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Holomorphic C-Existence Families

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Abstract. We define and construct holomorphic C-existence families and holomorphic integrated semigroups and discuss their relationships with holomorphic C-semigroups and each other. We prove simple Hille-Yosida type theorems, involving the rate of growth of $\|(w-A)^{-1}\|$ or $\|(w-A)^{-1}C\|$ in a sector, for an operator to generate any of these families of operators.

I. Introduction.

Some generalizations of strongly continuous semigroups, integrated semigroups (Definition 2.1), C-semigroups (Definition 2.2) and, most recently, C-existence families (Definition 2.3) have recently appeared. They have a relationship to the abstract Cauchy problem

$$u'(t, x)=A(u(t, x)) \qquad (t\geq 0), \quad u(0, x)=x, \tag{1.1}$$

that generalizes the well-known correspondence between A being a generator of a strongly continuous semigroup and (1.1) having a unique solution for all x in the domain of A (see Proposition 2.4).

The most comprehensive of these concepts is C-existence families. In this paper, we define holomorphic C-existence families (Definition 3.5) so as to generalize holomorphic strongly continuous semigroups, a class of semigroups that has found wide applicability. We generalize the Hille-Yosida type characterizations of densely defined generators of holomorphic strongly continuous semigroups (section IV). We give similar simple sufficient conditions for operators, that may not be densely defined, to have a holomorphic C-existence family (section V). Unlike generators of strongly continuous semigroups, generators of C-semigroups or integrated semigroups may not be densely defined.

As corollaries, we obtain Hille-Yosida type conditions for operators to generate holomorphic C-semigroups or integrated semigroups (sections IV and V).

In all proofs, we explicitly construct the desired family of operators.

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Our main results in sections IV and V may be summarized as follows. For an operator, A, to generate an exponentially bounded holomorphic k-times integrated semigroup, it is sufficient that the spectrum be contained in a sector of angle less than $\pi/2$, with $\|(w-A)^{-1}\|$ $O(w^{k-1-\varepsilon})$ outside that sector, for some positive ε (Theorem 4.6). If A is densely defined, it is necessary and sufficient that $\|(w-A)^{-1}\|$ be $O(w^{k-1})$ (Theorem 5.4). In order that there exist an exponentially bounded holomorphic C-existence family for A, it is sufficient that $\|A(w-A)^{-1}C\|$ be $O(1/w^{\varepsilon})$, for some positive ε (Theorems 4.1 and 4.2). If A is densely defined and $CA\subseteq AC$ (this is automatically true when A generates a C-semigroup), it is necessary and sufficient that $\|(A(w-A)^{-1}C\|$ be bounded (Theorems 5.1 and 5.2).

Section II contains preliminary material about C-existence families, C-semigroups, integrated semigroups and (1.1). In section III, we define holomorphic versions of these families of operators, and describe the relationships between them. Section VI contains stability results for holomorphic C-semigroups. We give examples in section VII.

All operators are linear, on a Banach space, X. C will always be a bounded operator. We will write $D(A)$ for the domain of the operator A, $\rho(A)$ for the resolvent set of A. We will write $B(X)$ for the set of all bounded operators from X into itself, $\mathrm{Im}(C)$ for the image of C.

II. Preliminaries.

N-times integrated semigroups were introduced, for $N=1$, in [1], and extended to arbitrary N in [22] (see also [2], [3], [9], [16], [18], [23] and [30]). C-semigroups were introduced, independently, in [4] and [7]; the generality of this paper is in [8] (see also [5], [9], [10], [17], [18], [19], [24], [28] and [29]). C-existence families were introduced in [11].

DEFINITION 2.1. The strongly continuous family of bounded operators $\{S(t)\}_{t\geq 0}$ is an *exponentially bounded n-times integrated semigroup, generated by A*, if there exists $w\geq 0$ such that $(w, \infty)\subseteq\rho(A)$, $\{S(t)\}$ is $O(e^{wt})$, $S(0)=0$, and

$$(r-A)^{-1}x=r^n\int_0^\infty e^{-rt}S(t)x\,dt\,,\qquad \forall r>w\,,\quad x\in X\,.$$

We will say 0-*times integrated semigroup* to mean a strongly continuous semigroup.

DEFINITION 2.2. Suppose C is injective. The strongly continuous family of bounded operators $\{W(t)\}_{t\geq 0}$ is a *C-semigroup* if $W(0)=C$, and $W(t)W(s)=CW(t+s)$, for all $s, t\geq 0$. The operator A *generates* $\{W(t)\}_{t\geq 0}$ if

$$Ax=C^{-1}\left(\lim_{t\to 0}\left[\frac{1}{t}(W(t)x-Cx)\right]\right),$$

with $D(A)\equiv\{x \mid \text{limit exists and is in } \mathrm{Im}(C)\}$.

DEFINITION 2.3. Suppose A is closed. The strongly continuous family of bounded operators $\{W(t)\}_{t\geq 0}$ is an *exponentially bounded mild C-existence family for A* if there exists $w>0$ such that $\|W(t)\|$ is $O(e^{wt})$, with

(1) $\int_0^t W(s)x\,ds \in D(A)$, $\forall t>0$, $x\in X$ and the map $t\mapsto A(\int_0^t W(s)x\,ds)$ continuous.

(2) $(r-A)(\int_0^\infty e^{-rt}W(t)x\,dt)=Cx$, $\forall r>w$, $x\in X$.

Note that (1), and the fact that A is closed, implies that $\int_0^\infty e^{-rt}W(t)x\,dt = r\int_0^\infty e^{-rt}(\int_0^t W(s)x\,ds)dt \in D(A)$, $\forall x\in X$, $r>w$.

The effect of these families of operators on (1.1) is in the next proposition. By a *solution* of (1.1) we mean $u\in C([0,\infty),[D(A)])\cap C^1([0,\infty),X)$, satisfying (1.1). By a *mild solution* we mean $u\in C([0,\infty),X)$ such that $\int_0^t u(s,x)\,ds\in D(A)$, $\forall t\geq 0$, satisfying $u(t,x)=A(\int_0^t u(s,x)\,ds)+x$, $\forall t\geq 0$.

PROPOSITION 2.4. (*see* [8], [22], *and* [11])

(a) *If A generates an exponentially bounded n-times integrated semigroup, then* (1.1) *has a unique solution, $\forall x\in D(A^{n+1})$. There exists M, $w>0$ such that* $\|u(t,x)\|\leq Me^{wt}\sum_{k=0}^n \|A^k x\|$, $\forall t\geq 0$, $x\in D(A^{n+1})$.

(b) *If an extension of A generates a C-semigroup, $W(t)$, that leaves $D(A)$ invariant, then* (1.1) *has a unique solution, $\forall x\in C(D(A))$, with* $\|u(t,x)\|\leq\|W(t)\|\,\|C^{-1}x\|$, $\forall t>0$, $x\in C(D(A))$.

(c) *If there exists an exponentially bounded mild C-existence family for A, then* (1.1) *has a unique exponentially bounded mild solution, $\forall x\in C(D(A))$. There exists M, $w>0$ such that* $\|u(t,Cx)\|\leqq Me^{wt}\|x\|$, $\forall x\in X$.

The relationship between integrated semigroups and C-semigroups is the following (see also [18]).

PROPOSITION 2.5. (*Theorem* 2.5 *in* [9]) *The following are equivalent.*

(a) *A generates an exponentially bounded n-times integrated semigroup.*

(b) *There exists $w>0$ such that $(w,\infty)\subseteq\rho(A)$, and A generates an exponentially bounded $(A-r)^{-n}$-semigroup, $\forall r>w$.*

PROPOSITION 2.6. (*from* [8]) *If A generates a C-semigroup, then*

(a) *A is closed.*

(b) $\forall x\in D(A)$, *$W(t)x$ is a differentiable function of t, $W(t)x\in D(A)$, with* $(d/dt)W(t)x=AW(t)x=W(t)Ax$, $\forall t\geq 0$.

(c) $\forall x\in X$, $t>0$, $\int_0^t W(s)x\,ds\in D(A)$, *with* $A(\int_0^t W(s)x\,ds)=W(t)x-Cx$.

In general, the generator of an exponentially bounded C-semigroup may have empty resolvent set. But it will have nontrivial *C-resolvent set.*

DEFINITION 2.7. We will say that $r\in\rho_C(A)$, the *C-resolvent set* of A, if $(r-A)$ is injective and $\mathrm{Im}(C)\subseteq\mathrm{Im}(r-A)$.

PROPOSITION 2.8. *(from [8]) If A generates a $O(e^{wt})$ C-semigroup, then* $\{r \mid \mathrm{Re}(r)>w\} \subseteq \rho_C(A)$.

