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Homologically Trivial Self-Maps

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0. Introduction.

In the previous paper [2], we investigated the image of the homology representation $H_X: [X, X] \to \mathrm{End}(H_*(X))$ for a complex X of the form $S^n \cup e^{n+2} \cup e^{n+4}$ where $[X, X]$ denotes the set of homotopy classes of self-maps of X. This problem was equivalent to characterizing a triple of integers (d_1, d_2, d_3) which can be obtained from a self-map as its degrees.

In this paper we investigate the case $(d_1, d_2, d_3)=(0, 0, 0)$, namely, homologically trivial self-maps of X. The homotopy groups $\pi_{n+4}(A)$ and $\pi_{n+3}(A)$, $A=S^n \cup e^{n+2}$, play an important role for our purpose. Then some differences exist between the case $n=2$ and the others, so we concentrate on the case $n \geqq 3$ in this paper. Our method is to construct short exact sequences containing $H_X^{-1}(0)$ in the category of sets with distinguished elements. If X is a suspended complex, this category turns out to be the category of groups, so the results become more clear. Here we state some results from our theorems. Let X be a complex as above.

(1) If $Sq^2(e^n) \neq 0$ $(n \geqq 6)$, or $Sq^2(e^n) \neq 0$ and $Sq^4(e^n) \neq 0$ $(n=4, 5)$, then homologically trivial self-maps of X are also homotopically trivial.

(2) If $Sq^2(e^n)=0$ and $Sq^2(e^{n+2}) \neq 0$ $(n \geqq 4)$, then the same conclusion as (1) holds with an exceptional case in which there exists only one homotopically non-trivial but homologically trivial self-map.

(3) For $n=3$, the set of homologically trivial self-maps contains countablly infinite many homotopically non-trivial ones.

(4) Let ξ be an n-dimensional real vector bundle over CP^2 and $T(\xi)$ be the Thom complex of ξ $(n \geqq 5)$. Suppose the second Stiefel-Whitney class $w_2(\xi)=0$. Then, for self-maps of $T(\xi)$, homological triviality is equal to homotopical triviality.

Throughout this paper we use the same notations as ones in [2] for generators of homotopy groups of spheres, and the following:

$$A=S^n \cup e^{n+2}, \quad X=A \cup e^{n+4}, \quad \text{and} \quad Y=X/S^n=S^{n+2} \cup e^{n+4},$$

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where attaching maps are α, β, γ respectively,

i: appropriate inclusion maps,

h_m: Hurewicz homomorphism at dimension m.

1. The fundamental sequence.

Let us consider a part of Puppe sequence associated with the cofibering $A \to X \to Y$ together with the homology representation, namely the following diagram:

$$\begin{tikzcd} & 0 \arrow[r] & \mathrm{Hom}(H_*(S^{n+4}), H_*(X)) \arrow[r] & \mathrm{Hom}(H_*(X), H_*(X)) \arrow[r] & \mathrm{Hom}(H_*(A), H_*(X)) \\ {[\Sigma A, X]} \arrow[r] & & {[S^{n+4}, X]} \arrow[u, "h_{n+4}"] \arrow[r] & {[X, X]} \arrow[u, "H_X"] \arrow[r] & {[A, X]} \arrow[u, "H_A"] \end{tikzcd}$$

Then this gives the exact sequence

$$0 \longrightarrow h_{n+4}^{-1}(0)/[\Sigma A, X]\Sigma\beta \longrightarrow H_X^{-1}(0) \longrightarrow H_A^{-1}(0)\,, \tag{1-1}$$

because we have the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \vee S_{n+4} \arrow[r] & S_{n+4} \vee S_{n+4} \arrow[d] \\ & S_{n+4} \arrow[ur] & \Sigma A \vee S_{n+4} \end{tikzcd}$$

where the arrows denote the well-known natural maps respectively.

