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Decompositions of Topological Dynamical Systems and Their Transformation Group C^*-Algebras

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I. Introduction.

Given a topological dynamical system $\Sigma=(X, \sigma)$ where X is a compact metric space with a homeomorphism σ, we sometimes meet the situation that the system Σ is decomposed into the disjoint union of subdynamical systems $\{\Sigma_\gamma=(X_\gamma, \sigma|X_\gamma) \mid \gamma\in\Gamma\}$ where each X_γ is an invariant closed subset of X. For instance, it is known that when σ is distal this is always the case in which Σ_γ turns out to be minimal. Thus, in such a case the transformation group C^*-algebra $A(\Sigma)$ associated to the system Σ may be considered as a connection of the family of transformation group C^*-algebras, $\{A(\Sigma_\gamma) \mid \gamma\in\Gamma\}$ of those topological dynamical subsystem, Σ_γ's.

It is the purpose of the present paper to show that, under what conditions, the algebra $A(\Sigma)$ becomes the algebra of all continuous operator fields over the fibred space $\{\Gamma|A(\Sigma_\gamma)\}$ in a fairly general setting for topological dynamical systems. There are recent results closely related to our present discussions, namely those by D. Williams [12] and M. A. Rieffel [13]. A main difference between their results and ours is the following; they start using the fibred space $\{Y, A(t), \mathscr{F}\}$ of C^*-algebras with a compatible action of a group G on the algebra of continuous operator fields on $\{Y, A(t), \mathscr{F}\}$ and ask whether the situation is still compatible or not in the fibered space $\{Y, A(t)\times_{\alpha_t}G\}$ of crossed product C^*-algebras $A(t)\times_{\alpha_t}G$'s, whereas we are interested in the conditions when such a starting situation arises from a given topological dynamical system. Throughout our arguments, a homomorphism between C^*-algebras always means a $*$-preserving homomorphism.

II. Analysis for decompositions of dynamical systems.

Since we are interested not only in usual dynamical systems as cited above but also in the topological dynamical systems arised in the theory of operator algebras such as shift dynamical systems (cf. [3], [11]), we consider, as a topological dynamical

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system, the system $\Sigma=(X, G, \alpha)$ where X is a compact Hausdorff space, G an amenable discrete group and the action α is a homomorphism from G into the group of all homeomorphisms on X. We note that no countability condition is imposed on X as well as on G in general. The action α of G on X naturally induces an action, denoted by the same α, on $C(X)$, the algebra of all complex valued continuous functions on X by the definition $\alpha_s(f)(x)=f(\alpha_s^{-1}x)$ for $f\in C(X)$. The C^*-algebra $A(\Sigma)$ is by definition the crossed product $C(X)\times_\alpha G$ with respect to the action α on $C(X)$. We denote δ_s the unitary operator of $A(\Sigma)$ corresponding to an element $s\in G$. Let ε be the canonical projection of norm one from $A(\Sigma)$ to $C(X)$. Since the group G is amenable, the map ε becomes a faithful positive map and moreover we can use the Fourier coefficients $\{a_s=\varepsilon(a\delta_s^*)\mid s\in G\}$ defined effectively for an element $a\in A(\Sigma)$.

Now suppose that $X=\bigcup_{\gamma\in\Gamma}X_\gamma$, where $\{X_\gamma\mid\gamma\in\Gamma\}$ is a family of disjoint closed invariant subsets. One may then consider the index set Γ as the quotient space of X with respect to the relation R defined by the family $\{X_\gamma\}$. Let q be the quotient map of X to Γ. We recall that a subset S in Γ is open if and only if the inverse image $q^{-1}(S)$ is open in X. Write $\Sigma_\gamma=(X_\gamma, G_\gamma, \alpha_\gamma=\alpha|X_\gamma)$, the induced dynamical system on X_γ and denote by $A(\Sigma_\gamma)$ the associated transformation group C^*-algebra. We notice that the same context for $A(\Sigma)$ as above is available for $A(\Sigma_\gamma)$ denoting ε_γ the canonical projection of norm one in $A(\Sigma_\gamma)$. The homomorphism ρ_γ: $C(X)\to C(X)|X_\gamma=C(X_\gamma)$ is compatible with the action α and it gives rise to the natural homomorphism from $A(\Sigma)$ to $A(\Sigma_\gamma)$ which we also denote by ρ_γ. The compatibility of the original homomorphism ρ_γ implies the equality $\rho_\gamma\circ\varepsilon=\varepsilon_\gamma\circ\rho_\gamma$. Define the ideal I_γ of $A(\Sigma)$ as the kernel of ρ_γ.

