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Star-Product of a Quadratic Poisson Structure

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Introduction. Using the results of Grabowski [1] and Omori-Maeda-Yoshioka [7] on the existence of a star-product of a Poisson structure, we show that "generical" quadratic Poisson structures in dimension n (cf. [2], [6]) do admit star-products and, in dimension 3, every quadratic Poisson structure does admit a star-product. Examples of star-product are given for the 13 first classes of the classification of Dufour-Haraki [2]. For the model #14 (the last one of the previous classification), we give an example of star-product when the polynomials Q (that generate this case) are the product of two nonconstant polynomials.

The present work is divided in two parts. In the first part, one can find all necessary notions that enable us to apply Grabowski [1] and Omori-Maeda-Yoshioka [7] results (These notions are developed in [1] and [3]) and the formula of the star-product derived from Drinfeld's [4]. In the second part, we recall the classification in [2] and we expose our results on the existence of a star-product of a quadratic Poisson structures.

1. Star-product of a Poisson structure.

11. Poisson structure. Let E be a commutative algebra with unit 1 over a field K of characteristic 0. Denote $M^{-1}(E)=E$ and $M^p(E)$ the vector space of $(p+1)$-linear maps from E^{p+1} to E $(p\geq 0)$. Put $M(E)=\bigoplus_{p>-2}M^p(E)$ and $A(E)=\bigoplus_{p>-2}A^p(E)$ where $A^{-1}(E)=E$ and $A^p(E)$ is the subspace of $(p+1)$-linear antisymmetric maps of $M^p(E)$. The interior product on $M(E)$ is defined by: $i: M(E)^2\to M(E)$

$$i(B)(A)=0 \qquad \text{if} \quad A\in M^{-1}(E)\,,$$

$$i(B)(A)(x_0,\cdots,x_{a+b})=\sum(-1)^{kb}A(x_0,\cdots,x_{k-1},B(x_k,\cdots,x_{k+b}),x_{k+b+1},\cdots,x_{a+b})$$
$$\text{if} \quad A\in M^a(E) \text{ and } B\in M^b(E)\,.$$

On $M(E)$, one defines a bracket Δ by:

$$A\Delta B=i(B)A+(-1)^{ab+1}i(A)B \qquad \text{with } A\in M^a(E) \text{ and } B\in M^b(E)\,.$$

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Applying the antisymmetrisation operator α one can then deduce a bracket $\bar{\Delta}$ on $A(E)$:

$$A\bar{\Delta}B=(a+b+1)!/(a+1)!(b+1)!\alpha(A\Delta B)\,,\qquad A\in A^a(E)\quad\text{and}\quad B\in A^b(E)\,.$$

PROPOSITION 111. *$(M(E),\Delta)$ and $(A(E),\bar{\Delta})$ are graded Lie algebras.*

PROPOSITION 112. *$c\in M^1(E)$ (resp. $c\in A^1(E)$) defines an associative algebra (resp. Lie algebra) on E if and only if $c\Delta c=0$ (resp. $c\bar{\Delta}c=0$).*

REMARK. A Lie bracket $\bar{\Delta}$ is nothing but the Richardson-Nijenhuis one [3] and Δ is a generalisation to $M(E)$ [3].

If V is a vector space over K, let's denote V_λ the space of the formal series with parameter λ and coefficients in V.

$A_\lambda=\sum_k \lambda^k A_k\in M^p(E)_\lambda$ is identified with an element A_λ of $M^p(E_\lambda)$ by

$$A_\lambda(x^0,\cdots,x^p)=\sum_k \lambda^k\left(\sum_{s+s_0+\cdots+s_p=k} A_s(x^0_{s_0},\cdots,x^p_{s_p})\right)$$

where $x^i=\sum_{k=0}^{+\omega}\lambda^k x^i_k$.