Lemma 1.1. *If $\Sigma\alpha \neq 0$, then $H_A^{-1}(0)$ is trivial. If $\Sigma\alpha = 0$, then $H_A^{-1}(0)$ contains only one non-trivial map which is given by the composite:*

$$A \longrightarrow A/S^n = S^{n+2} \xrightarrow[\eta_{n+1}]{} S^{n+1} \xrightarrow[\eta_n]{} S^n \xrightarrow[i]{} A\,.$$

Proof. Analogously to (1-1), from the cofibering

$$S^n \longrightarrow A \longrightarrow S^{n+2}\,,$$

we obtain a short exact sequence

$$\pi_{n+1}(S^n) \longrightarrow \pi_{n+2}(S^n) \longrightarrow H_A^{-1}(0) \longrightarrow 0\,.$$

Since $\Sigma\alpha \neq 0$ is equivalent to $\pi_{n+1}(S^n)\Sigma\alpha = \pi_{n+2}(S^n)$, the proof is completed. □

Lemma 1.2. *If $\Sigma\alpha = 0$, then the map in Lemma 1.1 has an extension contained in $H_X^{-1}(0)$ if and only if $\gamma = 0$.*

Proof. It is clear that the extendability satisfying the condition is equivalent to $i_{A*}(\eta_n\eta_{n+1}\gamma) = 0$ where i_A denotes the inclusion $S^n \to A$. On the other hand, i_{A*} is injective because $\partial\pi_{n+4}(A, S^n) = \alpha\pi_{n+3}(S^{n+1})$ for $n \geqq 3$ and we have the following diagram for $n = 2$:

$$\begin{array}{ccc} \pi_5(S^2) & \longrightarrow & \pi_5(A) \\ \text{inj.}\quad \Sigma \big\downarrow & & \Sigma \big\downarrow \\ \pi_6(S^3) & \longrightarrow & \pi_6(\Sigma A) = \pi_6(S^3 \vee S^5) \quad \text{inj.} \end{array} \qquad \square$$

From Lemmas 1.1 and 1.2 we obtain:

LEMMA 1.3. *In the sequence* (1-1), *we have that the image of the map* $H_X^{-1}(0) \to H_A^{-1}(0)$ *contains only one non-trivial map for the case* $\Sigma\alpha = 0 = \gamma$, *and is trivial for the other cases.*

REMARK. Since the condition $\gamma = 0$ gives a map $X \to S^{n+2}$ of degree 1, a non-trivial map contained in $H_X^{-1}(0)$ is given by the composite:

$$X \longrightarrow S^{n+2} \longrightarrow S^{n+1} \longrightarrow S^n \longrightarrow X .$$

Next we consider the following diagram of exact sequences:

$$\begin{array}{ccccccc} \pi_{n+5}(X, A) & \xrightarrow{\partial} & \pi_{n+4}(A) & \longrightarrow & h_{n+4}^{-1}(0) & \longrightarrow & 0 \\ & & (\Sigma\beta)^* \big\uparrow & & \big\uparrow (\Sigma\beta)^* & & \\ & & [\Sigma A, A] & \longrightarrow & [\Sigma A, X] & \longrightarrow & 0 \end{array} \qquad (1\text{-}2)$$

First of all, we note the following homotopy excision theorem in the case $n \geqq 3$.

LEMMA 1.4.

$$\partial\pi_{n+5}(X, A) = \bar{\beta}\pi_{n+4}(S^{n+3}) ,$$

$$\pi_{n+5}(X, A) \sim \begin{cases} \bar{\beta}\pi_{n+5}(D^{n+4}, S^{n+3}) & (n \geqq 3) , \\ \bar{\beta}\pi_7(D^6, S^5) + \{[\bar{\beta}, \iota_2]_r\} & (n = 2) , \end{cases}$$

where $\bar{\beta}$ *is the characteristic map for the* $n+4$*-cell of* X *and* $[\ ,\ ]_r$ *denotes the relative Whitehead product.*

LEMMA 1.5. *If* $\Sigma\alpha = 0$, *then we have*

$$[\Sigma A, A]\Sigma\beta = \{\pi_{n+1}(A)\delta\} + \{\pi_{n+3}(A)\Sigma\gamma\} ,$$

where $\Sigma\beta = \delta + \Sigma\gamma$ *in the decomposition*:

$$\pi_{n+4}(S^{n+1} \vee S^{n+3}) \sim \pi_{n+4}(S^{n+1}) + \pi_{n+4}(S^{n+3}) .$$

PROOF. Since $\Sigma\alpha = 0$ means $\Sigma A = S^{n+1} \vee S^{n+3}$, the proof is easy. $\square$