LEMMA 2.1. *I_γ is the closed linear span of the generating elements $\{f\delta_s\mid f\in C(X)$ with $f|X_\gamma=0,\ s\in G\}$.*

PROOF. Let J_γ be the closed linear span mentioned above. It then turns out to be an ideal because of the covariance relation between $C(X)$ and unitary elements $\{\delta_s\mid s\in G\}$ together with the invariance of X_γ. It is clearly contained in I_γ and by definition there is a natural embedding of $C(X_\gamma)$ into the quotient C^*-algebra $A(\Sigma)/J_\gamma$. Therefore, the universality of covariance relations for the crossed product $C(X_\gamma)\times_{\alpha_\gamma}G$ shows that there exists a homomorphism from $A(\Sigma_\gamma)$ to $A(\Sigma)/J_\gamma$, which combined with ρ_γ yields the quotient homomorphism: $A(\Sigma)\to A(\Sigma)/J_\gamma$. This completes the proof. □

LEMMA 2.2. *An element a in $A(\Sigma)$ belongs to I_γ if and only if all its Fourier coefficients a_s vanish on X_γ. Hence if $\rho_\gamma(a)=0$ for every $\gamma\in\Gamma$, $a=0$.*

PROOF. Let $\rho_\gamma(a)_s$ be the Fourier coefficients in $A(\Sigma_\gamma)$ for $s\in G$. We have then,

$$\begin{aligned}\rho_\gamma(a)_s&=\varepsilon_\gamma(\rho_\gamma(a)\rho_\gamma(\delta_s)^*)=\varepsilon_\gamma\circ\rho_\gamma(a\delta_s^*)\\&=\rho_\gamma\circ\varepsilon(a\delta_s^*)=\rho_\gamma(a_s)\,.\end{aligned}$$

Hence,

$$\begin{aligned} a \in I_\gamma &\leftrightarrow \rho_\gamma(a)_s = 0 \ \text{ for every } s \in G \\ &\leftrightarrow \rho_\gamma(a_s) = 0 \ \text{ for every } s \in G \\ &\leftrightarrow a_s | X_\gamma = 0 \ \text{ for every } s \in G\,. \end{aligned}$$

The last assertion is a consequence of standard results for reduced crossed products. □

In the following we shall eventually show when the algebras $A(\Sigma)$ is isomorphic to the algebra of all continuous operator fields over the fibered space $\{\Gamma | A(\Sigma_\gamma)\}$. For this purpose the conditions that we have to claim are the Hausdorff property of the space Γ and the continuity of the function: $\gamma \to \|\rho_\gamma(a)\|$ for each element $a \in A(\Sigma)$. The following propositions clarify the situations. Write $a(\gamma) = \rho_\gamma(a)$.

Proposition 2.3. *The following assertions are equivalent;*

(a) *Γ is Hausdorff,*

(b) *The map q is a closed map,*

(c) *The function: $\gamma \to \|a(\gamma)\|$ is upper semi-continuous for every element $a \in A(\Sigma)$.*

Proof. The implication (a) ⇒ (b); Since the map q is continuous it carries a closed hence compact subset of X to a compact subset of Γ, and if Γ is Hausdorff the latter set becomes closed in Γ.

The implication (b) ⇒ (c); We assert that for any $\varepsilon > 0$ the set $H = \{\gamma \in \Gamma \mid \|a(\gamma)\| < \varepsilon\}$ is open.

Take a point γ_0 in H. Then by Lemma 2.1 we can find an element $b = \sum f_s \delta_s$ of finite sum such that $f_s | X_{\gamma_0} = 0$ for every s and $\|a + b\| < \varepsilon$. On the other hand, by the compactness of X_{γ_0}, there exists a neighborhood U of X_{γ_0} such that

$$\sum_s |f_s(x)| < \varepsilon - \|a+b\| \qquad \text{for every} \quad x \in U\,.$$

Here by [5; Theorem 3.10(c)] the union of all members of X/R which are subsets of U is an open set containing X_{γ_0}. Hence its quotient image V becomes a neighborhood of γ_0 such that for every $\gamma \in V$ we have

$$\begin{aligned} \|a(\gamma)\| &\le \|a(\gamma) + b(\gamma)\| + \left\| \sum_s (f_s | X_\gamma) \rho_\gamma(\delta_s) \right\| \\ &\le \|a+b\| + \sup_{x \in U} \sum |f_s(x)| < \varepsilon\,. \end{aligned}$$

This completes the assertion.