PROPOSITION 2.9. *(from [24]) Suppose C is injective, $w \in \boldsymbol{R}$, $\{W(t)\}_{t\geq 0}$ is a strongly continuous $O(e^{wt})$ family of bounded operators and $(w, \infty) \subseteq \rho_C(A)$. Then the following are equivalent.*

(a) $\forall s>w$, $(s-A)^{-1}C^2=C(s-A)^{-1}C$ *and*

$$(s-A)^{-1}Cx=\int_0^\infty e^{-st}W(t)x\,dt\,,$$

$\forall x \in X$.

(b) *$W(t)$ is a C-semigroup generated by an extension of A.*

The next corollary, giving the relationship between C-semigroups and mild C-existence families, follows from Proposition 2.9 and 2.6 (c).

COROLLARY 2.10. *Suppose A is closed, C is injective, $w \in \boldsymbol{R}$, $(w, \infty) \subseteq \rho_C(A)$, $(s-A)^{-1}C^2=C(s-A)^{-1}C$, $\forall s>w$, $\{W(t)\}_{t\geq 0} \subseteq B(X)$ is $O(e^{wt})$ and $\int_0^t W(s)x\,ds \in D(A)$, $\forall x \in X$. Then the following are equivalent.*

(a) *$W(t)$ is a C-semigroup generated by an extension of A.*

(b) *$W(t)$ is a mild C-existence family for A.*

A partial converse of Proposition 2.6 (b) is the following.

PROPOSITION 2.11. *(Theorem 2.6 from [8]) Suppose $\{W(t)\}_{t\geq 0}$ is a strongly continuous family of bounded operators and A is an operator whose domain is invariant under $W(t)$, such that $\forall x \in D(A)$, $t \geq 0$, $W(t)Ax=AW(t)x$, and*

$$W(t)x=Cx+\int_0^t W(s)Ax\,ds\,.$$

Then $W(t)$ is a C-semigroup, generated by an extension of A, if either

(a) *$D(A)$ is dense, or*

(b) *$\rho(A)$ is nonempty.*

The following is essentially Theorem 3.6, from [11].

PROPOSITION 2.12. *Suppose A is closed, and there exists $w>0$ such that $(w, \infty) \subseteq \rho_C(A)$ and $\{W(t)\}_{t\geq 0}$ is an exponentially bounded strongly continuous family of bounded operators. Then the following are equivalent.*

(a) *$W(t)$ is a mild C-existence family for A.*

(b) $\int_0^t W(s)x\,ds \in D(A)$, *with* $A(\int_0^t W(s)x\,ds)=W(t)x-Cx$, $\forall x \in X$.

PROPOSITION 2.13. *(Proposition 2.9 from [10]) Suppose an extension of A generates a C-semigroup and $\rho(A)$ is nonempty. Then A generates the C-semigroup.*

III. Holomorphic exponentially bounded C-existence families, C-semigroups and integrated semigroups.

In this section, we present holomorphic versions of our families of operators, and the relationships between them that will be needed to unify our results.

DEFINITION 3.1. $S_\Theta \equiv \{re^{i\phi} \mid r > 0, |\phi| < \Theta\}$, $V_\Theta \equiv \{re^{i\phi} \mid r > 0, |\phi| \le \Theta\}$.

The following, when the domain of its generator is dense, may be shown ([29], Theorem 2) to be equivalent to "holomorphic semigroups of class (H_n)" ([26]); see also [25].

DEFINITION 3.2. Suppose $\pi/2 \ge \Theta > 0$. Then an n-times integrated semigroup $\{S(t)\}_{t \ge 0}$ is a *holomorphic n-times integrated semigroup of angle* Θ if it extends to a family of bounded operators $\{S(z)\}_{z \in S_\Theta}$ satisfying

(1) The map $z \mapsto S(z)$, from S_Θ into $B(X)$, is holomorphic.
(2) $\{(d/dz)^n S(z)\}_{z \in S_\Theta}$ is a semigroup.
(3) For all $\psi < \Theta$, $\{S(z)\}$ is strongly continuous on $\overline{S_\psi}$.

The following first appeared in [4].

DEFINITION 3.3. Suppose $\pi/2 \ge \Theta > 0$. The C-semigroup $\{W(t)\}_{t \ge 0}$ is a *holomorphic C-semigroup of angle* Θ if it extends to a family of bounded operators $\{W(z)\}_{z \in S_\Theta}$ satisfying

(1) The map $z \mapsto W(z)$, from S_Θ into $B(X)$, is holomorphic.
(2) $W(z)W(w) = CW(z+w)$, for all $z, w \in S_\Theta$.
(3) For all $\psi < \Theta$, $\{W(z)\}$ is strongly continuous on $\overline{S_\psi}$.

DEFINITION 3.4. The family of operators in Definition 3.2 (3.3) is *exponentially bounded* if, for all $\psi < \Theta$, there exists finite M_ψ, w_ψ, such that $\|S(z)\|(\|W(z)\|) \le M_\psi e^{w_\psi |z|}$, for all $z \in S_\psi$.

When $w_\psi = 0$, for all $\psi < \Theta$, then the family of operators is *bounded*. Note that M_ψ may get arbitrarily large as ψ gets close to Θ.

DEFINITION 3.5. Suppose $\pi/2 \ge \Theta > 0$. The exponentially bounded mild C-existence family $\{W(t)\}_{t \ge 0}$ is an *exponentially bounded holomorphic C-existence family of angle* Θ *for A* if it extends to a family of bounded operators $\{W(z)\}_{z \in S_\Theta}$ satisfying

(1) The map $z \mapsto W(z)$, from S_Θ into $B(X)$, is holomorphic.
(2) Whenever $|\phi| < \Theta$, $\{W(te^{i\phi})\}_{t \ge 0}$ is an exponentially bounded mild C-existence family for $e^{i\phi}A$.
(3) For all $\psi < \Theta$, $\{W(z)\}$ is strongly continuous on $\overline{S_\psi}$.

If $\|W(z)\|$ is bounded on $\overline{S_\psi}$, for all $\psi < \Theta$, then $\{W(z)\}_{z \in S_\Theta}$ is a *bounded holomorphic mild C-existence family*.

LEMMA 3.6. *Suppose* $\{W(z)\}_{z \in S_\Theta}$ *is an exponentially bounded holomorphic C-*

semigroup of angle Θ *generated by* A *and* $x \in X$. *Then* $W(z)x \in D(A^k)$, *for* $k=0, 1, 2, \cdots$, $z \in S_\Theta$, *with* $A^k W(z)x = (d/dz)^k W(z)x$.

PROOF. We will show this by induction. It is clearly true when $k=0$. Suppose the assertions of the theorem are true, for a fixed k. Then

$$\frac{1}{t}(W(t)(A^k W(z)x) - C(A^k W(z)x)) = C\left[\frac{1}{t}\left(\left(\frac{d}{dz}\right)^k W(t+z)x - \left(\frac{d}{dz}\right)^k W(z)x\right)\right],$$

which converges to $C(d/dz)^{k+1} W(z)x$, as $t \to 0$. Thus, $A^k W(z)x \in D(A)$, with $A(A^k W(z)x) = (d/dz)^{k+1} W(z)x$, completing the induction. ■

PROPOSITION 3.7. *Suppose* $\pi/2 \geq \Theta > 0$, *and* $|\phi| < \Theta$.

(a) *If* A *generates an exponentially bounded holomorphic n-times integrated semigroup of angle* Θ, $\{S(z)\}_{z \in S_\Theta}$, *then* $e^{i\phi}A$ *generates an exponentially bounded n-times integrated semigroup* $\{e^{-in\phi}S(te^{i\phi})\}_{t \geq 0}$.

(b) *If* A *generates an exponentially bounded holomorphic C-semigroup of angle* Θ, $\{W(z)\}_{z \in S_\Theta}$, *then an extension of* $e^{i\phi}A$ *generates an exponentially bounded C-semigroup* $\{W(te^{i\phi})\}_{t \geq 0}$.

PROOF. (a) There exists finite M, w such that $\|S(z)\| \leq Me^{w|z|}$, for all $z \in \overline{S_\phi}$. It is well known (see [22]) that $\{z \mid \mathrm{Re}(z) > w\} \subseteq \rho(A)$, with

$$(z-A)^{-1}x = z^n \int_0^\infty e^{-zt} S(t)x\,dt,$$

when $x \in X$, $\mathrm{Re}(z) > w$. Thus, for $r > w$, $(r - e^{i\phi}A)^{-1}$ exists, with

$$\begin{aligned}(r-e^{i\phi}A)^{-1}x &= e^{-i\phi}(re^{-i\phi} - A)^{-1}x \\ &= e^{-i\phi}(re^{-i\phi})^n \int_0^\infty e^{-re^{-i\phi}t} S(t)x\,dt \\ &= e^{-i\phi}(re^{-i\phi})^n \int_{e^{i\phi}[0,\infty)} e^{-rze^{-i\phi}} S(z)x\,dz,\end{aligned}$$

by a calculus of residues argument, since the integrand is holomorphic and exponentially decaying in S_ϕ,

$$\begin{aligned}&= e^{-i\phi}(re^{-i\phi})^n \int_0^\infty e^{-rt} S(te^{i\phi})x(e^{i\phi}dt) \\ &= r^n \int_0^\infty e^{-rt}(e^{-in\phi}S(te^{i\phi})x)\,dt.\end{aligned}$$

By Definition 2.1, this concludes the proof of (a).

