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Homogeneous Symplectic Manifolds and Dipolarizations in Lie Algebras

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Introduction.

A *parakähler manifold* is, by definition, a symplectic manifold with a pair of transversal Lagrangian foliations. A parakähler manifold was originally introduced by P. Libermann [10] from a different point of view (See also [3]). Let M be a parakähler manifold. By an *automorphism* of M we mean a symplectomorphism which preserves each of the two foliations. It turns out that the totality of automorphisms of M becomes a finite-dimensional Lie group (Section 1). If that group $\mathrm{Aut}\, M$ acts transitively on M, then M is called a *homogeneous parakähler manifold*. In our previous paper [3], we have introduced a class of homogeneous parakähler manifolds, called *parahermitian symmetric spaces*. A parahermitian symmetric space is a homogeneous parakähler manifold M which can be represented as an affine symmetric coset space with respect to the identity component of $\mathrm{Aut}\, M$. Under the assumption that the automorphism groups are semisimple, parahermitian symmetric spaces were classified up to local isomorphisms ([3, 4]). Under the same assumption, we have constructed a natural compactification $\tilde{M}$ of a parahermitian symmetric space M and have studied geometric properties of $\tilde{M}$ ([5]). It should be noted that this compactification has some applications to harmonic analysis on a parahermitian symmetric space M (cf. Ørsted [13]).

The first aim of this paper is to give a simple algebraic method of constructing homogeneous parakähler manifolds. First we introduce a *parakähler algebra* which is an intermediate algebraic interpretation of a homogeneous parakähler structure (Section 2). A parakähler algebra occupies the same situation as a Kähler algebra (Vinberg-Gindikin [12]) does for a homogeneous Kähler manifold. In Section 3, we introduce much simpler algebraic object, called a *weak dipolarization* and a *dipolarization* in a Lie algebra $\mathfrak{g}$. A homogeneous parakähler structure is perfectly described by a weak dipolarization (Theorem 3.6). A dipolarization is a stronger concept than a weak dipolarization. But, if the Lie algebra $\mathfrak{g}$ is semisimple, then a weak dipolarization is always a dipolarization. Our second aim is to study homogeneous parakähler manifolds

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which are obtained from semisimple graded Lie algebras. First of all we prove that a semisimple graded Lie algebra has a natural dipolarization, called the *canonical dipolarization* (Theorem 4.2). Let G be a connected semisimple Lie group with finite center and L be the Levi subgroup of a parabolic subgroup of G. We prove that the coset space G/L has a G-invariant parakähler structure corresponding to a canonical dipolarization coming from a gradation in the Lie algebra $\mathrm{Lie}\, G$ (Theorem 4.3). Finally we construct an equivariant compactification of the G-homogeneous parakähler manifold G/L (Theorem 4.7), which is a generalization of the compactification constructed in [5] for a parahermitian symmetric space.

We refer terminologies and basic facts on graded Lie algebras to our previous paper [7]. Throughout the present paper, Lie algebras are finite-dimensional. We abbreviate a "graded Lie algebra" as a GLA. $C^\infty(M)$ denotes the ring of smooth functions of class C^∞ on a manifold M.

1. Parakähler Manifolds.

DEFINITION 1.1. Let M be a symplectic manifold with symplectic form ω. Let (F^+, F^-) be a pair of transversal foliations on M. The triple $(M, \omega, F^\pm)$ is then called a *parakähler manifold*, if each leaf of $F^\pm$ is a Lagrangian submanifold of M.

Let $(M, \omega, F^\pm)$ be a $2n$-dimensional parakähler manifold. Let $p \in M$. Then there exist two Lagrangian leaves $F^+(p)$ of F^+ and $F^-(p)$ of F^- both passing through p. Note that $\dim F^\pm(p) = n$. Let $\hat{I}_p$ be the linear endomorphism of the tangent space T_pM at p to M such that $\hat{I}_p = \pm 1$ on the tangent spaces $T_pF^\pm(p)$, respectively. Then the tensor field $\hat{I} := (\hat{I}_p)_{p \in M}$ is a paracomplex structure [3] on M. Also $\hat{I}$ satisfies the integrability condition [3]:

$$[\hat{I}X, \hat{I}Y] = \hat{I}[\hat{I}X, Y] + \hat{I}[X, \hat{I}Y] - [X, Y], \tag{1.1}$$

where X and Y are vector fields on M. We need the following

LEMMA 1.2. *Let (M, ω) be a symplectic manifold and $F^\pm$ be two foliations on M. Suppose that the tangent bundle TM of M is expressed as the Whitney sum of F^+ and F^-. Let $\hat{I} = (\hat{I}_p)_{p \in M}$ be a (1,1)-tensor field on M such that $\hat{I}_p = \pm 1$ on the fibers $F^\pm_p$ of $F^\pm$ through a point $p \in M$. Then each leaf of $F^\pm$ is a Lagrangian submanifold of M if and only if we have the equality*

$$\omega(\hat{I}X, Y) + \omega(X, \hat{I}Y) = 0 \tag{1.2}$$

for any vector fields X, Y on M.