The implication (c) ⇒ (a); Take two different points γ_1 and γ_2 in Γ. Since X is compact, there exists a continuous function f such that $0 \le f \le 1$, $f | X_{\gamma_1} = 0$ and $f | X_{\gamma_2} = 1$. From the assumption the sets $V_1 = \{\gamma \in \Gamma \mid \|\rho_\gamma(f)\| < 1/2\}$ and $V_2 = \{\gamma \in \Gamma \mid \|\rho_\gamma(1-f)\| < 1/2\}$ are disjoint open subsets of Γ which contain γ_1 and γ_2, respectively. Hence, Γ is a Hausdorff space. □

PROPOSITION 2.4. *The following assertions are equivalent;*

(a) *q is an open map,*

(b) *The closure of any saturated subset of X is also saturated,*

(c) *The function: $\gamma\to\|a(\gamma)\|$ is lower semicontinuous for every element $a\in A(\Sigma)$,*

(d) *The space of ideals $\{I_\gamma \mid \gamma\in\Gamma\}$ with hull-kernel topology is homeomorphic to Γ.*

PROOF. The implication (a) $\Rightarrow$ (b); Take a set $S=\bigcup_{\gamma\in\Lambda}X_\gamma$ and a point $x\in\bar{S}$, the closure of S. Let y be a point of X equivalent to x and U be a neighborhood of y. The image $q(U)$ is then a neighborhood of $q(y)$, and $q(U)\cap\Lambda\neq\varnothing$ because $q(y)=q(x)\in\bar{\Lambda}$. Hence,

$$U\cap q^{-1}(\Lambda)=U\cap S\neq\varnothing\,,$$

which implies that $y\in\bar{S}$. Namely, $\bar{S}=R(\bar{S})$.

The implication (b) $\Rightarrow$ (c); We assert that for any $\varepsilon>0$ the set $F=\{\gamma\in\Gamma \mid \|a(\gamma)\|\leq\varepsilon\}$ is closed. Here we may assume that a is positive. Take a point $\gamma_0\in\bar{F}$ and suppose that $\|a(\gamma_0)\|>\varepsilon$. We consider a continuous function h on the real line defined as

$$h(t)=\begin{cases}0 & \text{if}\quad t\leq(\varepsilon+\|a(\gamma_0)\|)/2\\ 1 & \text{if}\quad t\geq\|a(\gamma_0)\|\\ \text{linear} & \text{otherwise}\,.\end{cases}$$

Then,

$$h(a)(\gamma)=h(\rho_\gamma(a))=0 \qquad \text{for every}\quad \gamma\in F\,,$$

hence by Lemma 2.2 every Fourier coefficient of $h(a)(\gamma)$ vanishes on X_γ. On the other hand, from the assumption for (b) the set X_{γ_0} is contained in the closure of $q^{-1}(F)$. Therefore every Fourier coefficient of $h(a)(\gamma_0)$ vanishes on X_{γ_0} and $h(a)(\gamma_0)=0$, whereas the property of the function $h(t)$ tells us that $h(a)(\gamma_0)\neq 0$. This is a contradiction and F is a closed set.

The implication (c) $\Rightarrow$ (d); Suppose that $\gamma_\alpha\to\gamma_0$ in Γ and take an element a in the intersection of the set $\{I_{\gamma_\alpha} \mid \alpha\geq\alpha_0\}$ for an arbitrary fixed index α_0. For any $\varepsilon>0$ the point γ_0 belongs to the closure of the set $\{\gamma\in\Gamma \mid \|a(\gamma)\|\leq\varepsilon\}$, hence $\|a(\gamma_0)\|\leq\varepsilon$ because the set is closed. It follows that $a(\gamma_0)=0$, that is, $a\in I_{\gamma_0}$. Thus the map: $\gamma\to I_\gamma$ is continuous. Conversely, let F be a closed set in Γ. We assert that the set $H=\{I_\gamma \mid \gamma\in F\}$ is hull-kernel closed. Thus take an ideal $I_{\gamma_0}\in\bar{H}$ and suppose that $\gamma_0\notin F$. Then, $X_{\gamma_0}\cap\overline{q^{-1}(F)}=\varnothing$ and we can find a continuous function f on X such that $f|X_\gamma=0$ for every $\gamma\in F$ and $f|X_{\gamma_0}=1$. It follows that f belongs to the intersection of any subset of H and $f\notin I_{\gamma_0}$, a contradiction.