LEMMA 1.6. *If* $\Sigma\alpha \neq 0$, *then we have*

$$[\Sigma A, A]\Sigma\beta = \begin{cases} 0 & (n \geqq 3) , \\ \pi_3(A)\beta' \quad \textit{for some } \beta' \textit{ of } \pi_6(S^3) & (n = 2) . \end{cases}$$

PROOF. Since $\Sigma\alpha\neq 0$ gives $\gamma=0$ because of $Sq^2 sq^2=0$, $\Sigma\beta$ is decomposed such as $S^{n+4}\to S^{n+1}\to\Sigma A$ for some β': $S^{n+4}\to S^{n+1}$. Hence we have

$$\pi_{n+1}(A)\beta'=[\Sigma A, A]\Sigma\beta .$$

Thus the lemma follows from $\pi_{n+1}(A)=0$ $(n\geqq 3)$. □

Especially, from Lemmas 1.1 and 1.6 we obtain:

PROPOSITION 1.7. *For $n\geqq 3$ and $\alpha\neq 0$, there exists an exact sequence*

$$0\longrightarrow h_{n+4}^{-1}(0)\longrightarrow H_X^{-1}(0)\longrightarrow 0 .$$

REMARK. In Prop. 1.7, if X is a suspended complex, the arrow in the middle of the sequence means an isomorphism of two groups.

Now, we investigate $\pi_{n+4}(A)$ to use the diagram (1-2).

2. The case $n\geqq 3$.

First we note the following well-known lemma.

LEMMA 2.1. *If $\alpha=0$, there exists an isomorphism*:

$$\pi_{n+4}(A)\sim\begin{cases}\pi_{n+4}(S^n)+\pi_{n+4}(S^{n+2}) & (n\geqq 4),\\ \pi_7(S^3)+\pi_7(S^5)+\{[\iota_3,\iota_5]\} & (n=3).\end{cases}$$

Next, suppose $\alpha\neq 0$ (i.e. $\alpha=\eta_n$) and consider a part of the homotopy exact sequence of the pair (A, S^n):

$$\pi_{n+5}(A, S^n)\xrightarrow[d_1]{}\pi_{n+4}(S^n)\xrightarrow[i]{}\pi_{n+4}(A)\xrightarrow[j]{}\pi_{n+4}(A, S^n)\xrightarrow[d_2]{}\pi_{n+3}(S^n)\,. \tag{2-1}$$

Here we quote the well-known facts about homotopy groups of spheres (cf. [3]):

$$\pi_9(S^5)\sim\mathbf{Z}/2\mathbf{Z}[\nu_5\eta_8]\,,\qquad [\iota_5\iota_5]=\nu_5\eta_8\,,\quad \eta_5\nu_6=0\,,$$

$$\pi_8(S^4)\sim\mathbf{Z}/2\mathbf{Z}[\nu_4\eta_7]+\mathbf{Z}/2\mathbf{Z}[\eta_4\nu_5]\,,\qquad \eta_4\nu_5=\Sigma(\omega\eta_6)=[\eta_4,\iota_4]\,,$$

$$\pi_7(S^3)\sim\mathbf{Z}/2\mathbf{Z}[\omega\eta_6]\,,\qquad \omega\eta_6=\eta_3\nu_4\,,\quad \eta_3\Sigma\omega=0\,.$$

Then, by applying the homotopy excision theorem to $\pi_*(A, S^n)$ in (2-1), we can calculate d_1, d_2, and obtain:

PROPOSITION 2.2. *If $\alpha\neq 0$, then we have*

$$\pi_{n+4}(A)\sim\begin{cases}\{0\} & (n\geqq 6),\\ \mathbf{Z}/2\mathbf{Z} & (n=4,5),\\ \mathbf{Z} & (n=3),\end{cases}$$

where $\mathbf{Z}/2\mathbf{Z}$ is generated by $\nu_n\eta_{n+3}$.

Now, to obtain $h_{n+4}^{-1}(0)$ we have to determine the subgroup $\beta\pi_{n+4}(S^{n+3})$ of $\pi_{n+4}(A)$ in the case $n=4$, 5 (see (1-2)).