PROPOSITION 113. *$M(V)_\lambda$ (resp. $A(V)_\lambda$) is a graded subalgebra of $(M(E_\lambda),\Delta)$ (resp. $(A(E_\lambda),\bar{\Delta})$).*

Let (V,c) be an associative algebra (resp. Lie algebra). A formal deformation of c is an element $c_\lambda=\sum_p \lambda^p c_p$ of $M^1(V)_\lambda$ (resp. $A^1(V)_\lambda$) such that $c_0=c$ and $c_\lambda\Delta c_\lambda=0$ (resp. $c_\lambda\bar{\Delta}c_\lambda=0$). This can be written $\sum_{i+j=p} c_j\Delta c_i=0$ (resp. $\sum_{i+j=p} c_j\bar{\Delta}c_i=0$), $\forall p\in\mathbf{N}$.

The product "$.$" on E induces on $M(E)$ an associative algebra structure:

$$A\,.\,B(x_0,\cdots,x_{a+b+1})=A(x_0,\cdots,x_a)\,.\,B(x_{a+1},\cdots,x_{a+b+1})\,,$$
$$A\in M^a(E)\ \text{and}\ B\in M^b(E)\,.$$

One defines on $A(E)$ the exterior-product "$\wedge$" by

$$A\wedge B=\frac{(a+b+1)!}{(a+1)!(b+1)!}\alpha(A\,.\,B)\,,\qquad A\in A^a(E)\ \text{and}\ B\in A^b(E)$$

which gives to $A(E)$ a graded commutative algebra structure.

The bracket $\bar{\Delta}$ satisfies:

$$(*)\qquad A\bar{\Delta}(B\wedge C)=(-1)^{a(c+1)}(A\bar{\Delta}B)\wedge C+B\wedge(A\bar{\Delta}C)\,,$$
$$\forall A\in A^a(E)\,,\ B\in A(E)\,,\ C\in A^c(E)\,.$$

On $M(E)$, linear operators are defined as follows: We define the map

$$\begin{aligned}\delta(x):M^0(E)&\longrightarrow M^0(E)\\ D&\longmapsto \delta(x)(D)=D\Delta H_x\end{aligned}\qquad\text{where }\begin{aligned}H_x:E&\longrightarrow E\\ y&\longmapsto xy\,.\end{aligned}$$

DEFINITION 114. $D\in M^0(E)$ is a linear differential operator of order $\leq r$ if

$\delta(x_0)\delta(x_1)\cdots\delta(x_r)D=0$ for all $x_0,\cdots,x_r\in E$. Denote by $\mathrm{Diff}_r(E)$ the vector space of linear differential operators of order $\leq r$ and $\mathrm{Diff}(E)=\bigcup_{r\geq 0}\mathrm{Diff}_r(E)$.

One has $\mathrm{Diff}_1(E)=\mathrm{Der}(E)\oplus\mathrm{Diff}_0(E)$ where $\mathrm{Der}(E)$ is the vector space of derivations of E. One can deduce that $\mathrm{Der}(E)=\{D\in\mathrm{Diff}_1(E)\mid D(1)=0\}$.

PROPOSITION 115. *$D\in M^0(E)$ is a differential operator of order $\leq r$ if and only if $\delta(x)^{r+1}(D)=0$ for all $x\in E$.*

One defines recurrently a differential operator on $M^n(E)$ by:

DEFINITION 116. $D\in M^n(E)$ is a differential operator of order $\leq r$ if $i_p(x)D$ is a differential operator of order $\leq r$ on $M^{n-1}(E)$ for all $p=0,1,\cdots,n$ and $x\in E$ where $i_p(x)D(x_0,\cdots,x_{n-1})=D(x_0,\cdots,x_{p-1},x,x_p,\cdots,x_{n-1})$.

In the same way, one denotes $\mathrm{Diff}^n_r(E)$ the space of the $(n+1)$-linear differential operators of order $\leq r$, $\mathrm{Diff}^n(E)=\bigcup_{r\geq 0}\mathrm{Diff}^n_r(E)$ and $\mathrm{Diff}(E)=\bigoplus_{p\geq -1}\mathrm{Diff}^p(E)$.