The implication (d) $\Rightarrow$ (a); Take an open set O in X and an arbitrary point $\gamma\in q(O)^c$, the complement of $q(O)$. Then, $X_\gamma\cap O=\varnothing$ which implies that the set $q^{-1}(q(O)^c)$ is contained in the closed set O^c. Therefore, for any point x of O there exists a continuous function f_x such that $f_x|q^{-1}(q(O)^c)=0$ and $f_x(x)=1$. This means that any ideal I_{γ_0} for $\gamma_0 \in q(O)$ does not belong to the closure of the set $\{I_\gamma \mid \gamma \in q(O)^c\}$. Hence, by assumption

$$q(O) \cap \overline{q(O)^c} = \varnothing$$

and $q(O)^c = \overline{q(O)^c}$. Thus, q is an open map. This completes all proofs. □

III. Main results and applications.

In order to state our main theorem we must recall the definition of the algebra of continuous operator fields over a fibred space $\{Y|A(t)\}$ where Y is a compact Hausdorff space and $A(t)$ is a C^*-algebra assigned for a point $t \in Y$ ([2], [10]). Let $\mathscr{F}$ be a $*$-algebra of cross sections on Y such that the function: $t \to \|b(t)\|$ is continuous for every element $b \in \mathscr{F}$ and at each point t the set $\{b(t) \mid b \in \mathscr{F}\}$ forms a dense $*$-subalgebra of $A(t)$. A cross section or an operator field a on $\{Y|A(t)\}$ is said to be continuous at a point t_0 with respect to $\mathscr{F}$ if for each $\varepsilon > 0$ there exist an element $b \in \mathscr{F}$ and a neighborhood U of t_0 such that $\|a(t) - b(t)\| < \varepsilon$ for every $t \in U$. The algebra of all continuous operator fields on $\{Y|A(t)\}$ then forms a C^*-algebra $C_{\mathscr{F}}(Y|A(t))$.

THEOREM 3.1. *Suppose that both conditions in Propositions* 2.3 *and* 2.4 *hold and let* $\mathscr{F}$ *be the family of cross sections* $a(\gamma)$*'s on* $\{\Gamma|A(\Sigma_\gamma)\}$ *coming from the elements of* $A(\Sigma)$. *Then* $A(\Sigma)$ *is isomorphic to the* C^**-algebra* $C_{\mathscr{F}}(\Gamma|A(\Sigma_\gamma))$.

PROOF. Define the map Φ from $A(\Sigma)$ into $C_{\mathscr{F}}(\Gamma|A(\Sigma_\gamma))$ by $\Phi(a) = \{a(\gamma)\}$. By Lemma 2.2, Φ is a $*$-isomorphism. We assert that Φ is an onto map. Take two different points γ_1 and γ_2 in Γ and a continuous function f on X such that $f|X_{\gamma_1}=0$ and $f|X_{\gamma_2}=1$. Then, $f \in I_{\gamma_1}$ and $1-f \in I_{\gamma_2}$, which means that $I_{\gamma_1} + I_{\gamma_2} = A(\Sigma)$. It follows that for an arbitrary pair of elements (c, d) in $(A(\Sigma_1), A(\Sigma_2))$ there exists an element $a \in A(\Sigma)$ such that $a(\gamma_1) = c$ and $a(\gamma_2) = d$. In fact, taking two elements a_1 and a_2 in $A(\Sigma)$ with $a_1(\gamma_1) = c$ and $a_2(\gamma_2) = d$ it suffices to put

$$a = a_1 - b_1 = a_2 - b_2$$

for an expression $a_1 - a_2 = b_1 - b_2$ where $b_1 \in I_{\gamma_1}$ and $b_2 \in I_{\gamma_2}$. Therefore, by the non-commutative Stone-Weierstrass theorem for continuous operator fields [10; Theorem 2.2] we see that $\Phi(A(\Sigma)) = C_{\mathscr{F}}(\Gamma|A(\Sigma_\gamma))$. □

REMARK 3.2. Actually what we have proved here is the fact that under the present conditions the set of hulls $\{h(I_\gamma) \mid \gamma \in \Gamma\}$ forms a continuous decomposition of the primitive ideal space Prim$(A(\Sigma))$ of $A(\Sigma)$ in the sense of [10; §3].