LEMMA 2.3. *Suppose $\alpha\neq 0$ and $n=4$, 5. If $Sq^4(e^n)\neq 0$ in $H_*(X, \mathbf{Z}/2\mathbf{Z})$, then $\beta\pi_{n+4}(S^{n+3})=i_*(\nu_n\eta_{n+3})$, and if $Sq^4(e^n)=0$, then $\beta\pi_{n+4}(S^{n+3})=0$.*

PROOF. Since $\alpha\neq 0$ implies $\gamma=0$ as stated in the proof of Lemma 1.6, there exists β' of $\pi_{n+3}(S^n)$ such that $i_*(\beta')=\beta$. Let β' be the element $N\nu_5$ for $n=5$ and $N\nu_4+m\Sigma\omega$ for $n=4$. Then from $i_*(\Sigma\omega\eta_7)=0$, we obtain $\beta\pi_{n+4}(S^{n+3})=0$ for an even N and $\beta\pi_{n+4}(S^{n+3})=i_*(\nu_n\eta_{n+3})$ for an odd N. Thus the proof is completed because $Sq^4(e^n)$ corresponds to N mod 2. □

By combining Lemmas 1.4, 2.3 and Prop. 2.2 with the diagram (1-2), we obtain:

PROPOSITION 2.4. *If $\alpha\neq 0$, then $h_{n+4}^{-1}(0)$ is equal to $\{0\}$ for $n\geqq 6$, to $\mathbf{Z}$ for $n=3$, to $\{0\}$ for $n=4$, 5 and $Sq^4(e^n)\neq 0$, and to $\mathbf{Z}/2\mathbf{Z}$ for $n=4$, 5 and $Sq^4(e^n)=0$.*

Next we suppose $\alpha\neq 0$ ($n\geqq 3$) and put $\beta=\beta_1+\gamma$ in the decomposition $\pi_{n+3}(A)\sim\pi_{n+3}(S^n)+\pi_{n+3}(S^{n+2})$. Then, by making use of (2-1), easy calculation gives a generator of $\beta\pi_{n+4}(S^{n+3})$ as follows:

LEMMA 2.5. *In the case $n\geqq 6$, $\beta\pi_{n+4}(S^{n+3})$ is $\{0\}$ for $\gamma=0$, and is generated by $\eta_{n+2}\eta_{n+3}$ for $\gamma\neq 0$.*

In the case $n=5$, we put $\beta_1=N\nu_5$ for an integer N. Then $\beta\pi_9(S^8)$ is $\{0\}$ for an even N and $\gamma=0$, and is generated by $\nu_5\eta_8$ for an odd N and $\gamma=0$, by $\eta_7\eta_8$ for an even N and $\gamma\neq 0$, and by $\nu_5\eta_8+\eta_7\eta_8$ for an odd N and $\gamma\neq 0$.

LEMMA 2.6. *In the case $n=4$, we put $\beta_1=N\nu_4+m\Sigma\omega$ for integers N and m. Then $\beta\pi_8(S^7)$ is*

(1) *for an even N and an even m, $\{0\}$ if $\gamma=0$ and generated by $\eta_6\eta_7$ if $\gamma\neq 0$,*

(2) *for an even N and an odd m, generated by $\eta_4\nu_5$ if $\gamma=0$ and $\eta_4\nu_5+\eta_6\eta_7$ if $\gamma\neq 0$,*

(3) *for an odd N and an even m, generated by $\nu_4\eta_7$ if $\gamma=0$ and $\nu_4\eta_7+\eta_6\eta_7$ if $\gamma\neq 0$,*

(4) *for an odd N and an odd m, generated by $\nu_4\eta_7+\eta_4\nu_5$ if $\gamma=0$ and $\nu_4\eta_7+\eta_4\nu_5+\eta_6\eta_7$ if $\gamma\neq 0$.*

LEMMA 2.7. *In the case $n=3$, we put $\beta_1=m\omega$ for an integer m. Then $\beta\pi_7(S^6)$ is*

(1) *for $m\equiv 0$ mod 2, $\{0\}$ if $\gamma=0$ and generated by $\eta_5\eta_6$ if $\gamma\neq 0$,*

(2) *for $m\equiv 1$ mod 2, generated by $\omega\eta_6$ if $\gamma=0$ and by $\omega\eta_6+\eta_5\eta_6$ if $\gamma\neq 0$.*