(b) It is clear from Definition 3.3 that $\{W(te^{i\phi})\}_{t \geq 0}$ is a C-semigroup. For

$x \in D(A)$, since $W(t)Ax = AW(t)x$, for all $t \geq 0$, and $W(z)x$ is a holomorphic function of z, it follows that $W(z)Ax = AW(z)x$, for all $z \in S_\Theta$. By Lemma 3.6, $(d/dt)W(te^{i\phi})x = W(te^{i\phi})(e^{i\phi}Ax)$, for all $x \in D(A)$, $t \geq 0$. This implies that an extension of $e^{i\phi}A$ *generates* $\{W(te^{i\phi})\}_{t \geq 0}$. ■

COROLLARY 3.8. *Suppose $\pi/2 \geq \Theta > 0$. If $\{S(z)\}_{z \in S_\Theta}$ is a holomorphic n-times integrated semigroup of angle Θ, and $\psi < \Theta$, then*

(1) *If $0 \leq k \leq n$, and $x \in D(A^k)$, then $(d/dz)^k S(z)x$ converges to 0, as z converges to 0 in $\overline{S_\psi}$.*

(2) *If $x \in D(A^n)$, then $(d/dz)^n S(z)x$ converges to x, as z converges to 0 in $\overline{S_\psi}$.*

PROOF. By Proposition 3.3, in [2], and Proposition 3.7 above, we have, for $0 \leq k \leq n$, $x \in D(A^k)$, $z \in \overline{S_\psi}$,

$$\left(\frac{d}{dz}\right)^k S(z)x = S(z)A^k x + \sum_{j=1}^{k} \frac{z^{n-j}}{(n-j)!} A^{k-j}x\,. \tag{*}$$

As $z \to 0$ in $\overline{S_\psi}$, $S(z)A^k x$ converges to 0, by Definition 3.2 (3) and the fact that $S(0) = 0$. Thus (*) yields both (1) and (2). ■

THEOREM 3.9. *Suppose A is closed, $\pi/2 \geq \Theta > 0$, $S_{(\Theta+\pi/2)} \subseteq \rho_C(A)$, C is injective and commutes with $(w-A)^{-1}C$, $\forall w \in \rho_C(A)$ and $\{W(z)\}_{z \in S_\Theta}$ is a subset of $B(X)$. Then the following are equivalent.*

(a) *$\{W(z)\}_{z \in S_\Theta}$ is an exponentially bounded holomorphic C-semigroup of angle Θ generated by an extension of A.*

(b) *$\{W(z)\}_{z \in S_\Theta}$ is an exponentially bounded holomorphic mild C-existence family of angle Θ for A.*

PROOF. (a)→(b). This will follow from Corollary 2.10 and Proposition 3.7 (b), once we verify that $\int_0^t W(se^{i\phi})x\,ds \in D(A)$, $\forall x \in X$, $|\phi| < \Theta$. For $\varepsilon > 0$, Lemma 3.6 and the fact that A is closed guarantees that $\int_\varepsilon^t W(se^{i\phi})x\,ds \in D(A)$, with $A(\int_\varepsilon^t W(se^{i\phi})x\,ds) = W(te^{i\phi})x - W(\varepsilon e^{i\phi})x$. The fact that A is closed and $s \mapsto W(se^{i\phi})x$ is continuous now gives the desired conclusion.

(b)→(a). By Definition 3.5 (2) and Proposition 2.9, $\{W(te^{i\phi})\}_{t \geq 0}$ is a C-semigroup, generated by an extension of $e^{i\phi}A$, whenever $|\phi| < \Theta$. The exponential boundedness in sectors, as in Definition 3.4, follows from (2) of Definition 3.5 and the fact that $\overline{S_\psi}$ equals the convex span of $e^{i\psi}[0, \infty) \cup e^{-i\psi}[0, \infty)$.

All that remains is to verify (2) of Definition 3.3. For all $x \in D(A)$, $(d/dt)W(te^{i\phi})x = e^{i\phi}AW(te^{i\phi})x = W(te^{i\phi})e^{i\phi}Ax$, so that $(d/dz)W(z)x = AW(z)x = W(z)Ax$. Thus, for $z, w \in S_\Theta$, $x \in D(A)$, $(d/dw)W(z-w)W(w)x = 0$, so that $CW(z+w)x = W(z)W(w)x$. For arbitrary $x \in X$, choose $r \in \rho_C(A)$. Then $(r-A)^{-1}Cx \in D(A)$, thus $(r-A)^{-1}C[CW(z+w)x] = CW(z+w)((r-A)^{-1}Cx) = W(z)W(w)((r-A)^{-1}Cx) = (r-A)^{-1}C[W(z)W(w)x]$, which implies (2) of Definition 3.3, since $(r-A)^{-1}C$ is injective. ■

THEOREM 3.10. *Suppose there exists real r such that* $[r, \infty) \subseteq \rho(A)$, *and* $\pi/2 \geq \Theta > 0$. *Then the following are equivalent.*

(a) *A generates an exponentially bounded holomorphic n-times integrated semigroup* $\{S(z)\}_{z \in S_\Theta}$, *of angle* Θ.

(b) *A generates an exponentially bounded holomorphic* $(A-r)^{-n}$-*semigroup*, $\{W(z)\}_{z \in S_\Theta}$, *of angle* Θ.

(c) *There exists a holomorphic semigroup* $\{T(z)\}_{z \in S_\Theta}$ *satisfying*

(1) *If* $z \in S_\Theta$ *and* $x \in D(A)$, *then* $T(z)x \in D(A)$, *with* $(d/dz)T(z)x = AT(z)x = T(z)Ax$.

(2) *If* $\psi < \Theta$ *and* $x \in D(A^n)$, *then* $T(z)x$ *converges to* x, *as* $z \to 0$ *in* $\overline{S_\Theta}$.

(3) *For all* $\psi < \Theta$, *there exists finite* M_ψ, w_ψ, *such that* $\|T(z)x\| \leq M_\psi e^{w_\psi |z|} \|(A-r)^n x\|$, *for all* $x \in D(A^n)$, $z \in S_\Theta$.

We then have $(d/dz)^n S(z) = T(z)$, $W(z) = (A-r)^{-n} T(z)$, *for* $z \in S_\Theta$.

PROOF. (c)$\to$(b). Let $W(0) \equiv (A-r)^{-n}$, and, for $z \in S_\Theta$, let $W(z) \equiv (A-r)^{-n} T(z)$. Using the facts that $T(z)Ax = AT(z)x$, for all $x \in D(A)$ and $T(z)$ is a semigroup, a short calculation shows that $W(z)W(w) = (A-r)^{-n} W(z+w)$, for all z, $w \in S_\Theta$. Strong continuity follows from (2), and the fact that $(A-r)^{-n} x \in D(A^n)$, for all $x \in X$. Condition (3) implies that $W(z)$ is exponentially bounded. It is also clear, since $(A-r)^{-n}$ is bounded, that the map $z \mapsto W(z)$ is holomorphic, for $z \in S_\Theta$.

To see that A generates $W(z)$, suppose $x \in D(A)$. By (1) and the fact that $(A-r)^{-n}$ is bounded, $(d/dz)W(z)x = AW(z)x = W(z)Ax$, for all $z \in S_\Theta$. Since $W(t)Ax$ is a continuous function of t, we have, for t nonnegative,

$$W(t)x = (A-r)^{-n} x + \int_0^t W(s)Axds.$$

By Proposition 2.13, A generates $\{W(t)\}$.

(b)$\to$(c). By Lemma 3.6, $W(z)x \in D(A^n)$, for all $x \in X$, $z \in S_\Theta$. Let $T(z) \equiv (A-r)^n W(z)$. $T(z)$ is clearly a semigroup, since $W(z)$ is an $(A-r)^{-n}$-semigroup. By Lemma 3.6, $T(z) = (d/dz - r)^n W(z)$, thus $T(z)$ is holomorphic.

If $x \in D(A)$, then $(d/dz)T(z)x = (d/dz)(d/dz - r)^n W(z)x = (d/dz - r)^n (d/dz) W(z)x = (A - r)^n A W(z)x = AT(z)x = T(z)Ax$, establishing (1). Conditions (2) and (3) are almost immediate, as in (c)$\to$(b).

(a)$\to$(b). In [9], it is shown that A generates an exponentially bounded $(A-r)^{-n}$-semigroup, $\{W(t)\}_{t \geq 0}$, given by $W(t)x \equiv (d/dt)^n S(t)(A-r)^{-n} x$. For $z \in S_\Theta$, $x \in X$, define $W(z)x \equiv (d/dz)^n S(z)(A-r)^{-n} x$. Since $S(z)$ is holomorphic and $(A-r)^{-n}$ is bounded, $W(z)x = (A-r)^{-n} (d/dz)^n S(z)x$, and is holomorphic. Since $(d/dz)^n S(z)$ is a semigroup, a short calculation implies (2) of Definition 3.3. Corollary 3.8 (2) implies (3) of Definition 3.3.