As an application of the above theorem we consider first the structure of the group C^*-algebra of 3-dimensional discrete Heisenberg group described as

$$H=\left\{\begin{pmatrix}1 & l & m\\ 0 & 1 & n\\ 0 & 0 & 1\end{pmatrix} \,\middle|\, l, m, n \in \boldsymbol{Z}\right\}.$$

As is well known, H is regarded as a semidirect product of the commutative group $\boldsymbol{Z}\times\boldsymbol{Z}$ by $\boldsymbol{Z}$ through the isomorphisms;

$$\left\{\begin{pmatrix}1 & 0 & m\\ 0 & 1 & n\\ 0 & 0 & 1\end{pmatrix} \,\middle|\, m, n \in \boldsymbol{Z}\right\} \simeq \boldsymbol{Z}\times\boldsymbol{Z},$$

$$\left\{\begin{pmatrix}1 & l & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \,\middle|\, l \in \boldsymbol{Z}\right\} \simeq \boldsymbol{Z}.$$

Hence the group is amenable and its group C^*-algebra $C^*(H)$ is isomorphic to the crossed product $C^*(\boldsymbol{Z}^2)\times_\alpha \boldsymbol{Z}$ where α is the action of $\boldsymbol{Z}$ on the group C^*-algebra $C^*(\boldsymbol{Z}^2)$ induced from that of $\boldsymbol{Z}$ on $\boldsymbol{Z}^2$. Furthermore, when we regard $C^*(\boldsymbol{Z}^2)$ as $C(\boldsymbol{T}^2)$, the algebra of all continuous functions on the 2-dimensional tori $\boldsymbol{T}^2$ the generating homeomorphism σ of $\boldsymbol{T}^2$ derived from the action α of $\boldsymbol{Z}$ on $C(\boldsymbol{T}^2)$ is defined as

$$\sigma(s, t)=(s, t-s) \qquad \text{(cf. [11; §4.4])}.$$

Thus, $\boldsymbol{T}^2=\bigcup_{s\in \boldsymbol{T}}(s, \boldsymbol{T})$ is a decomposition treated above where the parameter space Γ is naturally identified with the torus $\boldsymbol{T}$. It is then obvious that the condition (b) in Proposition 2.4 holds, and we have the following corollary.

COROLLARY 3.3. *The group C^*-algebra $C^*(H)$ is isomorphic to the C^*-algebra of all continuous fields $C_{\mathscr{F}}(T\,|\,A(\Sigma_s))$ over the fibred space $\{T|A(\Sigma_s)\}$ where $\Sigma_s=\{(s, \boldsymbol{T}), \sigma|(s, \boldsymbol{T})\}$. At each level s an irrational or a rational rotation C^*-algebra A_s appears as the fibre algebra $A_s=A(\Sigma_s)$ according to the conditions s being irrational or rational.* □

A similar topological reduction of the algebra $C^*(H)$ has been obtained by Anderson and Paschke [1; Theorem 1.1], but not as a decomposition of the above topological dynamical system. The result in the present form has been proved first by Tamura [7].

Next, consider the so-called non-commutative solid torus $D_\theta=D^2\times_\theta \boldsymbol{Z}$ which plays a basic rôle through the discussions in [6] and [7]. Here D^2 means the unit disk in the complex plane and the crossed product is defined by the θ-rotation in D^2 along the origin. One then easily sees that the relevant dynamical system on D^2 has the decomposition space [0, 1], the unit interval, and moreover satisfies the condition (b) in Proposition 2.4. Therefore this is also the case for which the theorem is applicable. Let A_θ be the rotation algebra $C(\boldsymbol{T})\times_\theta \boldsymbol{Z}$ with respect to the θ-rotation on the unit circle. It is then shown ([7; Proposition 6.4]) that we can further transform the resulting algebra of continuous operator fields into the algebra of all A_θ-valued continuous functions on $[0, 1]$, $C([0, 1], A_\theta)$ so that the algebra D_θ is realized as a C^*-subalgebra of $C([0, 1], A_\theta)$ with a restriction for the values at the point 0.