Using Proposition 114, $D\in\mathrm{Diff}^n_r(E)$ if and only if $\delta_p(x)^{r+1}(D)=0$ for all $x\in E$ and $p=0,1,\cdots,n$, where $\delta_p(x)D(x_0,\cdots,x_n)=D(x_0,\cdots,x_{p-1},x.x_p,\cdots,x_n)-x.D(X_0,\cdots,x_n)$. The space of the $(n+1)$-linear derivations on E is nothing but the set of the elements $D\in\mathrm{Diff}^n_1(E)$ such that $i_p(1)D=0$ for $p=0,1,\cdots,n$. Denote $\mathrm{Der}^n(E)$ the space of the $(n+1)$-linear derivations on E and $\mathrm{Der}^*(E)=\bigoplus_{p\geq -1}\mathrm{Der}^p(E)$.

The subspace of $\mathrm{Diff}^*(E)$, (resp. $\mathrm{Der}^*(E)$) formed by the multilinear skewsymmetric maps is denoted by $\mathrm{ADiff}^*(E)$ $(=\bigoplus_{p\geq -1}\mathrm{ADiff}^p(E))$ (resp. $\mathrm{ADer}^*(E)=\bigoplus_{p\geq -1}\mathrm{ADer}^p(E)$).

PROPOSITION 117. 1°) $\mathrm{Diff}^*(E)$, $\mathrm{Der}^*(E)$ *(resp.* $\mathrm{ADiff}^*(E)$, $\mathrm{ADer}^*(E)$) *are associative subalgebras of* $(M^*(E),.)$ *(resp.* $(A(E),\wedge)$).

2°) $\mathrm{Diff}^*(E)$ *(resp.* $\mathrm{ADiff}^*(E)$, $\mathrm{ADiff}^*_1(E)$, $\mathrm{ADer}^*(E)$) *is a graded Lie subalgebra of* $(M(E),\Delta)$ *(resp.* $(A(E),\bar{\Delta})$).

DEFINITION 118. Each differential operator $P\in\mathrm{ADiff}^1(E)$ such that $P\bar{\Delta}P=0$ and $P(1,.)=0$ is a Poisson structure on E.

THEOREM 119 ([1]). *If E has no nontrivial nilpotent element then every Poisson structure on E is of order* 1. *So a Poisson structure on E is necessarily an element of* $\mathrm{Der}^1(E)$. *Therefore if E is an algebra spanned by F, P is well defined by the images of the elements $(x,y)\in F^2$.*

12. Star-product of a Poisson structure. Let P be a Poisson structure on E (E is supposed without nilpotent element) and m the product on E. As $m\Delta P=0$, $m+\lambda P$ is a deformation of m of order 1.

DEFINITION 121. A formal deformation $A_\lambda=m+\lambda P+\sum_{n=2}^{+\infty}\lambda^n A_n$ satisfying

1°) $A_\lambda\in\mathrm{Diff}^1(E)_\lambda$,

2°) $A_k(u,v)=(-1)^k A_k(v,u)$,

3°) A_k vanishes on constants, i.e., $A_k(1,.)=0$
is called a star-product of P.

The condition 3°) implies that 1 is the unit of the algebra (E_t, A_t).

The following result gives a sufficient condition for the existence of star-product for the Poisson structure of the form: $P=\sum_{i,j} a_{ij}D_i . D_j$ with $a_{ij}=-a_{ji}$ and $D_i \in \mathrm{Der}(E)$. Let L be the Lie algebra $(\mathrm{Der}(E), \bar{\Lambda})$ and U its enveloping algebra.

DEFINITION 122. *$H \in \bigwedge^2 L$ is called a Poisson element if it satisfies the Yang-Baxter equation:*

$$[H_0, H_1]+[H_0, H_2]+[H_1, H_2]=0$$

where $[\,,]$ is the bracket induced by the associative structure on $\otimes^3 U$ and H_0, H_1 and H_2 are the respective images of H by the homomorphisms:

$$\psi_0 : \otimes^2 U \longrightarrow \otimes^3 U$$
$$u \otimes v \longmapsto 1 \otimes u \otimes v\,,$$
$$\psi_1 : \otimes^2 U \longrightarrow \otimes^3 U$$
$$u \otimes v \longmapsto u \otimes 1 \otimes v\,,$$
$$\psi_2 : \otimes^2 U \longrightarrow \otimes^3 U$$
$$u \otimes v \longmapsto u \otimes v \otimes 1\,.$$