These lemmas and Lemma 1.4 give rise to:

PROPOSITION 2.8. *Suppose $\alpha=0$ ($n\geqq 3$). Then $h_{n+4}^{-1}(0)$ is equal to*

(1) *for $n\geqq 6$, $\mathbf{Z}/2\mathbf{Z}$ if $Sq^2(e^{n+2})=0$ and $\{0\}$ if $Sq^2(e^{n+2})\neq 0$,*

(2) *for $n=5$, $\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$ if $Sq^2(e^7)=Sq^4(e^5)=0$ and $\mathbf{Z}/2\mathbf{Z}$ otherwise,*

(3) *for $n=4$, $\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$ if $N\equiv m\equiv 0$ mod 2 and $\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$ otherwise (see Lemma 2.6),*

(4) *for* $n=3$, $\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}$ *if* $m\equiv 0 \bmod 2$ *and* $\gamma=0$, *and* $\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}$ *otherwise* (*see Lemma* 2.7).

Finally we calculate $[\Sigma A, A]\Sigma\beta$ in the case $\alpha=0$, which is equal to

$$K_n=(\pi_{n+1}(A)\Sigma\beta_1 \cup \pi_{n+3}(A)\Sigma\gamma)+\pi_{n+3}(S^{n+2})\Sigma\gamma$$

for $\beta=\beta_1+\gamma$ in the direct sum decomposition

$$\pi_{n+3}(A)\sim\pi_{n+3}(S^n)+\pi_{n+3}(S^{n+2}) .$$

By (2-1) we can easily obtain:

Lemma 2.9. *If* $\alpha=0$ *and* $n\neq 4$, *then* K_n *is isomorphic to* $\{0\}$ *for* $\gamma=0$ *and to the torsion subgroup of* $\pi_{n+4}(A)$ *for* $\gamma\neq 0$.

Lemma 2.10. *If* $\alpha=0$ *and* $n=4$, *then* K_4 *is as follows*:
(1) *for* $\gamma=0$ *and* $Sq^4(e^4)=0$, $K_4=\{0\}$,
(2) *for* $\gamma=0$ *and* $Sq^4(e^4)\neq 0$, $K_4=\{\eta_4\nu_5\}$,
(3) *for* $\gamma\neq 0$ *and* $Sq^4(e^4)=0$, $K_4=\{\nu_4\eta_7\}+\{\eta_6\eta_7\}$,
(4) *for* $\gamma\neq 0$ *and* $Sq^4(e^4)\neq 0$, $K_4=\{\nu_7\eta_8\}+\{\eta_5\nu_8\}+\{\eta_7\eta_8\}$.

Now we state the theorems.

Theorem A. *Suppose* $\alpha\neq 0$ $(Sq^2(e^n)\neq 0)$.
(1) $H_X^{-1}(0)=\{0\}$ *for* $n\geqq 6$, *or* $n=4$, 5 *and* $Sq^4(e^n)\neq 0$.
(2) *There exists a one-one correspondence* $\mathbf{Z}/2\mathbf{Z}\to H_X^{-1}(0)$ *for* $n=4$, 5 *and* $Sq^4(e^n)=0$.
(3) *There exists an exact sequence* $0\to\mathbf{Z}\to H_X^{-1}(0)\to 0$ *for* $n=3$.

Theorem B. *Suppose* $\alpha=0$ *and* $\gamma\neq 0$ $(Sq^2(e^n)=0$ *and* $Sq^2(e^{n+2})\neq 0)$.
(1) $H_X^{-1}(0)=\{0\}$ *for* $n\geqq 5$, *or* $n=4$ *with the exception of the case* $N\equiv m\equiv 0 \bmod 2$.
(2) *There exists a one-one correspondence* $\mathbf{Z}/2\mathbf{Z}\to H_X^{-1}(0)$ *for* $n=4$ *and* $N\equiv m\equiv 0 \bmod 2$.
(3) *There exists a one-one correspondence* $\mathbf{Z}\to H_X^{-1}(0)$ *for* $n=3$, *namely, integer* k *corresponds to the composite*:

$$X\to S^7 \xrightarrow[k[\iota_3,\iota_5]]{} S^3\vee S^5\to X .$$

Theorem C. *Suppose* $\alpha=\gamma=0$ $(Sq^2(e^n)=Sq^2(e^{n+2})=0)$. *Then there exist the following exact sequences*:
(1) $0\to\mathbf{Z}/2\mathbf{Z}\to H_X^{-1}(0)\to\mathbf{Z}/2\mathbf{Z}\to 0$ *for* $n\geqq 6$, *or* $n=4$, 5 *and* $Sq^4(e^n)\neq 0$.
(2) $0\to\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}\to H_X^{-1}(0)\to\mathbf{Z}/2\mathbf{Z}\to 0$ *for* $n=5$ *and* $Sq^4(e^5)=0$, *or* $n=4$ *and* $N\equiv 0$ *and* $m\equiv 1 \bmod 2$.
(3) $0\to\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}\to H_X^{-1}(0)\to\mathbf{Z}/2\mathbf{Z}\to 0$ *for* $n=4$ *and* $N\equiv m\equiv 0 \bmod 2$.
(4) $0\to\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}\to H_X^{-1}(0)\to\mathbf{Z}/2\mathbf{Z}\to 0$ *for* $n=3$ *and* $m\equiv 1 \bmod 2$.

(5) $0 \to \mathbf{Z}+\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z} \to H_X^{-1}(0) \to \mathbf{Z}/2\mathbf{Z} \to 0$ *for* $n=3$ *and* $m \equiv 0 \bmod 2$.
Here N, m are the same ones as stated in Lemmas 2.6 *and* 2.7.

The proofs follow from Lemmas 1.6 and 2.5 for Theorem A, and Lemmas 1.3, 2.5–2.10 and Prop. 2.8 for Theorems B and C.

If X is a suspended complex, it is well-known that the set $[X, X]$ becomes a group and the sequence (1-1) is exact in the category of groups and homomorphisms. For example, X is always suspended if $n \geqq 5$.

REMARK. In the case $n=3$ in Theorems B and C, the part of $\mathbf{Z}$ is injective because integer k corresponds to the composite:

$$X \longrightarrow S^7 \xrightarrow[k[\iota_3, \iota_5]]{} S^3 \vee S^5 \longrightarrow X.$$

3. Examples.

(1) Generators. We define some self-maps belonging to $H_X^{-1}(0)$ as follows:

$$\xi_n = i\eta_n\eta_{n+1}q: X \to S^{n+2} \to S^{n+1} \to S^n \to X \quad \text{for} \quad n \geqq 3 \quad \text{and} \quad \alpha=\gamma=0,$$

$$\psi_n = i\eta_{n+2}\eta_{n+3}p: X \to S^{n+4} \to S^{n+3} \to S^{n+2} \to X \quad \text{for} \quad 3 \leqq n \leqq 5 \quad \text{and} \quad \alpha=\gamma=0,$$

$$\phi_n = i\nu_n\eta_{n+3}p: X \to S^{n+4} \to S^{n+3} \to S^n \to X \quad \text{for} \quad n=4, 5,$$

$$\left.\begin{array}{l} t_1 = i\lambda p: X \to S^7 \to A \to X \\ t_2 = i\omega\eta_6 p: X \to S^7 \to S^6 \to S^3 \to X \end{array}\right\} \quad \text{for} \quad n=3,$$

$$s = i\eta_4\nu_5 p: X \to S^8 \to S^5 \to S^4 \to X \quad \text{for} \quad n=4.$$

Here i denotes inclusions, p is the pinching map $X \to X/A$, q is a map of degree 1 at dimension $n+2$ and λ is a generator of $\pi_7(A)$ referred in Prop. 2.2.

Then these maps represent non-trivial self-maps corresponding to generators of groups in Theorems.

As an application we have:

(2) Let f be a self-map of X, g one of the self-maps stated in (1), and $d_k(f)$ the degree of f at dimension $n+2k$ ($k=0, 1, 2$). For any map g, it holds that

(a) $gf=0$ if $d_2(f)$ is even, and $gf=g$ if $d_2(f)$ is odd except the cases $g=t_1$ and $g=\xi_n$.

(b) $fg=0$ if $d_0(f)$ is even, and $fg=g$ if $d_0(f)$ is odd except the cases $g=t_1$ and $g=\psi_n$.

