_0^*, T_w^*)<\varepsilon_1$ for some $\varepsilon_1\in(0, \varepsilon_0]$, then there exists $\{v, \tilde{\theta}\}$ which solves (IP) for $0<t<\infty$.

(iv) In order to see (3.2) and (3.3), we go back to equations in (IP) and evaluate them by various $\|\cdot\|_p$-norms with use of (2.1), (4.1), (4.2), (5.4) and

$$\left.\begin{aligned} &t^{(1-\hat{\delta})/2}\|v(t)\|_{3/\hat{\delta}} \\ &t^{1/2}\|\nabla v(t)\|_3 \\ &t^{(\sigma-\gamma)/2}\|\tilde{\theta}(t)\|_{3/\gamma} \end{aligned}\right\}\le CE(v_0^*, \theta_0^*, T_w^*)\,. \tag{5.17}$$

Indeed, for each $3\le p<\infty$, we obtain

$$\begin{aligned} t^{(1-3/p)/2}\|v(t)\|_p &\le C\|v_0^*\|_3 \\ &\quad + Ct^{(1-3/p)/2}\|e\|_{3/(3-\sigma)}\int_0^t (t-s)^{(2\hat{\delta}-3+3/p)/2}\|\phi(s)\|_{3/(\sigma-2\hat{\delta})}ds \\ &\quad + Ct^{(1-3/p)/2}\int_0^t (t-s)^{-(\hat{\delta}+1-3/p)/2}\|v(s)\|_{3/\hat{\delta}}\|\nabla v(s)\|_3 ds \\ &\quad + Ct^{(1-3/p)/2}\|e\|_{3/(3-\sigma)}\int_0^t (t-s)^{(\sigma-\gamma-3+3/p)/2}\|\tilde{\theta}(s)\|_{3/\gamma}ds \\ &\le C\|v_0^*\|_3 + C[T_w^*]B((2\hat{\delta}-1+3/p)/2,\, 1-\hat{\delta}) \\ &\quad + CE(v_0^*, \theta_0^*, T_w^*)^2 B((1-\hat{\delta}+3/p)/2,\, \hat{\delta}/2) \\ &\quad + CE(v_0^*, \theta_0^*, T_w^*)B((\sigma-\gamma-1+3/p)/2,\, (\gamma-\sigma+2)/2) \\ &\le CE(v_0^*, \theta_0^*, T_w^*)\,. \end{aligned} \tag{5.18}$$

Next, for each $q$ given by (3.2) and (3.3), we can choose $p_i$, $\hat{p}_i$ ($\ge 3$ ; $i=0, 1$) such that

$$\sigma-\gamma-1<3/p_i<3/q-\gamma+1-i\,, \qquad 2\hat{\delta}-1<3/\hat{p}_i<3/q+2\hat{\delta}-\sigma+1-i\,.$$

Then by (4.1) and (4.2) together with (5.18),

$$t^{(\sigma+i-3/q)/2}\|\nabla^i\tilde{\theta}(t)\|_q, \qquad i=0, 1,$$

$$\leq C\|\theta_0^*\|_m + Ct^{(\sigma+i-3/q)/2}\int_0^t (t-s)^{(-\gamma-1-i-3/p_i+3/q)/2}\|v(s)\|_{p_i}\|\tilde{\theta}(s)\|_{3/\gamma}ds$$

$$+Ct^{(\sigma+i-3/q)/2}\int_0^t (t-s)^{(2\hat{\delta}-\sigma-1-i-3/\hat{p}_i+3/q)/2}\|v(s)\|_{\hat{p}_i}\|\phi(s)\|_{3/(\sigma-2\hat{\delta})}ds$$

$$\leq C\|\theta_0^*\|_m + CE(v_0^*, \theta_0^*, T_w^*)^2 B((-\gamma+1-i-3/p_i+3/q)/2, (\gamma-\sigma+1+3/p_i)/2)$$

$$+C[T_w^*]E(v_0^*, \theta_0^*, T_w^*)B((2\hat{\delta}-\sigma+1-i-3/\hat{p}_i+3/q)/2, (1-2\hat{\delta}+3/\hat{p}_i)/2)$$

$$\leq CE(v_0^*, \theta_0^*, T_w^*). \tag{5.19}$$

Corollary 1 immediately follows from (5.19) and Lemma 4.2.

To see the continuity at $t=0$ of the functions in (3.2) and (3.3), we have only to note that the left hand sides of (5.9)–(5.11) with $n=0$ tend to zero as $t\to 0$ and, therefore, the left hand sides of (5.17) have the same property (cf. [11; Section 4.3]).