As the third example, we consider the structure of the rational rotation algebra A_θ for the rotation $\theta = q/p$. In this case, every point of the torus $\boldsymbol{T}$ is a periodic point with the period p, and one easily sees that the orbit space is again homeomorphic with $\boldsymbol{T}$. Therefore, we obtain a topological reduction of the algebra A_θ over $\boldsymbol{T}$ by our theorem. Here every transformation group algebra $A(\Sigma_s)$ corresponding to the orbit $\{s + qi/p \mid 0 \leq i \leq p-1\}$ is isomorphic to the crossed product $l^\infty(p) \times_{\alpha_p} \boldsymbol{Z}$ with respect to the cyclic shift α_p on the set $\{0, 1, \cdots, p-1\}$. Moreover one can easily verify that the fibred space $\{\boldsymbol{T} | A(\Sigma_s)\}$ turns out to be a fibre bundle with the fibre $l^\infty(p) \times_{\alpha_s} \boldsymbol{Z}$ and A_θ is isomorphic to the algebra of all cross sections in this bundle. On the other hand, A_θ is known to be a p-homogeneous C^*-algebra (cf. [11; Corollary 4.1.9]), hence it is isomorphic to the algebra of all sections in the fibre bundle over $\boldsymbol{T}^2$ with the fibre M_p, $p \times p$ matrix algebra (cf. [11; Theorem 4.2.1] and [8; Theorem 5]). In the present context, this further structure of A_θ is obtained from the reduction of the fibre algebra $l^\infty(p) \times_{\alpha_p} \boldsymbol{Z}$ as a C^*-algebra combined with the first reduction before.

Now recall the case of a topological dynamical system $\Sigma = (X, \sigma)$ for a distal homeomorphism σ. As we said before, in this case X is always decomposed into the disjoint union of minimal closed subsets (cf. [11; Proposition 1.2.5]). Hence the condition (b) in Proposition 2.4 obviously holds for this decomposition, whereas it may not be a Hausdorff decomposition in general. We shall show that if the collection of homeomorphisms $\{\sigma^n \mid n \in \boldsymbol{Z}\}$ are equicontinuous (necessarily σ is distal) the decomposition becomes a Hausdorff decomposition, and we can apply our reduction theorem to the system Σ. Every fibre algebra $A(\Sigma_y)$ in this reduction is simple, because Σ_y is minimal. We proceed however our arguments keeping our general setting $\Sigma = (X, G, \alpha)$ on. Recall that a compact space X has a unique uniform structure which describes the original topology. We denote $\mathfrak{U} = \{\lambda, \mu, \cdots\}$ the family of indexes on X which determines the uniform structure on X. The family $\mathfrak{U}$ is a filter of open sets in the product space $X \times X$ satisfying the following condition;

(a) Every index λ contains the diagonal set Δ,

(b) If $\lambda = \{(x, y)\} \in \mathfrak{U}$, then $\lambda^{-1} = \{(y, x)\} \in \mathfrak{U}$,

(c) For each index λ there exists an index μ such that $\mu^2 \subset \lambda$, where μ^2 is defined as the set of $X \times X$, $\{(x, z)\}$ for whose pair (x, z) there exists $y \in X$ such that $(x, y) \in \mu$ and $(y, z) \in \mu$. For each point $x \in X$ the sets

$$\{\lambda(x) \mid \lambda \in \mathfrak{U}\} \qquad \text{where} \quad \lambda(x) = \{y \mid (x, y) \in \lambda\}$$

form a base of neighborhoods of x.

We recall the system Σ equicontinuous if the action α is equicontinuous. Namely, for any index λ there exists an index μ such that $(x, y) \in \mu$ implies that $(\alpha_s(x), \alpha_s(y)) \in \lambda$ for every $s \in G$. As in the case of usual topological dynamical systems, we say that Σ is minimal if all orbits of the points in X are dense in X.