PROOF. Suppose that leaves of $F^\pm$ are Lagrangian submanifolds, or equivalently, the fibers $F^\pm_p$, $p \in M$, are Lagrangian subspaces of the tangent space T_pM. Let X_p, $Y_p \in F^+_p$ (resp. F^-_p). Then $\omega(\hat{I}_pX_p, Y_p) = \omega(X_p, \hat{I}_pY_p) = \omega(X_p, Y_p) = 0$ (resp. $= -\omega(X_p, Y_p)$

$=0$). Suppose that $X_p \in F_p^+$ and $Y_p \in F_p^-$. Then $\omega(\hat{I}_p X_p, Y_p) = \omega(X_p, Y_p) = -\omega(X_p, \hat{I}_p Y_p)$. Thus we have (1.2). Conversely suppose that (1.2) is valid. Then it follows that $F_p^\pm$ are two totally isotropic subspaces of T_pM. Since $T_pM = F_p^+ \oplus F_p^-$ (direct sum), $F_p^\pm$ are Lagrangian subspaces of T_pM. Q.E.D.

Let $(M, \omega, F^\pm)$ be a parakähler manifold. We say that a symplectomorphism φ of M is an *automorphism* of $(M, \omega, F^\pm)$ if φ leaves the associated paracomplex structure $\hat{I}$ invariant (or equivalently, φ permutes respective leaves of the foliations $F^\pm$). We denote by $\mathrm{Aut}(M, \omega, \hat{I})$ the group of automorphisms of $(M, \omega, F^\pm)$. Then the group $\mathrm{Aut}(M, \omega, \hat{I})$ is a Lie group. In fact, if we put $g(X, Y) = \omega(\hat{I}X, Y)$ for vector fields X, Y on M, then it follows from Lemma 1.2 that g is an $\mathrm{Aut}(M, \omega, \hat{I})$-invariant pseudo-riemannian metric on M. Thus $\mathrm{Aut}(M, \omega, \hat{I})$ is a closed subgroup of the isometry group of M with respect to g. If the group $\mathrm{Aut}(M, \omega, \hat{I})$ acts transitively on M, then the parakähler manifold M is called *homogeneous*. Let G be a connected Lie group and H be a closed subgroup of G. Suppose that the coset space G/H has a parakähler structure $\{\omega, F^\pm\}$. Let $\hat{I}$ denote the paracomplex structure associated with $F^\pm$. If G leaves both ω and $\hat{I}$ invariant, then we say that G/H is a *parakähler coset space*.

EXAMPLES 1.3. (i) Let N be a complete simply connected Riemannian manifold whose sectional curvature is less than or equal to -1 everywhere. Let M be the smooth manifold of unit speed geodesics on N. Then M is a parakähler manifold (Kanai [2]). (ii) Parahermitian symmetric spaces are homogeneous parakähler manifolds ([3]).

2. Parakähler algebras.

DEFINITION 2.1. Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, I a linear endomorphism of $\mathfrak{g}$ and ρ be an alternating 2-form on $\mathfrak{g}$. Then the quadruple $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$ is called a *parakähler algebra*, if the following conditions (2.1)–(2.6) are satisfied:

$I(\mathfrak{h}) \subset \mathfrak{h}$ and $I^2 \equiv 1 \bmod \mathfrak{h}$. The ± 1-eigenspaces under the operator on $\mathfrak{g}/\mathfrak{h}$ induced by I are equi-dimensional, (2.1)

$$[X, IY] \equiv I[X, Y] \mod \mathfrak{h}, \qquad X \in \mathfrak{h}, \ Y \in \mathfrak{g}, \tag{2.2}$$

$$[IX, IY] \equiv I[IX, Y] + I[X, IY] - [X, Y] \mod \mathfrak{h}, \qquad X, Y \in \mathfrak{g}, \tag{2.3}$$

$$\rho(X, \mathfrak{g}) = 0 \quad \text{if and only if} \quad X \in \mathfrak{h}, \tag{2.4}$$

$$\rho(IX, IY) = -\rho(X, Y), \qquad X, Y \in \mathfrak{g}, \tag{2.5}$$

$$\rho([X, Y], Z) + \rho([Y, Z], X) + \rho([Z, X], Y) = 0, \qquad X, Y, Z \in \mathfrak{g}. \tag{2.6}$$

If the 2-form ρ is a coboundary df of a linear form f in the sense of the Lie algebra cohomology, then the parakähler algebra $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$ is said to be nondegenerate. In this case (2.4)–(2.6) can be replaced by

$$f([X, \mathfrak{g}])=0 \qquad \text{if and only if} \quad X\in\mathfrak{h}\,, \tag{2.7}$$

$$f([IX, IY])=-f([X, Y])\,, \qquad X, Y\in\mathfrak{g}\,. \tag{2.8}$$

PROPOSITION 2.2. *Let G be a connected Lie group and H be a closed subgroup of G. Let $\mathfrak{g}=\mathrm{Lie}\,G$ and $\mathfrak{h}=\mathrm{Lie}\,H$. Suppose that G/H is a parakähler coset space. Then there exist a linear endomorphism I of $\mathfrak{g}$ and an alternating 2-form ρ on $\mathfrak{g}$ such that $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$ is a parakähler algebra.*

