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# Irreducibility of the Linear Differential Equation Attached to Painlevé's First Equation

Keiji NISHIOKA

*Keio University*

**Abstract.** The linear homogeneous differential equation $z''=12yz$ with $y$ an arbitrary solution of Painlevé's first equation $y''=6y^2+x$ will be proved irreducible.

## 1. Introduction.

Let $K$ be a differential field of characteristic 0 with a single differentiation $D='$. Let $U$ be a universal extension of $K$, and $R$ be a differential field extension of $K$ with finite transcendence degree over $K$, $R\subset U$. Suppose that the field of constants of $R$ is algebraically closed in $U$. We adopt the usual notation of differential module of $R$ over $K$, $\Omega_K(R)$. Denote by $d_{R/K}$ the canonical mapping of $R$ to $\Omega_K(R)$. With $D$ there is associated an additive homomorphism of the differential module $D^1$ satisfying $D^1 d_{R/K}=d_{R/K}D$ on $R$. Since $R$ is finitely generated, the $\Omega_K(R)$ is an $R$-vector space of finite dimension. Let $\omega_1, \omega_2, \cdots, \omega_n$ be a base for it. Then they satisfy

$$D^1\omega_i=\sum_{j=1}^{n} a_{ij}\omega_j \qquad (1\leq i\leq n)\,,$$

where the $a_{ij}$ are elements of $R$. With this system of equations there associates the following system of linear differential equations

$$Dy_i=\sum_{j=1}^{n} a_{ij}y_j\,. \tag{1}$$

Thus we may consider the Picard-Vessiot group $\Gamma(R/K)$ for this system. The definition of $\Gamma(R/K)$ is independent of the choice of bases for $\Omega_K(R)$. It is clear that if $R/K$ has an intermediate differential field $S$ for which $0<\text{tr.deg}_K S<\text{tr.deg}_K R$ then $\Gamma(R/K)$ is reducible.

Equation (1) is directly derived from the defining equation of $y$ if $R$ has the description $R=K\langle y\rangle$. Let $F(y, y', \cdots, y^{(n)})=0$ be the defining equation of $y$ over $K$. Let $\alpha$ be a formal parameter with $\alpha^2=0$ and suppose $y+\alpha z$ satisfies the equation $F=0$. Then $z$ satisfies

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$$zF_y + z'F_{y'} + \cdots + z^{(z)}F_{y^{(n)}} = 0 .$$

This works as equation (1).

Here we will restrict ourselves to investigating the linear differential equation

$$z'' = 12yz \tag{2}$$

attached to the first equation of Painlevé

$$y'' = 6y^2 + x , \tag{3}$$

where $x' = 1$, $x \in K$. One of prominent properties of this equation is this: Let $y$ be the general solution of equation (3) over $K$. If $y$ satisfies a first order algebraic differential equation over a differential field extension $L$ of $K$ then it is algebraic over $L$. Therefore the differential field $R = K\langle y \rangle$ has no differential subfield $L$ with $\text{tr.deg}_K L = 1$. (cf. [2].)

We shall prove the following.

THEOREM. *Suppose that K contains the element x with $x' = 1$ and the field of constants of K is algebraically closed in U. Let y be the general solution of the first equation of Painlevé (3) over K. Then for the differential field extension $R = K\langle y \rangle$ of K the Picard-Vessiot group $\Gamma(R/K)$ is irreducible.*

It is reasonable furthermore to conjecture that $\Gamma(R/K)$ is $SL_2(C)$, provided $C$, the field of constants of $K$, is algebraically closed.