(c) $f\psi_n=0$ if $d_1(f)$ is even, and $f\psi_n=\psi_n$ if $d_1(f)$ is odd.

REMARK. It may be considered as $ft_1 = d_0(f)d_1(f)t_1$.

(3) Let X be the $(n-2)$-fold suspension of the complex projective 3-space CP^3 ($n \geqq 3$). Then we have that $H_X^{-1}(0)$ is isomorphic to $\{0\}$ ($n \geqq 6$), $\mathbf{Z}/2\mathbf{Z}$ ($n=4, 5$) and $\mathbf{Z}$ ($n=3$) because of $Sq^2(e^n) \neq 0$ and $Sq^4(e^n)=0$ ($n=4, 5$).

(4) Let X be the complex CP^{n+2}/CP^{n-1} $(n \geqq 2)$. Then we have that

(a) $H_X^{-1}(0)=\{0\}$ if n is odd.

(b) If n is even, there exists an exact sequence

$$0 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow H_X^{-1}(0) \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0\,,$$

because we know that $Sq^2(e^{2n})=Sq^2(e^{2n+2})=0$ if n is even, $Sq^4(e^4)\neq 0$ for $n=2$, and $Sq^2(e^{2n})\neq 0$ if n is odd.

(5) Let Aa be the complex $S^2 \cup e^4$ whose attaching map for the cell e^4 is given by $a\eta_3$ and ξ be an n-dimensional vector bundle over Aa $(n \geqq 5)$. Suppose the second Stiefel-Whitney class $w_2(\xi)=0$. Then it is clear that ξ is characterized by the first Pontryagin class $p_1(\xi)$. Since the Thom complex $T(\xi)$ is of the form $(S^n \vee S^{n+2}) \cup e^{n+4}$ with $\beta=J(\xi)+a\eta_{n+2}$, we know that

(a) For an odd a, $H_X^{-1}(0)=\{0\}$.

(b) For an even a, there exist exact sequences:

$$0 \to \mathbf{Z}/2\mathbf{Z} \to H_X^{-1}(0) \to \mathbf{Z}/2\mathbf{Z} \to 0 \qquad (n \geqq 6)\,,$$

$$0 \to \mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z} \to H_X^{-1}(0) \to \mathbf{Z}/2\mathbf{Z} \to 0 \qquad (n=5 \text{ and } p_1(\xi)\equiv 0 \bmod 4)\,,$$

$$0 \to \mathbf{Z}/2\mathbf{Z} \to H_X^{-1}(0) \to \mathbf{Z}/2\mathbf{Z} \to 0 \qquad (n=5 \text{ and } p_1(\xi)\equiv 2 \bmod 4)\,.$$

(6) Let X_r be the S^2-bundle over S^4 whose characteristic class is r times of a generator of $\pi_3\,(SO(3))\sim \mathbf{Z}$, and X be the $(n-2)$-fold suspension $\Sigma(n-2)X_r$ $(n \geqq 3)$. It always holds that $Sq^2(e^{n+2})=0$ because of the existence of the projection of the bundle, and it is clear that $Sq^2(e^n)=0$ for an even r, $Sq^2(e^n)\neq 0$ for an odd r and $Sq^4(e^{n+2})=0$ for $n=2, 3$. Since $\beta=ri_*(\Sigma(n-2)\omega)$, we have that

(a) If r is odd, then $H_X^{-1}(0)=\{0\}$ for $n \geqq 6$, $H_X^{-1}(0)\sim \mathbf{Z}/2\mathbf{Z}$ for $n=4, 5$ and $H_X^{-1}(0)\sim \mathbf{Z}$ for $n=3$.

(b) If r is even, then there exist exact sequences:

$$0 \to \mathbf{Z}/2\mathbf{Z} \to H_X^{-1}(0) \to \mathbf{Z}/2\mathbf{Z} \to 0 \qquad \text{for} \quad n \geqq 6\,,$$

$$0 \to \mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z} \to H_X^{-1}(0) \to \mathbf{Z}/2\mathbf{Z} \to 0 \qquad \text{for} \quad n=5\,,$$

$$0 \to \mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z} \to H_X^{-1}(0) \to \mathbf{Z}/2\mathbf{Z} \to 0 \qquad \text{for} \quad n=3, 4\,.$$

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