(v) The uniqueness part is derived by the following proposition which presents the continuous dependence of $\{v_0^*, \theta_0^*\}\to\{v, \tilde{\theta}\}$ from $L_\sigma^3(\Omega)\times L^m(\Omega)$ to $\mathscr{B}([0, \infty); L_\sigma^3(\Omega)\times L^m(\Omega))$:

**Proposition 5.1.** *Suppose that* $\{v_{0,i}^*, \theta_{0,i}^*\}$ $(i=1, 2)$ *and* $T_w^*(t)$ *satisfy* (H1) *and* (H2). *Then there exists a constant* $\varepsilon\in(0, \varepsilon_1]$ *such that if* $E(v_{0,i}^*, \theta_{0,i}^*, T_w^*)<\varepsilon$, *global solutions* $\{v_i, \tilde{\theta}_i\}$, *which correspond to* $\{v_{0,i}^*, \theta_{0,i}^*\}$ *and are of class* (3.2)–(3.3), *satisfy the following inequality*:

$$\|v_1(t)-v_2(t)\|_3+\|\tilde{\theta}_1(t)-\tilde{\theta}_2(t)\|_m\leq C\{\|v_{0,1}^*-v_{0,2}^*\|_3+\|\theta_{0,1}^*-\theta_{0,2}^*\|_m\},$$

*with a constant C independent of t.*

**Proof.** Set

$$\psi(t):=\max\{\sup_{0<s\leq t} s^{(1-\hat{\delta})/2}\|v_1(s)-v_2(s)\|_{3/\hat{\delta}},$$

$$\sup_{0<s\leq t} s^{1/2}\|\nabla v_1(s)-\nabla v_2(s)\|_3,$$

$$\sup_{0<s\leq t} s^{(\sigma-\gamma)/2}\|\tilde{\theta}_1(s)-\tilde{\theta}_2(s)\|_{3/\gamma}\}.$$

Then, by almost the same calculation as in showing (5.15) and (5.16) (see also [11; Section 4.4]), we get

$$\psi(t)\leq C\{\|v_{0,1}^*-v_{0,2}^*\|_3+\|\theta_{0,1}^*-\theta_{0,2}^*\|_m\}$$
$$+C\{E(v_{0,1}^*, \theta_{0,1}^*, T_w^*)+E(v_{0,2}^*, \theta_{0,2}^*, T_w^*)\}\psi(t).$$

Therefore, if $E(v_{0,i}^*, \theta_{0,i}^*, T_w^*)<\varepsilon$ for some $\varepsilon\in(0, \varepsilon_1]$, then

$$\psi(t) \le C\{\|v_{0,1}^* - v_{0,2}^*\|_3 + \|\theta_{0,1}^* - \theta_{0,2}^*\|_m\}.$$

We go back to equations in (IP) to complete the proof. Q.E.D.

## 6. Proof of Theorem 2.

We start with the following proposition:

PROPOSITION 6.1. *Suppose that* (H1) *and* (H2) *hold. Let* $\rho$ *satisfy* $1<\rho<\min\{2, \sigma, 3-2\hat{\delta}\}$, *where* $\hat{\delta}$ *is defined by* (5.2) *and* $\sigma = 3/m$. *Assume that*

$$\begin{aligned} &v_0^* \in L_\sigma^3(\Omega) \cap L^{3/\rho}(\Omega)^3, \\ &\theta_0^* \in L^m(\Omega) \cap L^{3/(\rho+\sigma-1)}(\Omega). \end{aligned} \tag{6.1}$$

*Then there exists a constant* $\varepsilon_* \in (0, \varepsilon]$ *such that if* $E(v_0^*, \theta_0^*, T_w^*) < \varepsilon_*$, *the solution* $\{v, \tilde{\theta}\}$ *in Theorem* 1 *satisfies*

$$\{t^{(\rho-1)/2}v, t^{(\rho-1)/2}\tilde{\theta}\} \in \mathscr{B}([0, \infty) ; L_\sigma^3(\Omega) \times L^m(\Omega)). \tag{6.2}$$