PROOF. Let $\dim G/H=2n$, and let $\hat{I}$ be the associated (G-invariant) paracomplex structure on G/H and ω be the symplectic form. Choose a local coordinate system $(u^1, \cdots, u^{2n}, u^{2n+1}, \cdots, u^m)$ around the unit element $e\in G$ satisfying the two conditions: (1) $u^i(e)=0$ $(1\le i\le m)$, (2) there exists a cubic neighborhood U of e with respect to $(u^1, \cdots, u^m)$ which satisfies

$$U\cap H=\{g\in U : u^1(g)=\cdots=u^{2n}(g)=0\}\,.$$

Let F be the set of elements $g\in U$ satisfying $u^i(g)=0$, $2n+1\le i\le m$. Let π be the natural projection of G onto G/H. The restriction $\pi|_F$ is a diffeomorphism of F onto an open neighborhood of the origin o in G/H. We identify $\mathfrak{g}$ with the tangent space T_eG. Let $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ corresponding to the tangent space T_eF under the above identification. Obviously we have $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ (a vector space direct sum). The differential π_{*e} is a linear surjection of $\mathfrak{g}$ onto the tangent space $T_o(G/H)$, whose kernel is $\mathfrak{h}$. We define a linear endomorphism I on $\mathfrak{g}$ by putting

$$I=\begin{cases} 0 & \text{on} \quad \mathfrak{h}\,, \\ ((\pi|_F)_{*e}^{-1}\hat{I}_o\pi_{*e} & \text{on} \quad \mathfrak{m}\,, \end{cases} \tag{2.9}$$

where $\hat{I}_o$ denotes the value of $\hat{I}$ at the point o. Then, making use of the same technique as in the case of a homogeneous complex structure (Fröhlicher [1]), we get

$$\pi_{*e}I=\hat{I}_o\pi_{*e}\,, \tag{2.10}$$

$$\pi_{*e}(I[X, Y])=\pi_{*e}([X, IY])\,, \qquad X\in\mathfrak{h}\,,\ Y\in\mathfrak{g}\,, \tag{2.11}$$

$$\pi_{*e}([IX, IY]-I[IX, Y]-I[X, IY]+[X, Y])=0\,, \qquad X, Y\in\mathfrak{g}\,. \tag{2.12}$$

In fact, (2.11) and (2.12) follow from the G-invariance of $\hat{I}$ and (1.1), respectively. It follows from (2.10)–(2.12) that I satisfies the conditions (2.1)–(2.3). The pull-back $\rho=\pi^*\omega$ is a G-invariant closed 2-form on G and hence it is viewed as an alternating 2-form on $\mathfrak{g}$. (2.4) and (2.5) are obtained from the nondegeneracy of ω and (1.2), respectively. Q.E.D.

As for the converse assertion of Proposition 2.2, we have the following

PROPOSITION 2.3. *Let G, H, $\mathfrak{g}$ and $\mathfrak{h}$ be the same as in Proposition 2.2. Suppose that the pair $\{\mathfrak{g}, \mathfrak{h}\}$ has the structure of a parakähler algebra $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$. Suppose further that*

$$[\mathrm{Ad}\, a, I] \equiv 0 \mod \mathfrak{h}, \qquad a \in H, \tag{2.13}$$

$$\rho((\mathrm{Ad}\, a)X, (\mathrm{Ad}\, a)Y) = \rho(X, Y), \qquad a \in H, \ X, Y \in \mathfrak{g}. \tag{2.14}$$

Then G/H has the structure of a parakähler coset space.

PROOF. We identify $\mathfrak{g}/\mathfrak{h}$ with the tangent space $T_o(G/H)$ to G/H at the origin $o \in G/H$. Let $\hat{I}_o$ be the linear endomorphism on $\mathfrak{g}/\mathfrak{h}$ induced by I (cf. (2.1)). Then (2.13) implies that $\hat{I}_o$ commutes with $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}\, a$, $a \in H$. Hence $\hat{I}_o$ extends to a G-invariant almost paracomplex structure on G/H, which will be denoted by $\hat{I}$. The torsion T of $\hat{I}$ is given by [3]

$$T(X, Y) = [\hat{I}X, \hat{I}Y] - \hat{I}[\hat{I}X, Y] - \hat{I}[X, \hat{I}Y] + [X, Y], \tag{2.15}$$

where X, Y are vector fields on G/H. We have to show that T vanishes identically on G/H ([3]). For this purpose we extend the original endomorphism I on $\mathfrak{g}$ to a left-invariant tensor field $\tilde{I}$ on G. Denoting the natural projection $G \to G/H$ by π, we have

$$\pi_* \tilde{I} = \hat{I} \pi_*. \tag{2.16}$$

Let us put

$$\tilde{T}(X, Y) = [\tilde{I}X, \tilde{I}Y] - \tilde{I}[\tilde{I}X, Y] - \tilde{I}[X, \tilde{I}Y] + [X, Y], \tag{2.17}$$

X and Y being vector fields on G. Then it follows that

$$\tilde{T}(X, \xi Y) = \xi \tilde{T}(X, Y) - (X\xi)(\tilde{I}^2 Y - Y), \tag{2.18}$$

where $\xi \in C^\infty(G)$. In view of (2.1), the equality (2.18) implies that $\tilde{T}(X, Y)$ is $C^\infty(G)$-bilinear in X and Y modulo $C^\infty(G)\mathfrak{h}$ (= the submodule, generated by $\mathfrak{h}$, of the $C^\infty(G)$-module of all vector fields on G). Consequently it follows from (2.3) that $\tilde{T}(X, Y) \in C^\infty(G)\mathfrak{h}$. Hence, as in the case of a homogeneous complex structure (Koszul [9]), one can conclude that T vanishes identically on G/H. We have thus proved that $\hat{I}$ is a (G-invariant) paracomplex structure ([3]). In other words, the ± 1-eigenspaces of $\hat{I}$ determine transversal foliations $F^\pm$ on G/H such that the Whitney sum $F^+ \oplus F^-$ is the whole tangent bundle of G/H. By (2.4), there exists a unique alternating 2-form ω_o on $\mathfrak{g}/\mathfrak{h}$ such that $\pi^*\omega_o = \rho$. ω_o is nondegenerate and $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} H$-invariant (cf. (2.4), (2.14)). Hence it extends to a G-invariant symplectic form ω on G/H (cf. (2.6)). (2.5) implies that ω satisfies (1.2), and so $F^\pm$ are Lagrangian foliations. Q.E.D.