(b)$\to$(a). For arbitrary continuous vector-valued functions, f, on $[0, \infty)$, define If by

$$(If)(t) \equiv f(t) - r\int_0^t f(s)ds\,.$$

In [9], it is shown that A generates an exponentially bounded n-times integrated semigroup

$$S \equiv I^n\left(W - h_r \sum_{j=0}^{n-1} \frac{p_j}{j!}(A-r)^{j-n}\right), \tag{*}$$

where $p_j(t) \equiv t^j$, $h_r(t) \equiv e^{rt}$.

For $0 \le \psi < \Theta$, f holomorphic and exponentially bounded on S_ψ, continuous on $\overline{S_\psi}$, If may be extended uniquely to an exponentially bounded holomorphic function on S_ψ, continuous on $\overline{S_\psi}$. Thus, $\{S(t)\}_{t\ge 0}$ may be extended to a holomorphic operator valued function $\{S(z)\}_{z\in S_\Theta}$, defined by (*) with z replacing t.

Conditions (1) and (3) of Definition 3.2 clearly follow from the corresponding conditions in Definition 3.3. Also by [9], $W(z)$ equals $(d/dz)^n S(z)(A-r)^{-n}$, which equals $(A-r)^{-n}(d/dz)^n S(z)$, since $S(z)$ is holomorphic and $(A-r)^{-n}$ is bounded. Thus $(d/dz)^n S(z) = (A-r)^n W(z)$, which is a semigroup, since $W(z)$ is an $(A-r)^{-n}$-semigroup. This establishes (2) of Definition 3.2. ■

IV. Sufficient conditions.

Theorems 4.1 and 4.2 provide sufficient conditions for an operator to have an exponentially bounded holomorphic mild C-existence family. A nonholomorphic version of Theorem 4.2 (b) is in [11]. Sufficient conditions for an extension of A to generate an exponentially bounded holomorphic C-semigroup that leaves $D(A)$ invariant (this is sufficient for the abstract Cauchy problem—see Proposition 2.4) follow almost immediately from Theorem 3.9 and Theorem 4.2 (Corollaries 4.3 and 4.4). Theorem 4.6 provides sufficient conditions for an operator to generate an exponentially bounded holomorphic n-times integrated semigroup. A nonholomorphic version of Theorem 4.6 (b) is in [3].

Corollaries 4.3 (a) and 4.4 (a), under some additional hypotheses, are in [5] and [29].

THEOREM 4.1. *Suppose A is closed and there exists $\pi > \psi > \pi/2$ such that $V_\psi \subseteq \rho_C(A)$, and $w \mapsto (w-A)^{-1}C$, from V_ψ into $B(X)$, is holomorphic. Then there exists a bounded holomorphic mild C-existence family of angle $(\psi - \pi/2)$ for A if either*

(a) *$\{x \in D(AC) \mid ACx \in \mathrm{Im}(C)\}$ is dense and $\|A(w-A)^{-1}C\|$ is bounded in V_ψ, or*

(b) *There exists $\varepsilon > 0$ such that $\|A(w-A)^{-1}C\|$ is bounded and $O(|w|^{-\varepsilon})$ in V_ψ.*

THEOREM 4.2. *Suppose A is closed and there exist $\pi > \psi > \pi/2$, $k > 0$ such that $(k + V_\psi) \subseteq \rho_C(A)$, and $w \mapsto (w-A)^{-1}C$, from $(k+V_\psi)$ into $B(X)$, is holomorphic. Then there* *exists an exponentially bounded holomorphic mild C-existence family of angle* $(\psi-\pi/2)$ *for A if either*

(a) $\{x\in D(AC)\mid ACx\in \mathrm{Im}(C)\}$ *is dense and* $\|A(w-A)^{-1}C\|$ *is bounded in* $(k+V_\psi)$,

or

(b) *There exists* $\varepsilon>0$ *such that* $\|A(w-A)^{-1}C\|$ *is* $O(|w|^{-\varepsilon})$ *in* $(k+V_\psi)$.

COROLLARY 4.3. *Suppose A is closed and there exists* $\pi>\psi>\pi/2$ *such that* $V_\psi\subseteq\rho_C(A)$, *the* map $w\mapsto(w-A)^{-1}C$ *is holomorphic and C is injective and commutes with* $(w-A)^{-1}C$, *for all* $w\in V_\psi$. *Then an extension of A generates a bounded holomorphic C-semigroup of angle* $(\psi-\pi/2)$ *that leaves* $D(A)$ *invariant if either*

(a) $D(A)$ *is dense and* $\|A(w-A)^{-1}C\|$ *is bounded in* V_ψ; *or*

(b) *There exists* $\varepsilon>0$ *such that* $\|A(w-A)^{-1}C\|$ *is bounded and* $O(|w|^{-\varepsilon})$ *in* V_ψ.

COROLLARY 4.4. *Suppose A is closed and there exist* $\pi>\psi>\pi/2$, $k>0$, *such that* $(k+V_\psi)\subseteq\rho_C(A)$, *the map* $w\mapsto(w-A)^{-1}C$ *is holomorphic and C is injective and commutes with* $(w-A)^{-1}C$, *for all* $w\in V_\psi$. *Then an extension of A generates an exponentially bounded holomorphic C-semigroup of angle* $(\psi-\pi/2)$ *that leaves* $D(A)$ *invariant if either*

(a) $D(A)$ *is dense and* $\|A(w-A)^{-1}C\|$ *is bounded in* $(k+V_\psi)$; *or*

(b) *There exists* $\varepsilon>0$ *such that* $\|A(w-A)^{-1}C\|$ *is* $O(|w|^{-\varepsilon})$ *in* $(k+V_\psi)$.

REMARK 4.5. In the preceding results, in order that the map $w\mapsto(w-A)^{-1}C$, from V_ψ into $B(X)$, be holomorphic, it is sufficient to have A closed and $\mathrm{Im}(C)\subseteq\mathrm{Im}((w-A)^3)$, for $w\in V_\psi$, with $\|(w-A)^{-1}(r-A)^{-1}(s-A)^{-1}C\|$ locally bounded. This may be shown with the identity $(r-A)^{-1}C-(s-A)^{-1}C=(s-r)(r-A)^{-1}(s-A)^{-1}C$.

THEOREM 4.6. *Suppose there exist* $\pi>\psi>\pi/2$, $k>0$, *such that* $(k+\overline{S_\psi})\subset\rho(A)$. *Then A generates an exponentially bounded holomorphic n-times integrated semigroup of angle* $(\psi-\pi/2)$ *if either*

(a) $D(A)$ *is dense and* $\|(w-A)^{-1}\|$ *is* $O(|w|^{n-1})$ *in* $(k+\overline{S_\psi})$; *or*

(b) *There exists* $\varepsilon>0$ *such that* $\|(w-A)^{-1}\|$ *is* $O(|w|^{n-1-\varepsilon})$ *in* $(k+\overline{S_\psi})$.

PROOF OF THEOREM 4.1. Note that, since $A(w-A)^{-1}C=w(w-A)^{-1}C-C$, either (b) or (a) imply that there exists finite M such that

$$\|(w-A)^{-1}C\|\le\frac{M}{|w|},\qquad \forall w\in V_\psi. \tag{*}$$

For $r>0$, let $\Gamma_r\equiv\{se^{\pm i\psi}\mid s\ge r\}\cup\{re^{i\theta}\mid -\psi\le\theta\le\psi\}$, oriented counterclockwise. Define, for $z\in S_{(\psi-\pi/2)}$,

$$W(z)\equiv\int_{\Gamma_r} e^{zw}(w-A)^{-1}C\frac{dw}{2\pi i}.$$

By Cauchy's theorem, this definition is independent of $r>0$.

Condition (1) of Definition 3.5 is clearly satisfied. We will verify that $\{W(z)\}_{z\in S_{(\psi-\pi/2)}}$

is bounded in the sense of Definition 3.5.

Fix $\phi<(\psi-\pi/2)$, $z\in S_\phi$. Letting $r\equiv 1/|z|$, then making the change of variables $y\equiv|z|w$, we have

$$2\pi\|W(z)\|=\left\|\int_{\Gamma_1} e^{\frac{zy}{|z|}}\left(\frac{y}{|z|}-A\right)^{-1}C\frac{dy}{|z|}\right\|$$

$$\leq\int_{\Gamma_1}|e^{\frac{zy}{|z|}}|M\frac{d|y|}{|y|},\qquad \text{by } (*),$$

$$\leq 2M\int_1^\infty e^{x\cos(\phi+\psi)}\frac{dx}{x}+2\psi Me\,.$$

Since $(\phi+\psi)>\pi/2$, this is finite, so that $\{W(z)\}_{z\in S_{\psi-\pi/2}}$ is bounded, as in Definition 3.5.