THEOREM 3.4. *If the system $\Sigma=(X, G, \alpha)$ is equicontinuous, the space X is decomposed into the disjoint union of closed invariant subsets $\{X_\gamma \mid \gamma\in\Gamma\}$ such that each system $\Sigma_\gamma=(X_\gamma, G, \alpha|X_\gamma)$ is minimal. This decomposition satisfies the conditions of Propositions* 2.3 *and* 2.4. *Hence the algebra $A(\Sigma)$ is isomorphic to $C_{\mathscr{F}}(\Gamma|A(\Sigma_\gamma))$.*

PROOF. Take an arbitrary point x_0 and denote by $O(x_0)$ the orbit of x_0. For the assertion it is enough to show that for any $y\in\overline{O(x_0)}$, $\overline{O(y)}=\overline{O(x_0)}$. Let λ be an index in $\mathfrak{U}$. By the equicontinuity, there exists an index μ such that $(x, z)\in\mu$ implies that $(\alpha_s(x), \alpha_s(z))\in\lambda$ for every $s\in G$. Choose an element $t\in G$ such that $\alpha_t(x_0)\in\mu^{-1}(y)$, that is, $(\alpha_t(x_0), y)\in\mu$. It follows that

$$(x_0, \alpha_t^{-1}(y))\in\lambda\,, \quad \text{that is}\,, \quad \alpha_t^{-1}(y)\in\lambda(x_0)\,.$$

As λ is arbitrary, $x_0\in\overline{O(y)}$ and $\overline{O(y)}=\overline{O(x_0)}$. Thus we have a decomposition $X=\bigcup_{\gamma\in\Gamma}X_\gamma$ where $X_\gamma=\overline{O(x)}$ for every $x\in X_\gamma$. Therefore, the decomposition satisfies the condition (b) in Proposition 2.4.

In order to show that Γ becomes a Hausdorff space, we shall show that the quotient map q is closed. Thus, let F be a closed subset of X and let $R(F)$ be its saturation by the relation R. Take an arbitrary net $\{x_\alpha\}$ in $R(F)$ converging to x_0. We may assume that $\{x_\alpha\}$ is not eventually contained in any set X_γ in $R(F)$. For, otherwise, the point x_0 belongs obviously to some set X_{γ_0} in $R(F)$. Let $x_\alpha\in X_{\gamma_\alpha}$ and choose an element y_α in $F\cap X_{\gamma_\alpha}$. One may then assume that $\{y_\alpha\}$ also converges to a point $y_0\in F$. We assert that $x_0 \underset{R}{\sim} y_0$ whence $x_0\in R(F)$. For an arbitrary index λ, choose an index μ such that $\mu^2\subset\lambda$, and furthermore an index ν such that $\nu^2\subset\mu$. Take an index ω such that

$$(x, y)\in\omega \text{ implies that } (\alpha_s(x), \alpha_s(y))\in\mu \text{ for every } s\in G\,.$$

As $\nu^{-1}(x_0)$ and $\omega(y_0)$ are neighborhood of x_0 and y_0 respectively, there exists an index α_0 for those converging nets such that $x_{\alpha_0}\in\nu^{-1}(x_0)$ and $y_{\alpha_0}\in\omega(y_0)$. Besides, since x_{α_0} belongs to $\overline{O(y_{\alpha_0})}$ there exists an element t of G such that $\alpha_t(y_{\alpha_0})\in\nu^{-1}(x_{\alpha_0})$, that is, $(\alpha_t(y_{\alpha_0}), x_{\alpha_0})\in\nu$. On the other hand, $(x_{\alpha_0}, x_0)\in\nu$ hence $(\alpha_t(y_{\alpha_0}), x_0)\in\mu$. Now since $(y_0, y_{\alpha_0})\in\omega$,

$$(\alpha_s(y_0), \alpha_s(y_{\alpha_0}))\in\mu \qquad \text{for every} \quad s\in G$$

and in particular $(\alpha_t(y_0), \alpha_t(y_{\alpha_0}))\in\mu$. Therefore, $(\alpha_t(y_0), x_0)\in\lambda$, that is, $\alpha_t(y_0)\in\lambda^{-1}(x_0)$. This means that $x_0\in\overline{O(y_0)}$ and $x_0\in R(F)$. This completes all proofs. □

In the above theorem, the fibre algebra $A(\Sigma_\gamma)$ may not be simple, even if Σ_γ is minimal, when G is non-commutative. It becomes simple if the action α satisfies the condition mentioned in [4; Theorem 4.4], which is necessarily satisfied when G is commutative.

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