REMARK 2.4. If H is connected, then the assertion of Proposition 2.3 holds without assuming (2.13) and (2.14).

3. Dipolarizations in Lie algebras.

DEFINITION 3.1. Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$ and ρ be an alternating 2-form on $\mathfrak{g}$. The triple $\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, \rho\}$ is called a *weak dipolarization* in $\mathfrak{g}$, if the following conditions are satisfied:

$$\mathfrak{g}=\mathfrak{g}^{+}+\mathfrak{g}^{-}, \tag{3.1}$$

$$\text{Put } \mathfrak{h}:=\mathfrak{g}^{+}\cap\mathfrak{g}^{-}. \text{ Then } \rho(X, \mathfrak{g})=0 \text{ if and only if } X\in\mathfrak{h}, \tag{3.2}$$

$$\rho(\mathfrak{g}^{+}, \mathfrak{g}^{+})=\rho(\mathfrak{g}^{-}, \mathfrak{g}^{-})=0, \tag{3.3}$$

$$\rho([X, Y], Z)+\rho([Y, Z], X)+\rho([Z, X], Y)=0, \qquad X, Y, Z\in\mathfrak{g}. \tag{3.4}$$

It follows from (3.1)–(3.3) that in the above definition $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$ are equidimensional (cf. Proof of Lemma 3.4).

DEFINITION 3.2. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$, and let f be a linear form on $\mathfrak{g}$. The triple $\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, f\}$ is called a *dipolarization* in $\mathfrak{g}$ if the following conditions are satisfied:

$$\mathfrak{g}=\mathfrak{g}^{+}+\mathfrak{g}^{-}, \tag{3.5}$$

$$\text{Put } \mathfrak{h}:=\mathfrak{g}^{+}\cap\mathfrak{g}^{-}. \text{ Then } f([X, \mathfrak{g}])=0 \text{ if and only if } X\in\mathfrak{h}, \tag{3.6}$$

$$f([\mathfrak{g}^{+}, \mathfrak{g}^{+}])=f([\mathfrak{g}^{-}, \mathfrak{g}^{-}])=0. \tag{3.7}$$

Note that a dipolarization $\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, f\}$ is a weak dipolarization just by taking df as ρ. We wish to find a relation between parakähler algebras and weak dipolarizations.

LEMMA 3.3. *Let* $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$ *be a parakähler algebra, and let*

$$\mathfrak{g}^{\pm}=\{X\in\mathfrak{g} : IX\equiv \pm X \mod \mathfrak{h}\}. \tag{3.8}$$

Then $\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, \rho\}$ *is a weak dipolarization in* $\mathfrak{g}$ *satisfying* $\mathfrak{g}^{+}\cap\mathfrak{g}^{-}=\mathfrak{h}$.

PROOF. We prove first that $\mathfrak{g}^{+}$ is a subalgebra of $\mathfrak{g}$. Let $X, Y\in\mathfrak{g}^{+}$. Then one can write

$$IX=X+h, \qquad IY=Y+h', \tag{3.9}$$

where $h, h'\in\mathfrak{h}$. By (2.1), (2.2), (2.3) and (3.9) we get

$$\begin{aligned} I[X, Y]&\equiv[IX, Y]+[X, IY]-I[IX, IY] \\ &=2[X, Y]+[h, Y]+[X, h']-I[X, Y]-I[X, h']-I[h, Y]-I[h, h'] \\ &\equiv 2[X, Y]+[h, Y]+[X, h']-I[X, Y]-[X, h']-[h, Y] \qquad \mod \mathfrak{h}. \end{aligned} \tag{3.10}$$

Therefore we have $I[X, Y]\equiv[X, Y] \mod \mathfrak{h}$, which implies that $\mathfrak{g}^{+}$ is a subalgebra of $\mathfrak{g}$. Similarly $\mathfrak{g}^{-}$ is a subalgebra of $\mathfrak{g}$. Let $\hat{I}_o$ be the linear endomorphism on $\mathfrak{g}/\mathfrak{h}$ induced by I. By (2.1) we have $\hat{I}_o^2 = 1$. Let $(\mathfrak{g}/\mathfrak{h})_\pm$ be the ± 1-eigenspaces in $\mathfrak{g}/\mathfrak{h}$ under $\hat{I}_o$. Then we have that $\mathfrak{g}^\pm$ coincide with the complete inverse images of $(\mathfrak{g}/\mathfrak{h})_\pm$ under the canonical projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{h}$, from which (3.1) follows. Let $X, Y \in \mathfrak{g}^+$ and write them in the form (3.9). We then have from (2.5) and (2.4) that $\rho(X, Y) = -\rho(IX, IY) = -\rho(X, Y)$, and hence we have (3.3). We next show that $\mathfrak{g}^+ \cap \mathfrak{g}^- = \mathfrak{h}$. Since $\mathfrak{g}^\pm$ are the complete inverse images of $(\mathfrak{g}/\mathfrak{h})_\pm$ under the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, $\mathfrak{h}$ is contained in $\mathfrak{g}^\pm$. By this and (3.3) we see $\rho(\mathfrak{h}, \mathfrak{g}) = 0$. Let $Z \in \mathfrak{g}$ and write $Z = Z^+ + Z^-$, $Z^\pm \in \mathfrak{g}^\pm$. Choose $X \in \mathfrak{g}^+ \cap \mathfrak{g}^-$. Then, since ρ satisfies (3.3), one has $\rho(X, Z) = \rho(X, Z^+) + \rho(X, Z^-) = 0$. Z being arbitrary, we conclude by (2.4) that $X \in \mathfrak{h}$. Thus we have proved (3.2). Q.E.D.