## 2. Poincaré field.

Let $K$ be a differential field of characteristic 0 with the element $x$, $x' = 1$. Let $y$ be the general solution of the first equation of Painlevé (3) and $R = K\langle y \rangle$. The polynomial algebra $K[y, y']$ is a differential ring extension of $K$. We divide the differentiation $D$ into three parts

$$D = \xi + \eta + \zeta ,$$

where

$$\xi = y'\partial/\partial y + 6y^2\partial/\partial y' , \qquad \zeta = x\partial/\partial y'$$

and $\eta$ indicates the derivation of $K[y, y']$ over $K$ with $\eta y = \eta y' = 0$. Let $\gamma = y'^2 - 4y^3$, $A = K[\gamma]$ and $L = K(\gamma)$. Then $K[y, y'] = A[y] \oplus y'A[y]$. The derivation operator $\xi$ of $R$ over $L$ can be represented by $\xi = y'd/dy$. We thus have the so-called Poincaré field $R = L\langle y \rangle$ over $L$ with respect to $\xi$:

$$y'^2 = 4y^3 + \gamma , \quad \xi y = y' , \quad \xi L = 0.$$

Note $\xi^2 y = \xi y' = 6y^2$.

PROPOSITION 1. *The field of constants of R with respect to $\xi$ is the same as L.*

PROOF. Every element of $R=L(y, y')$ has the form

$$a+y'b\,, \qquad a, b\in L(y)\,.$$

Suppose $\xi(a+y'b)=0$. Then we have

$$\frac{da}{dy}=(4y^3+\gamma)\frac{db}{dy}+6y^2b=0\,.$$

This implies $b=0$.

Let us define a differential operator of Lamé type:

$$\lambda=\xi^2-12y=(4y^3+\gamma)d^2/dy^2+6y^2d/dy-12y\in R[\xi]\,,$$

which is seen to be reducible (cf. Proposition 3). Clearly $\lambda A[y]\subset A[y]$, $\deg_y\lambda a\leq \deg_y a+1$ $(a\in A[y])$.

PROPOSITION 2. *For $a, b\in L(y)$, we have*

$$\lambda(a+y'b)=f+y'g\,,$$

*where $f, g$ are elements of $L(y)$ with*

$$f=\lambda a\,, \qquad g=\mu b=y'^2\frac{d^2b}{dy^2}+18y^2\frac{db}{dy}\,.$$

PROOF. In fact

$$\xi^2(a+y'b)=\xi\left(6y^2b+y'^2\frac{db}{dy}+y'\frac{da}{dy}\right)$$
$$=6y^2\frac{da}{dy}+y'^2\frac{d^2a}{dy^2}+y'\frac{d}{dy}\left(6y^2b+y'^2\frac{db}{dy}\right).$$

PROPOSITION 3. *Let $w$ be an element of some extension of $R$ with $\xi w=y'^{-2}$. Then $y'$ and $wy'$ constitute a fundamental system of solutions for $\lambda z=0$. The element $w$ does not belong to $R$. Every element of $R$ satisfying $\lambda z=0$ belongs to $y'L$.*

PROOF. It is straightfoward that $\lambda(wy')=0$. Suppose that we write $w=\alpha+y'b$, $a, b\in L(y)$. We then have

$$y'\frac{da}{dy}+6y^2b+y'^2\frac{db}{dy}=y'^{-2}\,,$$

hence

$$\frac{da}{dy}=0\,, \qquad f^2\frac{db}{dy}+\frac{1}{2}f\frac{df}{dy}b=1\,,$$

where $f=y'^2$. From the last equality we see that $b$ has a pole of at most 1 order at

$y=e$, with $e$ satisfying $4e^3+\gamma=0$, whence $c=fb \in L[y]$. The element $c$ must satisfy

$$(4y^3+\gamma)\frac{dc}{dy}-9y^2c=1\,.$$

This is, however, impossible.

From this the operator $\lambda$ is seen to be reducible.