PROOF. Set

$$V(t) := \sup_{0<s\le t} s^{(\rho-1)/2}\|v(s)\|_3,$$

$$\Theta_\omega(t) := \sup_{0<s\le t} s^{(\rho+\sigma-\omega-1)/2}\|\tilde{\theta}(s)\|_{3/\omega},$$

where $\omega$ ($\le \rho+\sigma-1$) is a positive parameter. Then, by virtue of (6.1), it follows from (2.1), (5.4) and (5.17) that

$$\begin{aligned} &t^{(\rho-1)/2}\|v(t)\|_3 \\ &\le C\|v_0^*\|_{3/\rho} + Ct^{(\rho-1)/2}\|e\|_{3/(2+\rho-\sigma)}\int_0^t (t-s)^{(2\hat{\delta}-\rho-1)/2}\|\phi(s)\|_{3/(\sigma-2\hat{\delta})}ds \\ &\quad + Ct^{(\rho-1)/2}\int_0^t (t-s)^{-1/2}\|v(s)\|_3\|\nabla v(s)\|_3 ds \\ &\quad + Ct^{(\rho-1)/2}\|e\|_{3/(3-\sigma)}\int_0^t (t-s)^{(\sigma-\omega-2)/2}\|\tilde{\theta}(s)\|_{3/\omega}ds \\ &\le C\|v_0^*\|_{3/\rho} + C[T_w^*]t^{(\rho-1)/2}\int_0^t (t-s)^{(2\hat{\delta}-\rho-1)/2}s^{-\hat{\delta}}ds \\ &\quad + CE(v_0^*, \theta_0^*, T_w^*)V(t)t^{(\rho-1)/2}\int_0^t (t-s)^{-1/2}s^{-\rho/2}ds \\ &\quad + C\Theta_\omega(t)t^{(\rho-1)/2}\int_0^t (t-s)^{(\sigma-\omega-2)/2}s^{(\omega-\rho-\sigma+1)/2}ds. \end{aligned}$$

Note that $e(x)=x/|x|^3\in L^{3/(2+\rho-\sigma)}(\Omega)^3$ because of $0<2+\rho-\sigma<2$. If $\omega$ satisfies

$$\{\rho+\sigma-3\}^+<\omega<\sigma\,,$$

then

$$\begin{aligned}V(t)\le & C\|v_0^*\|_{3/\rho}+C[T_w^*]B((2\hat{\delta}-\rho+1)/2,\,1-\hat{\delta})\\ &+CE(v_0^*,\theta_0^*,T_w^*)B(1/2,\,1-\rho/2)V(t)\\ &+CB((\sigma-\omega)/2,\,(\omega-\rho-\sigma+3)/2)\Theta_\omega(t)\,,\qquad t>0\,.\end{aligned}\tag{6.3}$$

Similarly, we get the following estimate by using (4.1), (5.4) and (5.19) ($\lambda$ is a positive parameter determined later):

$$\begin{aligned}&t^{(\rho+\sigma-\omega-1)/2}\|\tilde{\theta}(t)\|_{3/\omega}\\ &\le C\|\theta_0^*\|_{3/(\rho+\sigma-1)}+Ct^{(\rho+\sigma-\omega-1)/2}\int_0^t(t-s)^{(\omega-\lambda-2)/2}\|v(s)\|_3\|\tilde{\theta}(s)\|_{3/\lambda}ds\\ &\quad+Ct^{(\rho+\sigma-\omega-1)/2}\int_0^t(t-s)^{(\omega-\sigma+2\hat{\delta}-2)/2}\|v(s)\|_3\|\phi(s)\|_{3/(\sigma-2\hat{\delta})}ds\\ &\le C\|\theta_0^*\|_{3/(\rho+\sigma-1)}+CE(v_0^*,\theta_0^*,T_w^*)V(t)t^{(\rho+\sigma-\omega-1)/2}\int_0^t(t-s)^{(\omega-\lambda-2)/2}s^{(\lambda-\rho-\sigma+1)/2}ds\\ &\quad+C[T_w^*]V(t)t^{(\rho+\sigma-\omega-1)/2}\int_0^t(t-s)^{(\omega-\sigma+2\hat{\delta}-2)/2}s^{(1-\rho-2\hat{\delta})/2}ds\,.\end{aligned}$$

For each $\omega$ satisfying

$$\sigma-2\hat{\delta}<\omega\le\rho+\sigma-1\,,$$

we take $\lambda$ such that

$$\rho+\sigma-3<\lambda<\omega\quad\text{and}\quad\lambda\le\sigma\,.\tag{6.4}$$

Then we obtain

$$\begin{aligned}\Theta_\omega(t)\le & C\|\theta_0^*\|_{3/(\rho+\sigma-1)}+CE(v_0^*,\theta_0^*,T_w^*)B((\omega-\lambda)/2,\,(\lambda-\rho-\sigma+3)/2)V(t)\\ &+C[T_w^*]B((\omega-\sigma+2\hat{\delta})/2,\,(3-\rho-2\hat{\delta})/2)V(t)\,,\qquad t>0\,.\end{aligned}\tag{6.5}$$