Conversely we have

LEMMA 3.4. *Let $\mathfrak{g}$ be a real Lie algebra and let $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ be a weak dipolarization in $\mathfrak{g}$. Put $\mathfrak{h} := \mathfrak{g}^+ \cap \mathfrak{g}^-$. Then the pair $\{\mathfrak{g}, \mathfrak{h}\}$ has the structure of a parakähler algebra.*

PROOF. Let π be the natural projection of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{h}$. Then by (3.1), $\mathfrak{g}/\mathfrak{h} = \pi(\mathfrak{g}^+) + \pi(\mathfrak{g}^-)$. The right-hand side is a direct sum of the vector spaces, since $\pi^{-1}(\pi(\mathfrak{g}^\pm)) = \mathfrak{g}^\pm$ holds. Define an alternating 2-form ω_o on $\mathfrak{g}/\mathfrak{h}$ by putting $\omega_o(\pi(X), \pi(Y)) = \rho(X, Y)$, $X, Y \in \mathfrak{g}$. (3.2) implies that ω_o is well-defined and non-degenerate on $\mathfrak{g}/\mathfrak{h}$. It follows from (3.2) and (3.3) that $\pi(\mathfrak{g}^\pm)$ are maximal totally isotropic subspaces with respect to ω_o. This implies that $\pi(\mathfrak{g}^+)$ and $\pi(\mathfrak{g}^-)$ are equi-dimensional. Define a linear endomorphism $\hat{I}_o$ on $\mathfrak{g}/\mathfrak{h}$ by setting $\hat{I}_0 = \pm 1$ on $\pi(\mathfrak{g}^\pm)$, respectively. Let I be a linear endomorphism on $\mathfrak{g}$ satisfying $\pi I = \hat{I}_0 \pi$. Then I satisfies (2.1). On the other hand, it is easily seen that, with respect to the endomorphism I, $\mathfrak{g}^\pm$ are given by

$$\mathfrak{g}^\pm = \{X \in \mathfrak{g} : IX \equiv \pm X \mod \mathfrak{h}\}. \tag{3.11}$$

In order to prove (2.2), one can assume, in view of (3.1), that Y in (2.2) lies either in $\mathfrak{g}^+$ or in $\mathfrak{g}^-$. Suppose first that $Y \in \mathfrak{g}^+$. One can then write $IY = Y + h'$, where $h' \in \mathfrak{h}$. Therefore, if $X \in \mathfrak{h}$, then $[X, IY] = [X, Y + h'] \equiv [X, Y] \mod \mathfrak{h}$. Since $[X, Y]$ lies in $\mathfrak{g}^+$, we have $I[X, Y] \equiv [X, Y] \mod \mathfrak{h}$ (cf. (3.11)). Thus (2.2) is valid for $Y \in \mathfrak{g}^+$. Similarly (2.2) is valid for $Y \in \mathfrak{g}^-$. Next we wish to prove that the linear endomorphism I satisfies (2.3). We break up into three cases: (i) $X, Y \in \mathfrak{g}^+$, (ii) $X \in \mathfrak{g}^+$, $Y \in \mathfrak{g}^-$, and (iii) $X, Y \in \mathfrak{g}^-$. Let us first consider the case (i). By (3.11) one can write X, Y in the form (3.9). Thus, by using (3.11) and (2.2) just proved, we have

$$\begin{aligned} [IX, IY] &= [X + h, Y + h'] \\ &\equiv [X, Y] + [X, h'] + [h, Y] \qquad \mod \mathfrak{h}, \end{aligned} \tag{3.12}$$

and so

$$
\begin{aligned}
I[IX, Y]+I[X, IY]-[X, Y]& \\
&=I[X+h, Y]+I[X, Y+h']-[X, Y]\\
&=I[X, Y]+I[h, Y]+I[X, Y]+I[X, h']-[X, Y]\\
&\equiv[X, Y]+[h, IY]+[X, Y]+[IX, h']-[X, Y]\\
&=[X, Y]+[h, Y+h']+[X+h, h']\\
&\equiv[X, Y]+[h, Y]+[X, h']\equiv[IX, IY] \qquad \mathrm{mod}\, \mathfrak{h}\,.
\end{aligned} \tag{3.13}
$$

By similar arguments, one can prove (2.3) for the two remaining cases. We shall show (2.5). In the case where $X, Y\in\mathfrak{g}^{\pm}$, it follows from (3.3) that both sides of (2.5) are zero. Suppose that $X\in\mathfrak{g}^{+}$ and $Y\in\mathfrak{g}^{-}$. Then, by (3.11), we have $IX=X+h$, $IY=-Y+h'$, where $h, h'\in\mathfrak{h}$. Therefore, by (3.2),

$$
\begin{aligned}
\rho(IX, IY)&=\rho(X+h, -Y+h')\\
&=-\rho(X, Y)+\rho(X, h')-\rho(h, Y)+\rho(h, h')=-\rho(X, Y)\,,
\end{aligned}
$$

which proves (2.5). Q.E.D.