To establish (3) of Definition 3.5, we need to consider

$$2\pi i(W(z)x-Cx)=\int_{\Gamma_r} e^{zw}\left((w-A)^{-1}C-\frac{1}{w}C\right)xdw\,,$$

by a calculus of residues argument,

$$=\int_{\Gamma_r} e^{zw}A(w-A)^{-1}Cx\frac{dw}{w}\,.$$

If (b) holds, then, since

$$\||w|^\varepsilon A(w-A)^{-1}C\| \text{ is bounded in } S_\psi, \tag{**}$$

dominated convergence implies that, for $z\in S_{\psi-\pi/2}$,

$$2\pi i\lim_{z\to 0}(W(z)x-Cx)=\int_{\Gamma_r} A(w-A)^{-1}Cx\frac{dw}{w}\,.$$

The following calculus of residues argument shows that this integral equals zero. For any $N>r$, let $\theta_N\equiv\{w\in\Gamma_r\,|\,|w|\leq N\}\cup\gamma_N$, where $\gamma_N\equiv\{Ne^{i\theta}\,|\,-\psi\leq\theta\leq\psi\}$. By (**), $\lim_{N\to\infty}\int_{\gamma_N}A(w-A)^{-1}Cx(1/w)dw=0$. Thus,

$$\int_{\Gamma_r} A(w-A)^{-1}Cx\frac{dw}{w}=\lim_{N\to\infty}\int_{\theta_N-\gamma_N} A(w-A)^{-1}Cx\frac{dw}{w}$$

$$=\lim_{N\to\infty}\int_{\theta_N} A(w-A)^{-1}Cx\frac{dw}{w}=0\,,$$

by Cauchy's theorem.

This establishes (3) of Definition 3.5, under hypothesis (b).

Suppose (a) holds and $x\in\mathscr{D}\equiv\{x\in D(AC)\,|\,ACx\in\operatorname{Im}(C)\}$. The same argument, using (*) in place of (**) and the fact that $A(w-A)^{-1}Cx=(w-A)^{-1}ACx=(w-A)^{-1}Cy$, for some $y\in X$, shows that $W(z)x$ converges to Cx, as $z\to 0$ in $S_{(\psi-\pi/2)}$. Since $\mathscr{D}$ is dense, the fact that $\{W(z)\}_{z\in S_{\psi-\pi/2}}$ is bounded in the sense of Definition 3.5 now implies that the same is true for all $x\in X$.

All that remains is to verify (2) of Definition 3.5. Suppose $|\phi|<\theta$. By what we have previously shown, $\{W(te^{i\phi})\}_{t\geq 0}$ is bounded and strongly continuous, when $W(0)\equiv C$. We will apply Proposition 2.12. Note that, for $x\in X$, $t>0$,

$$2\pi i\int_0^t W(se^{i\phi})x\,ds=\int_{\Gamma_r}\left[\int_0^t e^{se^{i\phi}w}ds\right](w-A)^{-1}Cx\,dw$$

$$=\int_{\Gamma_r}(e^{te^{i\phi}w}-1)(w-A)^{-1}Cx\frac{dw}{e^{i\phi}w}\,.$$

Under hypothesis (b), since A is closed, (**) implies that $\int_0^t W(se^{i\phi})x\,ds\in D(A)$, with

$$2\pi iA\left(\int_0^t W(se^{i\phi})x\,ds\right)=\int_{\Gamma_r}(e^{te^{i\phi}w}-1)A(w-A)^{-1}Cx\frac{dw}{e^{i\phi}w}$$

$$=\int_{\Gamma_r}e^{te^{i\phi}w}A(w-A)^{-1}Cx\frac{dw}{e^{i\phi}w}-\int_{\Gamma_r}A(w-A)^{-1}Cx\frac{dw}{e^{i\phi}w}$$

$$=\int_{\Gamma_r}e^{te^{i\phi}w}(A-w+w)(w-A)^{-1}Cx\frac{dw}{e^{i\phi}w}$$

since, as argued previously, the second integral is zero,

$$=-\int_{\Gamma_r}e^{te^{i\phi}w}Cx\frac{dw}{e^{i\phi}w}+\int_{\Gamma_r}e^{te^{i\phi}w}(w-A)^{-1}Cx\frac{dw}{e^{i\phi}}\,,$$

$$=2\pi ie^{-i\phi}(W(te^{i\phi})x-Cx)\,,$$

again by calculus of residues.

By Proposition 2.12, $\{W(te^{i\phi})\}_{t\geq 0}$ is a bounded mild C-existence family for $e^{i\phi}A$, as desired.

Under hypothesis (a), the same argument, for $x\in D(A)$, using (*) and the fact that $A(w-A)^{-1}Cx=(w-A)^{-1}CAx$, implies that

$$e^{i\phi}A\left(\int_0^t W(se^{i\phi})x\,ds\right)=W(te^{i\phi})x-Cx\,. \qquad (***)$$

Now let $x\in X$ be arbitrary. Choose $\{x_n\}\subset D(A)$ such that $x=\lim_{n\to\infty}x_n$. Then $W(se^{i\phi})x_n$ converges uniformly to $W(se^{i\phi})x$, on $[0, t]$, thus, since A is closed, (***) is valid, for all $x\in X$, so that, again by Proposition 2.12, (2) of Definition 3.5 is verified, concluding the proof. ■

PROOF OF THEOREM 4.2. Let $B \equiv (A-k)$. A short calculation shows that

$$B(w-B)^{-1}C = A(w+k-A)^{-1}C - \frac{k}{(w+k)}(C + A(w+k-A)^{-1}C), \tag{*}$$

for $w \in V_\psi$.

If (a) holds, then, since $k>0$, $k/(w+k)$ is bounded on V_ψ, so that (*) implies that $\|B(w-B)^{-1}C\|$ is bounded on V_ψ. By Theorem 4.1, there exists a bounded holomorphic mild C-existence family, $W(z)$, of angle $(\psi-\pi/2)$, for B, so that $e^{kz}W(z)$ is an exponentially bounded holomorphic mild C-existence family, of angle $(\psi-\pi/2)$, for A.

If (b) holds, then, as argued above, (*) implies that $\|B(w-B)^{-1}C\|$ is bounded and $O(|w+k|^{-\varepsilon})$ in V_ψ. Since $k>0$, $w/(w+k)$ is bounded on V_ψ, thus $\|B(w-B)^{-1}C\|$ is $O(|w|^{-\varepsilon})$ in V_ψ. Theorem 4.1 again yields the desired result. ■

PROOF OF COROLLARIES 4.3 and 4.4. By Theorem 3.9, all that needs to be shown is that $D(A)$ is left invariant by $W(z)$. Since A is closed, this is clear from the definition of $W(z)$. ■

To prove Theorem 4.6, we will first need the following lemma.

LEMMA 4.7. *Suppose k and r are nonnegative, $\psi > \pi/2$, $(k+\overline{S_\psi}) \subseteq \rho(A)$, $|\arg(z)| < (\psi-\pi/2)$, and n is a nonnegative integer. Let $\Gamma_{r,k} \equiv k + \Gamma_r$, where Γ_r is defined in the proof of Theorem* 4.1. *Then, for $x \in D(A^n)$,*

$$\int_{\Gamma_{r,k}} e^{zw}(w-A)^{-1}x\,dw = \int_{\Gamma_{r,k}} e^{zw}(w-A)^{-1}A^n x\frac{dw}{w^n} + 2\pi i\sum_{j=0}^{n-1}\frac{z^j}{j!}A^j x.$$

PROOF. This follows by induction, from the following calculation, where all integrals are taken over $\Gamma_{r,k}$ and $0 \le j < n$.

$$\begin{aligned}\int e^{zw}(w-A)^{-1}A^j x\frac{dw}{w^j} &= \int e^{zw}(w-A)^{-1}A^j(w-A+A)x\frac{dw}{w^{j+1}}\\ &= \left(\int e^{zw}\frac{dw}{w^{j+1}}\right)A^j x + \int e^{zw}(w-A)^{-1}A^{j+1}x\frac{dw}{w^{j+1}}\\ &= (2\pi i)\frac{z^j}{j!}A^j x + \int e^{zw}(w-A)^{-1}A^{j+1}x\frac{dw}{w^{j+1}}.\end{aligned}$$ ■

PROOF OF THEOREM 4.6. With $\Gamma_{r,k}$ as in Lemma 4.7, $x \in X$, define

$$T(z)x \equiv \int_{\Gamma_{r,k}} e^{zw}(w-A)^{-1}x\frac{dw}{2\pi i},$$

for $|\arg(z)| < (\psi-\pi/2)$, $r>0$.

We will show the following.

(1) $\forall x \in D(A)$, $z \in S_{\psi-\pi/2}$, $(d/dz)T(z)x = AT(z)x = T(z)Ax$.