THEOREM 123 ([1], [4]). *Let $P=\sum_{i,j} a_{ij}D_i . D_j$ (with $a_{ij}=-a_{ji}$ and $D_i \in \mathrm{Der}(E)$) be a Poisson structure on E. If $\bar{P}=\sum_{i,j} a_{ij}D_i \otimes D_j$ is a Poisson element, then P admits a star-product.*

Let $\bar{P}=\sum_{1\le i,j\le p} a_{ij}D_i \otimes D_j$ be a Poisson element where $(a_{ij})_{i,j}$ is an invertible matrix. Let $(b_{ij})_{i,j}$ be the inverse matrix of $(a_{ij})_{i,j}$, G the Lie group of the Lie algebra L, $(h_{ij})_{i,j}$ the matrix of the adjoint action of G in the basis $(D_i)_{1\le i,j\le p}$, $L'=L\otimes Ku$ a central extension of L whose bracket $[\,,]'$ is defined by: $[D_i, D_j]'=[D_i, D_j]+b_{ij}u$, and $\varphi \in (L')^*$ such that $\varphi(L)=0$ and $\varphi(u)=1$.

PROPOSITION 124. *Let $P=\sum_{i,j} a_{ij}D_i . D_j$ be a Poisson structure on E such that $P=\sum_{i,j} a_{ij}D_i \otimes D_j$ is a Poisson element. Then a star-product of P is given by:*

$$(**)\qquad (u,v) \longmapsto m(u,v)+\sum_{n=1}^{+\infty}\Bigg(\sum_{|\alpha_i|+|\beta_i|>1,\,1\le i\le n} \varphi(c'_{\alpha_1\beta_1})\cdots \cdots \varphi(c'_{\alpha_n\beta_n})\times(2\lambda)^{\sum|\alpha_i|+|\beta_i|-n}.H^{\sum\alpha_i}H^{\sum\beta_i}\Bigg)$$

where $c'_{\alpha\beta}$ (α, $\beta \in \mathbf{N}^p$) are the coefficients of the term $t_\alpha s_\beta$ in the Campbell-Baker-Hausdorff series

$$CH'(\sum t_k D_k, \sum s_k D_k) = \sum t_k D_k + \sum s_j D_j + \tfrac{1}{2}\sum t_k s_j [D_k, D_j]' + \cdots$$

and $H^\nu = (H_1^{\nu_1} \circ \cdots \circ H_p^{\nu_p})(e)$ $(\nu = (\nu_1, \nu_2, \cdots, \nu_p))$ *with* $H_i = \sum_{j,k} h_{ik}\alpha_{jk} D_k$, $h_{ik}(e) = \delta_i^k$, e *is the unit of* G.

In particular,

PROPOSITION 125. *Let* $P = \sum_{i,j} a_{ij} D_i . D_j$ *be a Poisson structure on* E *such that* D_i *commute pairwise, then* $\exp \lambda P$ *is a star-product of* P *and one has*: $\exp \lambda P = \sum_{k=0}^{\infty} (1/k!) \lambda^k P^k$ *where*

$$\frac{1}{k!} P^k = \sum_{\substack{1 \le i,j \le p \\ r_{ii} = 0, \sum r_{ij} = k}} \frac{(-1)^{k - \sum_{i<j} r_{ij}}}{\prod_{ij} r_{ij}!} \prod_{ij} a_{ij}^{r_{ij} + r_{ji}}$$
$$\times (D_1^{\sum_{i=1}^p r_{1i}} \circ \cdots \circ D_p^{\sum_{i=1}^p r_{pi}}) . (D_1^{\sum_{i=1}^p r_{i1}} \circ \cdots \circ D_p^{\sum_{i=1}^p r_{ip}}) .$$

2. Star-product of a quadratic Poisson structure.

21. Quadratic Poisson structure. From now on, we suppose that E $(= S^*(F))$ is the symmetric algebra of a vector space F of finite dimension n over $K = \mathbf{R}$ or $\mathbf{C}$. A basis $(x_1, \cdots, x_n)$ of F is fixed.

DEFINITION 211. A quadratic Poisson structure on E is a Poisson structure $P = \sum_{i,j} a_{ij} D_i \wedge D_j$ $(a_{ij} = -a_{ji}$ and $D_i \in \mathrm{Der}(E))$ on E such that $P(F, F) \subseteq S^2(F)$.