Let $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$ and $\{\mathfrak{g}', \mathfrak{h}', I', \rho'\}$ be two parakähler algebras. They are said to be *isomorphic* if there exists a Lie isomorphism φ of $\mathfrak{g}$ onto $\mathfrak{g}'$ satisfying the conditions:

$$
\begin{aligned}
&\varphi(\mathfrak{h})=\mathfrak{h}'\,,\\
&\varphi I\equiv I'\varphi \qquad \mathrm{mod}\,\mathfrak{h}'\,,\\
&\varphi^{*}\rho'=\rho\,,
\end{aligned} \tag{3.14}
$$

where φ^* denotes the isomorphism, induced by φ, between the tensor algebras on $\mathfrak{g}$ and $\mathfrak{g}'$. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras, and let $\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, \rho\}$ and $\{\mathfrak{g}'^{+}, \mathfrak{g}'^{-}, \rho'\}$ be weak dipolarizations in $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. They are said to be *isomorphic* if there exists a Lie isomorphism φ of $\mathfrak{g}$ onto $\mathfrak{g}'$ satisfying the conditions

$$
\begin{gathered}
\varphi(\mathfrak{g}^{+})=\mathfrak{g}'^{+}\,, \qquad \varphi(\mathfrak{g}^{-})=\mathfrak{g}'^{-}\,,\\
\varphi^{*}\rho'=\rho\,.
\end{gathered} \tag{3.15}
$$

Combining Lemmas 3.3 and 3.4, we finally obtain

THEOREM 3.5. *Let $\mathfrak{g}$ be a real Lie algebra. Then there exists a bijection between the set of isomorphism classes of parakähler algebra structures on $\mathfrak{g}$ and the set of isomorphism classes of weak dipolarizations in $\mathfrak{g}$.*

PROOF. Let $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}$ be a parakähler algebra and let $\mathfrak{g}^{\pm}$ be the ones given in (3.8). By Lemma 3.3, $\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, \rho\}$ is a weak dipolarization in $\mathfrak{g}$. In view of Lemmas 3.3 and 3.4 the correspondence $\{\mathfrak{g}, \mathfrak{h}, I, \rho\}\mapsto\{\mathfrak{g}^{+}, \mathfrak{g}^{-}, \rho\}$ induces a bijection between the respective isomorphism classes. Q.E.D.

We finally have

THEOREM 3.6. *Let G be a connected Lie group and H be a closed subgroup of G. Let $\mathfrak{g}=\mathrm{Lie}G$ and $\mathfrak{h}=\mathrm{Lie}H$. Suppose that G/H is a parakähler coset space. Then $\mathfrak{g}$ admits a weak dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ such that*

$$\mathfrak{h}=\mathfrak{g}^+\cap\mathfrak{g}^- . \tag{3.16}$$

Conversely, suppose that there exists a weak dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ in $\mathfrak{g}$ satisfying the conditions (3.16) *and*

$$(\mathrm{Ad}_{\mathfrak{g}} H)\mathfrak{g}^{\pm}\subset\mathfrak{g}^{\pm} , \tag{3.17}$$

$$\rho \textit{ is } \mathrm{Ad}_{\mathfrak{g}} H\textit{-invariant} . \tag{3.18}$$

Then G/H has the structure of a parakähler coset space.

PROOF. The first assertion is immediate from Proposition 2.2 and Lemma 3.3. Suppose that $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ is a weak dipolarization in $\mathfrak{g}$ satisfying (3.16)–(3.18). Let $a\in H$. Then, under the notations in the proof of Lemma 3.4, we have that $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}a$ leaves $\pi(\mathfrak{g}^{\pm})$ stable and that $[\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}a, \hat{I}_o]=0$. This implies that $[\mathrm{Ad}_{\mathfrak{g}}a, I]\equiv 0 \mod \mathfrak{h}$, or equivalently, (2.13) is valid. Therefore the second assertion follows from Lemma 3.4 and Proposition 2.3. Q.E.D.

The above manifold G/H is called the *parakähler coset space corresponding to a weak dipolarization* $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$.

4. Parakähler manifolds associated with graded Lie algebras.

4.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra and B be the Killing form of $\mathfrak{g}$. Note that a weak dipolarization in $\mathfrak{g}$ is always a dipolarization, since the second cohomology group of $\mathfrak{g}$ vanishes.

LEMMA 4.1. *Let $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ be a dipolarization in $\mathfrak{g}$. Then $\mathfrak{h}:=\mathfrak{g}^+\cap\mathfrak{g}^-$ coincides with the centralizer $\mathfrak{c}(Z)$ in $\mathfrak{g}$ of an element $Z\in\mathfrak{g}$.*

PROOF. Let $Z\in\mathfrak{g}$ be a unique element satisfying

$$B(Z, X)=f(X) , \qquad X\in\mathfrak{g} . \tag{4.1}$$

Choose an element $X\in\mathfrak{h}$. Then for any element $Y\in\mathfrak{g}$, we have

$$B([Z, X], Y)=B(Z, [X, Y])=f([X, Y]) . \tag{4.2}$$

The last member of (4.2) is zero by (3.6) and consequently $[Z, X]=0$ or equivalently $\mathfrak{h}\subset\mathfrak{c}(Z)$. The converse inclusion follows from (4.2) and (3.6). Q.E.D.