(2) $\forall \phi < (\psi - \pi/2)$, $\exists$ finite M_ϕ such that

$$\| T(z)x \| \leq M_\phi e^{(k+1)|z|} \|(A-k)^n x\| ,$$

$\forall z \in S_\phi$, $x \in D(A^n)$.

(3) $\forall \phi < (\psi - \pi/2)$, $x \in D(A^n)$, $T(z)x$ converges to x, as z converges to 0, in S_ϕ.

To prove (1), note that, since the integrand is a holomorphic function of both z and w that decays exponentially, we may differentiate $T(z)$ as follows, where all integrals are taken over $\Gamma_{r,k}$.

$$\begin{aligned}2\pi i \frac{d}{dz} T(z)x &= \int e^{zw} w(w-A)^{-1} x\,dw \\ &= \left(\int e^{zw} dw \right) x + \int e^{zw} (w-A)^{-1} Ax\,dw \\ &= 2\pi i T(z)Ax = 2\pi i A T(z)x ,\end{aligned}$$

by calculus of residues and the fact that A is closed.

To prove (2), we need some more notation. Let $\Theta_r \equiv \{ s e^{\pm i\psi} \mid s \geq r \}$, $\mathbf{F}_r \equiv \{ r e^{i\xi} \mid -\psi \leq \xi \leq \psi \}$. (Note that $\Gamma_r = \Theta_r \cup \mathbf{F}_r$.)

Fix positive $\phi < (\psi - \pi/2)$. For $z \in S_\phi$ and $x \in D(A^n)$, we will apply Lemma 4.7. First, we will obtain an upper bound for the norm of the integral in Lemma 4.7.

$$\begin{aligned}|e^{-zk}| \left\| \int_{\Gamma_{r,k}} e^{zw} (w-A)^{-1} A^n x \frac{dw}{w^n} \right\| &= \left\| \int_{\Gamma_r} e^{zy} (y+k-A)^{-1} A^n x \frac{dy}{(y+k)^n} \right\| \\ &= \left\| \int_{\Gamma_{r|z|}} e^{\frac{zw}{|z|}} \left(\frac{w}{|z|} + k - A \right)^{-1} \frac{dw}{|z| \left(\frac{w}{|z|} + k \right)^n} \right\| \\ &= \left\| \int_{\Gamma_1} e^{\frac{zw}{|z|}} \left(\frac{w}{|z|} + k - A \right)^{-1} A^n x \frac{dw}{|z| \left(\frac{w}{|z|} + k \right)^n} \right\| ,\end{aligned}$$

by Cauchy's theorem,

$$\begin{aligned}&\leq \Bigg[\int_{\Theta_1} e^{|w| e^{i(\psi-\phi)}} K \left| \frac{w}{|z|} + k \right|^{n-1} \frac{d|w|}{|z| \left| \frac{w}{|z|} + k \right|^n} \\ &\quad + \int_{\mathbf{F}_1} eK \left| \frac{w}{|z|} + k \right|^{n-1} \frac{d|w|}{|z| \left| \frac{w}{|z|} + k \right|^n} \Bigg] \|A^n x\| ,\end{aligned}$$

for some constant K, by hypothesis (a) or (b),

$$= K\|A^n x\| \left(\int_{\Theta_1} e^{|w|\cos(\psi-\phi)} \frac{d|w|}{|w+k|z||} \right.$$

$$\left. + e \int_{F_1} \frac{d|w|}{|w+k|z||} \right)$$

$$\leq \|A^n x\| \frac{K}{\sin(\psi)} \left(\int_{\Theta_1} e^{|w|\cos(\psi-\phi)} d|w| + 2e\psi \right);$$

note that the quantities in parentheses are finite, since $\psi < \pi$, and $(\psi - \phi) > \pi/2$.

We also have

$$\left\| \sum_{j=0}^{n-1} \frac{z^j}{j!} A^j x \right\| \leq e^{|z|} \sup_{0 \leq j \leq n} \|A^j x\| .$$

Since $(A-k)$ is invertible, $\|(A-k)^n x\|$ and $(\sup_{0 \leq j \leq n} \|A^j x\|)$ are equivalent norms on $D(A^n)$. Thus, our estimates above, combined with Lemma 4.7, yield assertion (2).

To prove assertion (3), first suppose we are under hypothesis (b). Lemma 4.7 implies that, for $x \in D(A^n)$,

$$2\pi i(T(z)x - x) = \int_{\Gamma_{r,k}} e^{zw}(w-A)^{-1} A^n x \frac{dw}{w^n} + 2\pi i \sum_{j=1}^{n-1} \frac{z^j}{j!} A^j x .$$

Since $\int_{\Gamma_{r,k}} \|(w-A)^{-1}\| (1/|w|^n) d|w|$ is finite, the same arguments used in the proof of Theorem 4.1 show that, for $z \in S_\phi$,

$$\lim_{z \to 0} \int_{\Gamma_{r,k}} e^{zw}(w-A)^{-1} A^n x \frac{dw}{w^n} = \int_{\Gamma_{r,k}} (w-A)^{-1} A^n x \frac{dw}{w^n} = 0 .$$

Clearly, $\lim_{z \to 0} \sum_{j=1}^{n-1} (z^j/j!) A^j x$ equals zero. Thus, as z converges to zero in S_ϕ, $T(z)x$ converges to x, under hypothesis (b).

Under hypothesis (a), use Lemma 4.7 again, for $x \in D(A^{n+1})$:

$$2\pi i(T(z)x - x) = \int_{\Gamma_{r,k}} e^{zw}(w-A)^{-1} A^{n+1} x \frac{dw}{w^{n+1}} + 2\pi i \sum_{j=1}^{n} \frac{z^j}{j!} A^j x .$$

Since $\int_{\Gamma_{r,k}} \|(w-A)^{-1}\| (1/|w|^{n+1}) d|w|$ is finite, the same arguments show that $T(z)x$ converges to zero, as z converges to zero in S_ϕ, for $x \in D(A^{n+1})$.

This is saying that, for all $x \in D(A)$, $(A-k)^{-n} T(z)x$ converges to $(A-k)^{-n}x$, as z converges to zero in S_ϕ. By (2), $\|(A-k)^{-n} T(z)\|$ is bounded, for z near zero, in S_ϕ. Thus, since $D(A)$ is dense, $(A-k)^{-n} T(z)x$ converges to $(A-k)^{-n}x$, as z converges to zero in S_ϕ, for all $x \in X$, which is equivalent to (3).

This establishes assertions (1)–(3).

Assertions (1) and (3) imply that, for all $x \in D(A^n)$, z, $w \in S_{\psi - \pi/2}$, $T(z)T(w)x = T(z+w)x$. If $x \in X$, then, since $(A-k)^{-n}x \in D(A^n)$, $(A-k)^{-n} T(z)T(w)x = T(z)T(w)(A-$

$k)^{-n}x = T(z+w)(A-k)^{-n}x = (A-k)^{-n}T(z+w)x$, so that $T(z)T(w)x = T(z+w)x$, that is, $T(z)$ is a holomorphic semigroup on $S_{\psi-\pi/2}$.

By Theorem 3.10 and assertions (1)–(3), A generates an exponentially bounded holomorphic n-times integrated semigroup of angle $(\psi-\pi/2)$. ■

V. Characterization.

Theorem 5.4 (a) $\leftrightarrow$ (b) is essentially in [26], where semigroups of class (H_n) are considered (see [29], Theorem 2).

THEOREM 5.1. *Suppose A is closed, $\{x \in D(AC) \mid ACx \in \mathrm{Im}(C)\}$ is dense, $\pi/2 \geq \Theta > 0$ and $S_{(\pi/2+\Theta)} \subseteq \rho_C(A)$. Then the following are equivalent.*

(a) *There exists a bounded holomorphic mild C-existence family of angle Θ for A.*

(b) *$\forall \psi < (\pi/2+\Theta)$, $\|A(w-A)^{-1}C\|$ is bounded in S_ψ.*

THEOREM 5.2. *Suppose A is closed, $\{x \in D(AC) \mid ACx \in \mathrm{Im}(C)\}$ is dense, $\pi/2 \geq \Theta > 0$ and $\forall \psi < (\pi/2+\Theta)$, there exists $k_\psi \in \boldsymbol{R}$ such that $(k_\psi + S_\psi) \subseteq \rho_C(A)$. Then the following are equivalent.*

(a) *There exists an exponentially bounded holomorphic mild C-existence family of angle Θ for A.*

(b) *$\forall \psi < (\pi/2+\Theta)$, $\|A(w-A)^{-1}C\|$ is bounded in $(k_\psi + S_\psi)$.*

REMARK 5.3. The preceding theorems, along with Theorem 3.9, may be used to obtain characterization theorems for holomorphic C-semigroups, since $CA \subseteq AC$, when A generates a C-semigroup. Under the additional hypothesis that $\mathrm{Im}(C)$ is dense, similar characterizations appear in [5] and [29].