## 3. Proof of Theorem.

Putting $u=z'/z$ in equation (2), we have the Riccati equation

$$u'+u^2=12y\,. \tag{4}$$

Suppose that (4) has a rational solution $u$ over $R$. Write $u=f/g$, $f, g \in K[y, y']$, where $f$ and $g$ are coprime. Then we have

$$f'g-fg'+f^2=12yg^2\,,$$

or

$$f(f-g')=g(12yg-f')\,.$$

Since $f, g$ are coprime it follows that $f-g'$ is divisible by $g$, namely, there is an $h \in K[y \cdot y']$ with

$$f=g'+gh, \qquad u=g'/g+h\,.$$

By (4),

$$g''+2g'h+gh^2=12yg\,. \tag{5}$$

We here use the weight function $w$ of $K[y, y']$ in [2], which is defined as $w(F)=\max\{2i+3j;\ a_{ij}\neq 0\}$ for any nonzero element $F=\sum a_{ij}y^i y'^j$ of $K[y, y']$. For this weight function we know $w(F')\leq w(F)+1$ if $F'\neq 0$.

If $w(h)\geq 2$ then $w(gh^2)=w(g)+2w(h)\geq w(g)+4$. By the way

$$w(gh^2)=w(12yg-g''-2g'h)\leq \max\{w(g)+2,\ w(g)+w(h)+1\}\,.$$

This is a contradiction. We thus have $h=0$ or $w(h)\leq 1$, and hence $h \in K$ (cf. [3]). Enlarging $K$, if necessary, we may assume $K$ has a nonzero element $e$ of $U$ with $e'=he$. Set $P=eg \in K[y, y']$. This polynomial satisfies equation (2). We shall prove that there does not exist such a polynomial. Let $H_i$ denote the vector space consisting of the zero polynomial and all polynomials with weight $i$: $K[y, y']=\sum_{i=0}^{\infty} H_i$. Let us assume the polynomial $P$ has the decomposition: $P=\sum_{i=0}^{n} P_i$, $n=w(P)$. By $D^2P=12yP$ we have

$$\lambda P_i=-2\xi\eta P_{i+1}-\eta^2 P_{i+2}-(\xi\zeta+\zeta\xi)P_{i+4}-(\eta\zeta+\zeta\eta)P_{i+5}-\zeta^2 P_{i+8}\,. \tag{6}$$

When $i=n$ this reads $\lambda P_n=0$. By Proposition 3, we have

$$P_n = ay'\gamma^r, \qquad n = 6r+3, \quad a \in K.$$

In particular this implies $n \geq 3$.

When $i = n-1$ equation (6) reads

$$\lambda P_{n-1} = -2\xi\eta P_n = -2\xi(a'y'\gamma^r) = -12a'y^2\gamma^r.$$

Taking the weight into account, by Proposition 3 we see

$$P_{n-1} = 2a'y\gamma^r.$$

When $i = n-2$ equation (6) reads

$$\lambda P_{n-2} = -2\xi\eta P_{n-1} - \eta^2 P_n = -2\xi(2a''y\gamma^r) - \eta^2(ay'\gamma^r),$$

hence

$$\lambda P_{n-2} = -5a''y'\gamma^r.$$

On putting $P_{n-2} = f + y'g$, $f, g \in L(y)$, by Proposition 2 we have

$$\lambda f = 0, \qquad \mu g = -5a''\gamma^r,$$

where $\mu$ indicates the derivative operator with the expression

$$\mu = (4y^3 + \gamma)d^2/dy^2 + 18yd/dy.$$

From the second equation it follows that $dg/dy = 0$, and so that $a'' = 0$. This shows

$$P_{n-2} = 0.$$

When $i = n-3$ equation (6) reads

$$\lambda P_{n-3} = -2\xi\eta P_{n-2} - \eta^2 P_{n-1} = -\eta^2(a'y\gamma^r) = 0,$$

hence by Proposition 3,

$$P_{n-3} = 0.$$

If $n = 3$, then $P = ay' + 2a'y$ and

$$\begin{aligned} D^2P &= D(a(6y^2 + x) + 3a'y') \\ &= a(12yy' + 1) + 4a'(6y^2 + x) \\ &= 12yP + 4a'(6y^2 + x) + a \end{aligned}$$

which does not equal $12yP$. Therefore $n > 3$, whence $n \geq 9$.