EXAMPLES 212. 1°) If $D_i \in \mathrm{Der}(E)$ $(i = 1, 2, \cdots . p)$ commute pairwise and satisfy $D_i . D_j(F^2) \subseteq S^2(F)$, then $P = \sum_{i,j} D_i \wedge D_j$ is a quadratic Poisson structure on E. This is a consequence of the graded distributivity of "$\wedge$" relative to $\bar{\Delta}$ (see $(*)$).

2°) If $D_i \in \mathrm{Der}(E)$, $i = 1, 2, \cdots, p$, satisfies $D_i . D_j(F^2) \subseteq S^2(F)$ and $\bar{P} = \sum_{i,j} a_{ij} D_i \otimes D_j$ satisfies the Yang-Baxter equation then $P = \sum_{i,j} a_{ij} D_i \wedge D_j$ is a quadratic Poisson structure. Indeed

$$[\bar{P}_0, \bar{P}_1] + [\bar{P}_0, \bar{P}_2] + [\bar{P}_1, \bar{P}_2]$$
$$= \sum_{i,j,k,l} a_{ij} a_{kl} ([1 \otimes (D_i \otimes D_j - D_j \otimes D_i), D_k \otimes 1 \otimes D_l - D_l \otimes 1 \otimes D_k]$$
$$+ [1 \otimes (D_i \otimes D_j - D_j \otimes D_i), D_k \otimes D_l \otimes 1 - D_l \otimes D_k \otimes 1]$$
$$+ [D_i \otimes 1 \otimes D_j - D_j \otimes 1 \otimes D_i, D_k \otimes D_l \otimes 1 - D_l \otimes D_k \otimes 1])$$
$$= \sum_{i,j,k,l} a_{ij} a_{kl} (D_k \otimes D_i \otimes [D_j, D_l] - D_k \otimes D_j \otimes [D_i, D_l] - D_l \otimes D_i \otimes [D_j, D_k]$$
$$+ D_l \otimes D_j \otimes [D_i, D_k] + D_k \otimes [D_i, D_l] \otimes D_j - D_k \otimes [D_j, D_l] \otimes D_i$$
$$- D_l \otimes [D_i, D_k] \otimes D_j + D_l \otimes [D_j, D_k] \otimes D_i + [D_i, D_k] \otimes D_l \otimes D_j$$

$$-[D_j, D_k]\otimes D_l \otimes D_i - [D_i, D_l]\otimes D_k \otimes D_j + [D_j, D_l]\otimes D_k \otimes D_i)$$

$$=4 \sum_{i,j,k,l} a_{ij}a_{kl}(D_k \otimes D_i \otimes [D_j, D_l] + D_k \otimes [D_i, D_l]\otimes D_j + [D_i, D_k]\otimes D_l \otimes D_j)$$

and $P\bar{\Delta}P = 4\sum_{i,j,k,l} a_{ij}a_{kl}D_k \wedge D_i \wedge [D_j, D_l]$. Therefore $[\bar{P}_0, \bar{P}_1]+[\bar{P}_0, \bar{P}_2]+[\bar{P}_1, \bar{P}_2]=0$ implies $P\bar{\Delta}P=0$.

NOTATION. Denote ∂_i $(i=1, \cdots, n)$ the derivation of E defined by $\partial_i(x_j)=\delta_{ij}$. Therefore every quadratic Poisson structure P on E can be written $P=\sum_{i,j} P_{ij}\partial_i \wedge \partial_j$ where $P_{ij} \in S^2F$ and $P_{ij}=-P_{ji}$.

At every quadratic Poisson structure P on E, one can associate $R_P \in \mathrm{Der}(E)$ called the rotational of P ([2], [6]) and which verifies $P\bar{\Delta}R_P=0$: If $P=\sum_{i,j} P_{ij}\partial_i \wedge \partial_j$ then $R_P=\sum_{i,j} \partial_j(P_{ij}).\partial_i$.