THEOREM 4.2. *Let $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$ be a semisimple GLA of the ν-th kind, and $Z\in\mathfrak{g}$*

be its characteristic element. Let $\mathfrak{g}^{\pm}=\sum_{k=0}^{\nu}\mathfrak{g}_{\pm k}$. *Define a linear form* f *on* $\mathfrak{g}$ *by* $f(X)=B(Z, X)$, $X\in\mathfrak{g}$. *Then* $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ *is a dipolarization in* $\mathfrak{g}$.

PROOF. (3.5) is trivially satisfied. Note that $\mathfrak{g}^+\cap\mathfrak{g}^-=\mathfrak{g}_0=\mathfrak{c}(Z)$. Let $X\in\mathfrak{g}_0$. Then we have $[Z, X]=0$. Consequently, by (4.2) we have $f([X, \mathfrak{g}])=0$. Conversely, let $X\in\mathfrak{g}$ and suppose that $f([X, Y])=0$ for any $Y\in\mathfrak{g}$. Then, by (4.2) we have $X\in\mathfrak{c}(Z)=\mathfrak{g}_0$. Thus (3.6) is valid. Next we claim

$$[\mathfrak{g}^+, \mathfrak{g}^+]=[\mathfrak{g}_0, \mathfrak{g}_0]+\mathfrak{g}_1+\cdots+\mathfrak{g}_\nu\,. \tag{4.3}$$

Indeed, the inclusion $\subset$ is trivial. We have $[\mathfrak{g}^+, \mathfrak{g}^+]\supset[\mathfrak{g}_0, \mathfrak{g}_0]+\sum_{k=1}^{\nu}[Z, \mathfrak{g}_k]=[\mathfrak{g}_0, \mathfrak{g}_0]+\sum_{k=1}^{\nu}\mathfrak{g}_k$, which shows (4.3). By using (4.3) we have

$$\begin{aligned} f([\mathfrak{g}^+, \mathfrak{g}^+])&=B(Z, [\mathfrak{g}^+, \mathfrak{g}^+]) \\ &=B([Z, \mathfrak{g}_0], \mathfrak{g}_0)+\sum_{k=1}^{\nu} B(Z, \mathfrak{g}_k)\,. \end{aligned} \tag{4.4}$$

The first term of the third member of (4.4) is zero. By a well-known property $B(\mathfrak{g}_p, \mathfrak{g}_q)=0$ for $p+q\neq 0$, it follows that $B(Z, \mathfrak{g}_k)=0$ for $k>0$. Hence, by (4.4) we obtain $f([\mathfrak{g}^+, \mathfrak{g}^+])=0$. Similarly we have $f([\mathfrak{g}^-, \mathfrak{g}^-])=0$. Q.E.D.

The dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ in Theorem 4.2 is called the *canonical dipolarization* in the GLA $\mathfrak{g}$.

THEOREM 4.3. *Let* $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$ *be a semisimple GLA of the* ν*-th kind with characteristic element* Z. *Let* G *be a connected Lie group generated by* $\mathfrak{g}$ *and* $C(Z)$ *be the centralizer of* Z *in* G. *Then* $M:=G/C(Z)$ *has the structure of a parakähler coset space.*

PROOF. Let $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ be the canonical dipolarization in the GLA $\mathfrak{g}$. We have $\mathfrak{g}^+\cap\mathfrak{g}^-=\mathfrak{g}_0=\mathrm{Lie}\, C(Z)$. Since $\mathrm{Ad}_{\mathfrak{g}}C(Z)$ consists of grade-preserving automorphisms of $\mathfrak{g}$, the subalgebras $\mathfrak{g}^{\pm}$ are stable under $\mathrm{Ad}_{\mathfrak{g}}C(Z)$. By using (4.1), we see that f is $\mathrm{Ad}_{\mathfrak{g}}C(Z)$-invariant. Therefore the assertion follows from Theorem 3.6. Q.E.D.

The above parakähler coset space $G/C(Z)$ is called a *semisimple parakähler coset space* (*of the* ν*-th kind*). If G is simple, then it is called a *simple parakähler coset space*.

REMARK 4.4. (1) The space $G/C(Z)$ is the coadjoint orbit of G through f, and so it is a Hamiltonian G-space in the sense of Kostant [8]. (2) Let $G/C(Z)$ be a semisimple parakähler coset space. One can assume that the center of G is finite. Then the subgroup $C(Z)$ can be characterized as the Levi subgroup of a parabolic subgroup of G. (3) A semisimple parakähler coset space of the ν-th kind is a parahermitian symmetric space if and only if $\nu=1$ ([3]).

4.2. Let $\mathfrak{g}$ be a real semisimple Lie algebra. A gradation $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$ is said to be *of type* α_0, if $\mathfrak{m}^+=\sum_{k=1}^{\nu}\mathfrak{g}_k$ and $\mathfrak{m}^-=\sum_{k=1}^{\nu}\mathfrak{g}_{-k}$ are generated by $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, respectively. The subalgebra $\mathfrak{g}^+=\sum_{k=0}^{\nu}\mathfrak{g}_k$ is called the *parabolic part* of the GLA $\mathfrak{g}$. $\mathfrak{m}^{\pm}$ are called the *positive* and *negative parts* of $\mathfrak{g}$, respectively.