When A generates a bounded holomorphic C-semigroup of angle Θ, then $S_{\pi/2+\Theta} \subseteq \rho_C(A)$; this may be seen using Proposition 2.8 and the proof of Lemma 5.5 (b). When there exists a bounded holomorphic mild C-existence family of angle Θ for A, then it may be shown that $\mathrm{Im}(C) \subseteq \mathrm{Im}(w-A)$, $\forall w \in S_{\pi/2+\Theta}$.

THEOREM 5.4. *Suppose $D(A)$ is dense. Then the following are equivalent.*

(a) *A generates an exponentially bounded n-times integrated semigroup of angle Θ.*

(b) *$\forall \psi < (\pi/2+\Theta)$, there exists $k_\psi > 0$ such that $(k_\psi + S_\psi) \subseteq \rho(A)$ and $\|(A-r)^{-n}A(w-A)^{-1}\|$ is bounded in $(k_\psi + S_\psi)$, for some $r \in \rho(A)$.*

(c) *$\forall \psi < (\pi/2+\Theta)$, there exists $k_\psi > 0$ such that $(k_\psi + S_\psi) \subseteq \rho(A)$ and $\|(w-A)^{-1}\|$ is $O(|w|^{n-1})$ in $(k_\psi + S_\psi)$.*

We will need the following lemma.

LEMMA 5.5. *Suppose $\Theta < \pi/2$.*

(a) *If A generates an exponentially bounded n-times integrated semigroup, then there exists $k > 0$ such that $(k + S_\Theta) \subseteq \rho(A)$ and $\|(w-A)^{-1}\|$ is $O(|w|^{n-1})$ in $(k + S_\Theta)$.*

(b) *If there exists a bounded mild C-existence family for A and $S_\Theta \subseteq \rho_C(A)$, then*

$\|A(w-A)^{-1}C\|$ *is bounded on* S_Θ.

PROOF. (a) Let $S(t)$ be the n-times integrated semigroup generated by A. By hypothesis, there exist finite, positive M, s so that $\|S(t)\| \le Me^{st}$, $\forall t \ge 0$. It is wellknown (see [22]) that $\{w \mid \mathrm{Re}(w) > s\} \subseteq \rho(A)$, with

$$(w-A)^{-1}x = \int_0^\infty w^n e^{-wt} S(t)x\,dt\,,$$

when $x \in X$, $\mathrm{Re}(w) > s$, so that

(1) $\|(w-A)^{-1}/w^n\| \le M/(\mathrm{Re}(w)-s)$, for $\mathrm{Re}(w) > s$.

If $w \in (s+S_\Theta)$, then we may write $w = s + re^{i\phi}$, where $r > 0$, $|\phi| < \Theta$. Thus, $|w-s|/(\mathrm{Re}(w)-s) = r/r\cos(\phi)$, so that

(2) $|(w-s)/(\mathrm{Re}(w)-s)| \le 1/\cos(\Theta)$, for $w \in (s+S_\Theta)$.

Finally, choose any $k > s$. There exists finite c such that

(3) $|w/(w-s)| \le c$, for $w \in (k+S_\Theta)$.

Putting (1), (2) and (3) together gives

$$\left\|\frac{(w-A)^{-1}}{w^n}\right\| \le \left(\frac{Mc}{\cos(\Theta)}\right)\frac{1}{w}\,,$$

for $w \in (k+S_\Theta)$.

(b) Let $W(t)$ be the bounded mild C-existence family for A, $M \equiv \sup_{t \ge 0}\|W(t)\|$. The same argument as that given in the proof of Theorem 3.6, in [11], shows that

$$(w-A)^{-1}Cx = \int_0^\infty e^{-wt}W(t)x\,dt\,, \qquad \forall w \in S_\Theta\,, \quad x \in X\,.$$

Thus, we have, as in the proof of (a),

(4) $\|(w-A)^{-1}C\| \le M/\mathrm{Re}(w)$,

(5) $|w/\mathrm{Re}(w)| \le 1/\cos(\Theta)$,

for $w \in S_\Theta$, so that

$$\|A(w-A)^{-1}C\| = \|w(w-A)^{-1}C - C\| \le \left(\frac{M}{\cos(\Theta)} + \|C\|\right),$$

for $w \in S_\Theta$. ■

PROOF OF THEOREM 5.1. (a)→(b). Fix $\psi < (\pi/2 + \Theta)$. Choose $\phi > 0$ such that $(\psi - \pi/2) < \phi < \Theta$. There exist bounded mild C-existence families for both $e^{i\phi}A$ and $e^{-i\phi}A$. Since $(\psi-\phi) < \pi/2$, Lemma 5.5 implies that $\|A(e^{\pm i\phi}y - A)^{-1}C\| = \|A(y - e^{\pm i\phi}A)^{-1}C\|$ is bounded, for $y \in S_{(\psi-\phi)}$. Since S_ψ is the union of $e^{i\phi}S_{(\psi-\phi)}$ with $e^{-i\phi}S_{(\psi-\phi)}$, this implies that $\|A(w-A)^{-1}C\|$ is bounded, for $w \in S_\psi$.

(b)→(a) follows from Theorem 4.1 (a). ■

PROOF OF THEOREM 5.2. (a)→(b). Fix $\psi<(\pi/2+\Theta)$. Choose ϕ so that $(\psi-\pi/2)<\phi<\Theta$. Let $W(z)$ be the mild C-existence family for A. There exists positive k and M such that $\|W(z)\|\leq M|e^{kz}|$, $\forall z\in S_\phi$. Let $B\equiv(A-k)$. Then $e^{-kz}W(z)$ is a bounded holomorphic mild C-existence family of angle ϕ, for B. By Theorem 5.1, since $\psi<(\pi/2+\phi)$, $\|(y-B)^{-1}C\|=(1/|y|)\|C+B(y-B)^{-1}C\|$ is $O(1/|y|)$, for $y\in S_\psi$. This is saying that $\|(w-A)^{-1}C\|$ is $O(1/|w-k|)$, for $w\in(k+S_\psi)$. Let $k_\psi\equiv(k+1)$. Since $|w/(w-k)|$ is bounded in $(k_\psi+S_\psi)$, $\|(w-A)^{-1}C\|$ is $O(1/|w|)$ in $(k_\psi+S_\psi)$, so that $\|A(w-A)^{-1}C\|=\|w(w-A)^{-1}C-C\|$ bounded in $(k_\psi+S_\psi)$, as desired.

(b)→(a) follows from Theorem 4.2 (a). ■

PROOF OF THEOREM 5.4. (c)→(a) follows from Theorem 4.6 (a).

(a)→(c). Fix $\psi<(\pi/2+\Theta)$. Choose $\phi>0$ such that $(\psi-\pi/2)<\phi<\Theta$. By Proposition 3.7 (a), both $e^{i\phi}A$ and $e^{-i\phi}A$ generate exponentially bounded n-times integrated semigroups.

Since $(\psi-\phi)<\pi/2$, Lemma 5.5 implies that there exists $k>0$ such that $\|(e^{\pm i\phi}y-A)^{-1}\|=\|(y-e^{\pm i\phi}A)^{-1}\|$ is $O(|y|^{n-1})$, for $y\in(k+S_{(\psi-\phi)})$. There exists k_ψ so that $(k_\psi+S_\psi)$ is contained in the union of $e^{i\phi}(k+S_{(\psi-\phi)})$ with $e^{-i\phi}(k_\psi+S_{(\psi-\phi)})$. Thus, $\|(w-B)^{-1}\|$ is $O(|w|^{n-1})$, for $w\in(k_\psi+S_\psi)$, as desired.

(a)→(b). Fix $\psi<(\pi/2+\Theta)$. As in (a)→(c), there exists $k_\psi>0$ such that $(k_\psi+S_\psi)\subseteq\rho(A)$. By Theorem 3.10, there exists $r>k_\psi$ such that A generates an exponentially bounded holomorphic $(A-r)^{-n}$-semigroup of angle Θ. Theorems 5.2 and 3.9 now imply (b).

(b)→(a). By Theorems 5.2, 3.9 and 2.13, A generates an exponentially bounded holomorphic $(A-r)^{-n}$-semigroup of angle Θ. Theorem 3.10 thus implies (a). ■

VI. Stability.

For a bounded holomorphic C-semigroup generated by A to be stable it is sufficient that the range of A be dense (Corollary 6.2). When the range of C is dense, it is also necessary (Theorem 6.6). This is equivalent to the solutions of the corresponding abstract Cauchy problem (6.3) being stable.

THEOREM 6.1. *Suppose $W(t)$ is a bounded holomorphic C-semigroup generated by A. Then* $\lim_{t\to\infty}W(t)x=0$, $\forall x\in\overline{\mathrm{Im}(A)}$.

COROLLARY 6.2. *Suppose* $\mathrm{Im}(A)$ *is dense and $W(t)$ is a bounded holomorphic C-semigroup generated by A. Then* $\lim_{t\to\infty}W(t)x=0$, $\forall x\in X$.