We are now ready to show the uniform boundedness of $V(t)$ and $\Theta_\sigma(t)$. We first fix $\omega$ such that $\sigma-2\hat{\delta}<\omega<\sigma$ and choose $\lambda$ satisfying (6.4). Then by (6.3) and (6.5), there exists a constant $\varepsilon_*\in(0,\varepsilon]$ such that, if $E(v_0^*,\theta_0^*,T_w^*)<\varepsilon_*$, then

$$V(t)\le C\{\|v_0^*\|_{3/\rho}+\|\theta_0^*\|_{3/(\rho+\sigma-1)}+[T_w^*]\}\,,\qquad t>0\,.\tag{6.6}$$

To accomplish the proof of (6.2), we next take $\omega=\sigma$ in (6.5) and choose $\lambda$ satisfying $\rho+\sigma-3<\lambda<\sigma$; then (6.6) yields

$$\Theta_\sigma(t)\le C\{\|v_0^*\|_{3/\rho}+\|\theta_0^*\|_{3/(\rho+\sigma-1)}+[T_w^*]\}\,,\qquad t>0\,,$$

which completes the proof of Proposition 6.1. Q.E.D.

PROOF OF THEOREM 2. Proposition 6.1 enables us to show the decay property in $L_\sigma^3(\Omega)\times L^m(\Omega)$ of solutions along the same idea as in Kato [16; Noted added in p. 480] (see also Kozono [18]); we give the proof for completeness. For $\{v_0^*, \theta_0^*\}\in L_\sigma^3(\Omega)\times L^m(\Omega)$ satisfying $E(v_0^*, \theta_0^*, T_w^*)<\varepsilon_*$, let $\{v, \tilde{\theta}\}$ be the corresponding solution in Theorem 1. For any $\zeta>0$, we take $\{v_{0,\zeta}^*, \theta_{0,\zeta}^*\}$ such that it satisfies (6.1) and $\|v_{0,\zeta}^*-v_0^*\|_3+\|\theta_{0,\zeta}^*-\theta_0^*\|_m<\zeta$. By taking $\zeta$ small enough, $E(v_{0,\zeta}^*, \theta_{0,\zeta}^*, T_w^*)<\varepsilon_*$ holds; so, Theorem 1 and Proposition 6.1 assert that for $\{v_{0,\zeta}^*, \theta_{0,\zeta}^*\}$, there exists a global solution $\{v_\zeta, \tilde{\theta}_\zeta\}$ which enjoys (6.2). By virtue of Proposition 5.1, we get

$$\begin{aligned}\|v(t)\|_3+\|\tilde{\theta}(t)\|_m&\le C\zeta+\|v_\zeta(t)\|_3+\|\tilde{\theta}_\zeta(t)\|_m\\&\le C\zeta+C_\zeta t^{(1-\rho)/2}\,.\end{aligned}$$

Hence

$$\limsup_{t\to\infty}\{\|v(t)\|_3+\|\tilde{\theta}(t)\|_m\}\le C\zeta\,,$$

which implies (3.4) and (3.5). Q.E.D.

PROOF OF COROLLARIES 2 AND 3. For each $3<p<\infty$, we combine (3.4) with (5.18) to get

$$\begin{aligned}t^{(1-3/p)/2}\|v(t)\|_p&\le\|v(t)\|_3^{3/2p}\{t^{(p-3)/(2p-3)}\|v(t)\|_{2p-3}\}^{1-3/2p}\\&\le C_pE(v_0^*, \theta_0^*, T_w^*)^{1-3/2p}\|v(t)\|_3^{3/2p}\to0\qquad\text{as}\quad t\to\infty\,,\end{aligned}$$

which implies (3.6). Similarly, for each $q$ given by (3.2), we fix $q_0\in(q, m_0)$ arbitrarily (we may take $q_0=\infty$ if $3/2<m<3$). With use of $\beta\in(0, 1)$ satisfying $q=\beta m+(1-\beta)q_0$, it follows from (3.5) and (5.19) that

$$\begin{aligned}t^{3(1/m-1/q)/2}\|\tilde{\theta}(t)\|_q&\le\|\tilde{\theta}(t)\|_m^{\beta m/q}\{t^{3(1/m-1/q_0)/2}\|\tilde{\theta}(t)\|_{q_0}\}^{(1-\beta)q_0/q}\\&\le C_qE(v_0^*, \theta_0^*, T_w^*)^{(1-\beta)q_0/q}\|\tilde{\theta}(t)\|_m^{\beta m/q}\to0\qquad\text{as}\quad t\to\infty\,,\end{aligned}$$

which implies (3.7). Corollary 3 is an immediate consequence of Corollary 2 and Lemma 4.2. Q.E.D.

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*Present Address*:
DEPARTMENT OF MATHEMATICS, WASEDA UNIVERSITY
OHKUBO, SHINJUKU-KU, TOKYO 169–50, JAPAN