LEMMA 4.5. *Let $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$ be a gradation of $\mathfrak{g}$ which is not of type α_0. Then there exists a gradation of type α_0 of $\mathfrak{g}$ with the same parabolic and positive parts as those for the original gradation.*

PROOF. Let Π be the restricted fundamental system of roots for $\mathfrak{g}$. It is known [7] that every gradation of $\mathfrak{g}$ is described by a partition $\Pi=\Pi_0\cup\Pi_1\cup\cdots\cup\Pi_n$. Put $\Pi'_1=\Pi_1\cup\cdots\cup\Pi_n$. Then the gradation of $\mathfrak{g}$ corresponding to the partition $\Pi=\Pi_0\cup\Pi'_1$ is of type α_0 (Theorem 2.6 [7]) and satisfies the required properties. Q.E.D.

Under the notations and assumptions in Theorem 4.3, we assume further without loss of generality that the center of G is finite. Let us consider the subgroups of G

$$U^{\pm}=C(Z)\exp\mathfrak{m}^{\pm}\,, \tag{4.5}$$

where $\mathfrak{m}^{\pm}$ are the positive and negative parts of $\mathfrak{g}$, respectively. Then we have the R-spaces $M^{\pm}=G/U^{\pm}$ which can be expressed as one and the same coset space of a maximal compact subgroup of G. $M^{\pm}$ are not symmetric R-spaces in general. If G is complex semisimple, then $M=G/C(Z)$ has the natural G-invariant complex structure, and $M^{\pm}=G/U^{\pm}$ are Kähler C-spaces in the sense of H. C. Wang.

PROPOSITION 4.6. *The semisimple parakähler coset space $M=G/C(Z)$ is diffeomorphic to the cotangent bundle of the R-space $M^{+}=G/U^{+}$ (or $M^{-}=G/U^{-}$). If G is complex semisimple, then $G/C(Z)$ is holomorphically equivalent to the cotangent bundle of the Kähler C-space G/U^{+} (or G/U^{-}).*

PROOF. By Lemma 4.5, one can assume that the gradation $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$ corresponding to M is of type α_0, and hence the corresponding partition of Π is given by $\Pi=\Pi_0\cup\Pi_1$ (Theorem 2.6 [7]). Thus the characteristic element Z of the gradation is determined by ([7])

$$B(Z,\alpha_i)=\begin{cases}0\,, & \alpha_i\in\Pi_0\,,\\ 1\,, & \alpha_i\in\Pi_1\,.\end{cases} \tag{4.6}$$

Therefore the first assertion follows from a result of Takeuchi [11]. Note that if G is complex semisimple, then everything is done within the complex category. Q.E.D.

Let us consider the product manifold

$$\tilde{M}=M^{-}\times M^{+}\,. \tag{4.7}$$

The group G acts on $\tilde{M}$ diagonally, that is, $g(p,q)=(gp,gq)$, where $g\in G$ and $(p,q)\in\tilde{M}$. Let $o^{\pm}$ denote the origins of the coset spaces $M^{\pm}$, respectively.

THEOREM 4.7. *Let $M=G/C(Z)$ be a semisimple parakähler coset space. Then M is equivariantly imbedded in $\tilde{M}$ as the G-orbit through the point (o^{-},o^{+}) under the diagonal*

G-action. The image of M is open and dense in $\tilde{M}$. In particular, $\tilde{M}$ is viewed as a G-equivariant compactification of M. If G is complex semisimple, then the above imbedding is holomorphic.

PROOF. The isotropy subgroup of G at (o^-, o^+) is given by $U^- \cap U^+ = C(Z)$ (cf. (4.5)), which implies the first assertion. That the image of M is dense in $\tilde{M}$ can be proved in the same way as for Lemma 3.4 [6], and so we can omit the details. We only note that $(\exp \mathfrak{m}^-)C(Z)(\exp \mathfrak{m}^+)$ is open and dense in G (see also Takeuchi [11]).

Q.E.D.

EXAMPLE 4.8. Let $\mathfrak{g} = \mathfrak{su}(p, q)$, $p \leq q$. Under the notations in [7], consider the gradation of $\mathfrak{g}$ of the second kind corresponding to $\Pi_1 = \{\alpha_k\}$, $1 \leq k \leq p$. The simple parakähler coset space (of the second kind) corresponding to this gradation is given by

$$M_k = U(p, q)/GL(k, \boldsymbol{C}) \times U(p-k, q-k)\,, \qquad 1 \leq k \leq p\,. \tag{4.8}$$

By Proposition 4.6, M_k is diffeomorphic to the cotangent bundle of the R-space

$$\begin{aligned} M_k^+ &= U(p) \times U(q)/U(k) \times U(p-k) \times U(q-k) \\ &= G_{k,p-k}(\boldsymbol{C}) \times V_{k,q}(\boldsymbol{C})\,, \end{aligned} \tag{4.9}$$

where $G_{k,p-k}(\boldsymbol{C})$ denotes the complex Grassmannian of k-dimensional subspaces in $\boldsymbol{C}^p$ and $V_{k,q}(\boldsymbol{C})$ denotes the complex Stiefel manifold of unitary k-frames in $\boldsymbol{C}^q$. Note that $M_p^+ = U(q)/U(q-p) = V_{p,q}(\boldsymbol{C})$ is the Silov boundary of the bounded classical symmetric domain of type $I_{p,q}$, and that M_1^+ is the hermitian quadric of index $p-1$ in the complex projective $(p+q-1)$-space. M_k is parahermitian symmetric if and only if $k = p = q$, in which case $M_p = U(p, p)/GL(p, \boldsymbol{C})$ ([5]).

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Added in Proof. The infinitesimal classification of simple parakähler coset spaces of the second kind has been given in [15, 16].

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