This immediately yields information about the following stable abstract Cauchy problem.

$$u'(t,x)=A(u(t,x))\ (t\geq 0)\,,\quad \lim_{t\to\infty}\|u(t,x)\|=0\,. \tag{6.3}$$

COROLLARY 6.4. *Suppose* $\mathrm{Im}(A)$ *is dense and* A *generates a bounded holomorphic* C*-semigroup. Then* (6.3) *has a unique solution*, $\forall x \in C(D(A))$.

THEOREM 6.5. *Suppose* $W(t)$ *is a* C*-semigroup generated by* A *and* $\lim_{t\to\infty} W(t)x = 0$, $\forall x \in D(A)$. *Then* $\overline{C(D(A))} \subseteq \overline{\mathrm{Im}(A)}$.

THEOREM 6.6. *Suppose* $\mathrm{Im}(C)$ *is dense and* $W(t)$ *is a bounded holomorphic* C*-semigroup generated by* A. *Then the following are equivalent.*

(a) $\lim_{t\to\infty} W(t)x = 0$, $\forall x \in X$.
(b) $\lim_{t\to\infty} W(t)x = 0$, $\forall x \in D(A)$.
(c) $\mathrm{Im}(A)$ *is dense.*
(d) (6.3) *has a unique solution*, $\forall x \in C(D(A))$.

PROOF OF THEOREM 6.1. Exactly as with strongly continuous holomorphic semigroups (see [27], Theorem 5.2), one may use the analyticity of $W(t)$ to show that $AW(t)$ is bounded, $\forall t > 0$, and $\|AW(t)\| = \|(d/dt)W(t)\|$ is $O(1/t)$, as $t \to \infty$. Thus, $\lim_{t\to\infty} W(t)x = 0$, $\forall x \in \mathrm{Im}(A)$. Since $W(t)$ is bounded, the same is true, $\forall x \in \overline{\mathrm{Im}(A)}$. ■

PROOF OF COROLLARY 6.4. Since A generates a C-semigroup, $W(t)$, (6.3), without the stability condition, has the unique solution $u(t, Cx) = W(t)x$, when $x \in D(A)$ (Proposition 2.4). Corollary 6.2 now implies that this is a solution of (6.3). ■

PROOF OF THEOREM 6.5. Suppose $\phi \in X^*$ annihilates $\mathrm{Im}(A)$. Then, $\forall x \in D(A)$, $t \geq 0$, $0 = (d/dt)\phi(W(t)x)$. Thus $\phi(W(t)x) = \phi(Cx)$, $\forall t \geq 0$, so that $\phi(Cx) = 0$, that is, ϕ annihilates $C(D(A))$. ■

PROOF OF THEOREM 6.6. (b)$\to$(c). By Proposition 2.6 (c), $\forall x \in X$, $t > 0$, $(1/t)\int_0^t W(s)x\,ds \in D(A)$. Since $\lim_{t\to 0} C((1/t)\int_0^t W(s)x\,ds) = C^2x$, $C(D(A))$ is dense in $\mathrm{Im}(C^2)$, which is dense, since Im(C) is dense. Thus $C(D(A))$ is dense, so that, by Theorem 6.5, $\mathrm{Im}(A)$ is dense.

(c)$\to$(a) is Corollary 6.2.

(d)$\leftrightarrow$(b). Since A generates a C-semigroup, $W(t)$, without the stability condition, has the unique solution $u(t, Cx) = W(t)x$, $\forall x \in D(A)$. ■

VII. Examples.

We apply section IV to matrices of operators, acting on the products of (possibly different) Banach spaces,

$$A \equiv \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \ddots & & \vdots \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix},$$

on $\times_{j=1}^{n} X_j$, where A_{ij} maps a subspace of X_j into X_i. Another approach to (1.1), with such an A, may be found in [12], [13], [14], [20] and [21].

LEMMA 7.1. (*Lemma* 7.4 *from* [11]) *Suppose B is an injective operator, from X_1 into X_2, and for $i=1,2$, there exists injective $D_i \in B(X_i)$ such that D_2B and D_1B^{-1} are bounded. Then B is closable and $\bar{B}$ is injective.*

EXAMPLE 7.2. *Suppose $n, m \in N \cup 0$, $s \in \rho(G_1) \cap \rho(G_2)$,*

$$\begin{bmatrix} G_1 & B \\ 0 & G_2 \end{bmatrix}, \quad D(A) \equiv D(G_1) \times [D(B) \cap D(G_2)],$$

where

(1) *G_i generates a strongly continuous holomorphic semigroup, for $i=1,2$.*
(2) *$(s-G_1)^{-m}B$ is bounded.*
(3) *$D(G_2^n) \subset D(B)$.*
(4) *B is closed.*

Then A is closable and there exists an exponentially bounded holomorphic mild C-existence family for $\bar{A}$, where

$$C \equiv \begin{bmatrix} I & 0 \\ 0 & (s-G_2)^{-n} \end{bmatrix}.$$

PROOF. By (3) and (4), $[B(s-G_2)^{-n}] \in B(X_2, X_1)$. For $r \in \rho(G_1) \cap \rho(G_2)$, $(r-A)$ is injective. By Lemma 7.1, with $D_i \equiv \begin{bmatrix} (s-G_1)^{-m} & 0 \\ 0 & (s-G_2)^{-n} \end{bmatrix}$, for $i=1,2$, A is closable, and $(r-\bar{A})$ is injective. Since

$$(r-A)^{-1} \begin{bmatrix} I & 0 \\ 0 & (s-G_2)^{-n} \end{bmatrix} = \begin{bmatrix} (r-G_1)^{-1} & (r-G_1)^{-1}[B(s-G_2)^{-n}](r-G_2)^{-1} \\ 0 & (r-G_2)^{-1}(s-G_2)^{-n} \end{bmatrix},$$

$\|\bar{A}(r-\bar{A})^{-1}C\|$ is bounded in a sector $(k+V_\psi)$, for some $k>0$, $\psi>\pi/2$. It is also not hard to see that $D(G_1) \times D(G_2) \subseteq \{x \in D(AC) \mid ACx \in \mathrm{Im}(C)\}$. Thus the result follows from Theorem 4.2 (a). ■

EXAMPLE 7.3. *Suppose $N_2, \cdots N_n, M_1, \cdots, M_{n-1} \in N$,*

$$A \equiv \begin{bmatrix} G_1 & B_{1,2} & \cdots & B_{1,n} \\ 0 & G_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{n-1,n} \\ 0 & \cdots & 0 & G_n \end{bmatrix},$$

$$D(A) \equiv D(G_1) \mathop{\times}_{j=2}^{n} D(G_j^{N_j}),$$

where

(1) $D(G_j^{N_j}) \subseteq D(B_{i,j})$, *for* $1<j\leq n$.

(2) G_i *generates a strongly continuous holomorphic semigroup, for* $1\leq i\leq n$.

(3) *There exists* $s\in \boldsymbol{C}$ *such that* $(s-G_i)^{N_i}B_{i,j}(s-G_j)^{-N_j}\in B(X_j, X_i)$, *for* $1\leq i<j\leq n$. $(N_1\equiv 0)$

(4) *There exists* $s\in \boldsymbol{C}$ *such that* $(s-G_i)^{-M_i}B_{i,j}(s-G_j)^{M_j}$ *is bounded, for* $1<i<j\leq n$. *Then* A *is closable and there exists an exponentially bounded holomorphic* $(s-A)^{-1}$ C-*existence family for* $\bar{A}$, *where*

$$C\equiv\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & (s-G_2)^{-N_2} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (s-G_n)^{-N_n}\end{bmatrix}.$$

PROOF. As in Example 7.9, in [11], it may be shown that A is closable, $\bigcap_{k=1}^{n}\rho(G_k)\subseteq\rho_C(\bar{A})$ and $\|(r-\bar{A})^{-1}C\|$ is $O(1/|r|)$ in a sector $(k+V_\psi)$, for some $k>0$, $\psi>\pi/2$, so that this follows from Theorem 4.2 (b). ■

EXAMPLE 7.4. As argued in Example 7.11, in [11], we may choose, in Examples 7.2 and 7.3, $X_j\equiv L^p(\boldsymbol{R}^N)$ $(1\leq p\leq\infty)$, G_j equal to a constant coefficient differential operator $p_j(D)$, where $D=(i(\partial/\partial x_1), \cdots, i(\partial/\partial x_n))$, p_j is an elliptic nonconstant polynomial such that $\{p_j(x)\mid x\in\boldsymbol{R}^N\}\subseteq(k-V_\phi)$, for some $k\in\boldsymbol{R}$, $0\leq\phi<\pi/2$ (such as G_j equal to the Laplacian), and $B_{i,j}$ equal to a linear partial differential operator, $B_{i,j}\equiv\sum_{|\alpha|\leq m_{i,j}}h_{\alpha,i,j}D^\alpha$, of arbitrary order, where $h_{\alpha,i,j}$ is infinitely differentiable with bounded derivatives.

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