When $i = n-4$ equation (6) reads

$$\begin{aligned} \lambda P_{n-4} &= -2\xi\eta P_{n-3} - \eta^2 P_{n-2} - (\xi\eta + \zeta\xi) P_n \\ &= -(\xi\eta + \zeta\xi)(ay'\gamma^r) \\ &= -\xi(ax\gamma^r + 2raxy'^2\gamma^{r-1}) - \zeta(6ay^2\gamma^r) \end{aligned}$$

$$= -36ray^2y'\gamma^{r-1}\,.$$

On putting $P_{n-4}=f+y'g$, $f, g\in L(y)$, we have

$$\lambda f=0\,, \qquad \mu g=-36raxy^2\gamma^{r-1}\,.$$

This implies $f=0$, $g=-2axy\gamma^{r-1}$, hence $P_{n-4}=-2axyy'\gamma^{r-1}$.

When $i=n-5$, equation (6) reads

$$\begin{aligned}\lambda P_{n-5} &= -2\xi\eta P_{n-4}-\eta^2P_{n-3}-(\xi\eta+\zeta\xi)P_{n-1}-(\eta\zeta+\zeta\eta)P_n\\ &= -2\xi\eta(-2raxyy'\gamma^{r-1})-(\xi\eta+\zeta\xi)(2a'y\gamma^r)-(\eta\zeta+\zeta\eta)(ay'\gamma^r)\,.\end{aligned}$$

We calculate each term in the third member. The first term is

$$\begin{aligned}&4r\xi[(a'x+a)yy'\gamma^{r-1}]=4r(a'x+a)(y'^2+6y^3)\gamma^{r-1}\\ &\quad=40r(a'x+a)y^3\gamma^{r-1}+4r(a'x+a)\gamma^r\,,\end{aligned}$$

the second term is

$$\begin{aligned}&\xi(4ra'xyy'\gamma^{r-1})+\zeta(2a'y'\gamma^r)\\ &\quad=4ra'x(y'^2+6y^3)\gamma^{r-1}+2a'x\gamma^r+4ra'xy'^2\gamma^{r-1}\\ &\quad=56ra'xy^3\gamma^{r-1}+2(4r+1)a'x\gamma^r\,,\end{aligned}$$

and the third term is

$$\begin{aligned}&\eta(ax\gamma^r+raxy'^2\gamma^{r-1})+\zeta(a'y'\gamma^r)\\ &\quad=(a'x+a)\gamma^r+r(a'x+a)y'^2\gamma^{r-1}+a'x\gamma^r+ra'xy'^2\gamma^{r-1}\\ &\quad=4r(2a'x+a)y^3\gamma^r+[2(r+1)a'x+(r+1)a]\gamma^r\,.\end{aligned}$$

Hence

$$\lambda P_{n-5}=-12r(2a'x+3a)y^3\gamma^{r-1}+[(-6r-4)a'x+(3r-1)a]\gamma^r\,.$$

If we set $P_{n-5}=f+y'g$, $f, g\in L(y)$, we have

$$\begin{aligned}&\lambda f=-12r(2a'x+3a)y^3\gamma^{r-1}+[(-6r-4)a'x+(3r-1)a]\gamma^r\,,\\ &\mu g=0\,.\end{aligned}$$

From the first equation, noting $n-5=6(r-1)+4$, we have the expression $f=by^2$, $b\in L$, so that

$$\begin{aligned}\lambda f&=b[2(4y^3+\gamma)+12y^2-12y^23]\\ &=28by^3+2b\gamma\,.\end{aligned}$$

This yields

$$2b=-3r(2a'x+3a)\gamma^{r-1}=[(-6r-4)a'x+(3r-1)a]\gamma^{r-1}\,,$$

hence $4a'x=(12r-1)a$. This contradicts the fact that $a''=0$. Thus the proof of the Theorem is completed.

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*Present Address*:
FACULTY OF ENVIRONMENTAL INFORMATION, KEIO UNIVERSITY AT SFC
ENDOH, FUJISAWA 252, JAPAN