The restriction of R_P to F is an endomorphism of E so it can be reduced to its Jordan form. If two quadratic Poisson structures are isomorphic then their rotationals are isomorphic too. Hence, the Jordan form of the rotational is an invariant for the classification by linear isomorphisms of quadratic Poisson structures. By convention, let's call eigenvalue of P an eigenvalue of its rotational R_P.

THEOREM 213 ([2]). *Every quadratic Poisson structure which has eigenvalues λ_i verifying no relation of the form $\lambda_i+\lambda_j=\lambda_r+\lambda_s$ with $r \neq s$ and $\{i, j\} \neq \{r, s\}$ is isomorphic to a structure of the form $\sum_{i,j} a_{ij}x_i x_j \partial_i \wedge \partial_j$ where $a_{ij} \in K$ with $a_{ij}=-a_{ji}$.*

In what follows let's replace x_1, x_2, x_3 by x, y, z respectively if $n=3$.

THEOREM 214 ([2]). *In dimension 3, every quadratic Poisson structure is isomorphic to one of the following 14 models:*

1°) $axy\partial_1 \wedge \partial_2 + byz\partial_2 \wedge \partial_3 + czx\partial_3 \wedge \partial_1$

2°) $b(x^2+y^2)\partial_1 \wedge \partial_2 + z(2bx-ay)\partial_2 \wedge \partial_3 + z(ax+2by)\partial_3 \wedge \partial_1$

3°) $x^2\partial_1 \wedge \partial_2 + (-ayz+2xz)\partial_2 \wedge \partial_3 + axz\partial_3 \wedge \partial_1$

4°) $axy\partial_1 \wedge \partial_2 + (x^2+cyz)\partial_2 \wedge \partial_3 + axz\partial_3 \wedge \partial_1$

5°) $ax^2\partial_1 \wedge \partial_2 + (yz+(1+2a)xz)\partial_2 \wedge \partial_3 - xz\partial_3 \wedge \partial_1$ $(a \neq -1/2)$

6°) $-1/2x^2\partial_1 \wedge \partial_2 + byz\partial_2 \wedge \partial_3 - bzx\partial_3 \wedge \partial_1$

7°) $a(x^2+y^2)\partial_1 \wedge \partial_2 + (byz+(2a+c)xz)\partial_2 \wedge \partial_3 + ((2a+c)yz-bxz)\partial_3 \wedge \partial_1$

8°) $(((a+b)/2)(x^2+y^2) \pm z^2)\partial_1 \wedge \partial_2 + axz\partial_2 \wedge \partial_3 + ayz\partial_3 \wedge \partial_1$

9°) $-\frac{1}{3}x^2\partial_1 \wedge \partial_2 + (ax^2-\frac{1}{3}y^2+\frac{1}{3}xz)\partial_2 \wedge \partial_3 + \frac{1}{3}xy\partial_3 \wedge \partial_1$

10°) $-(2b+1)x^2\partial_1 \wedge \partial_2 + (by^2-(1+4b)xz)\partial_2 \wedge \partial_3 + (2b+1)xy\partial_3 \wedge \partial_1$

11°) $(cx^2+dz^2)\partial_1 \wedge \partial_2 + (2c+1)xz\partial_2 \wedge \partial_3$

12°) $(cx^2+dz^2)\partial_1 \wedge \partial_2 + (x^2+(2c+1)xz)\partial_2 \wedge \partial_3$

13°) $(cx^2+dz^2+2xz)\partial_1 \wedge \partial_2 + (ax^2+z^2+(2c+1)xz)\partial_2 \wedge \partial_3$

14°) $\partial_3(Q)\partial_1 \wedge \partial_2 + \partial_1(Q)\partial_2 \wedge \partial_3 + \partial_2(Q)\partial_3 \wedge \partial_1$ *where* $Q \in S^3F$.

REMARK 215. In the theorem the cases 1°) to 4°) correspond to nonvanishing diagonal rotationals; In the cases 2°) and 3°) there is a nonvanishing double eigenvalue; The case 4°) corresponds to one vanishing eigenvalue. The cases 5°) and 6°) correspond to a nontrivial 2×2 jordan block and a nonvanishing double eigenvalue. The case 7°) corresponds to a pure complex eigenvalue with nonvanishing real part. The case 8°) corresponds to a pure imaginary eigenvalue. The cases 9°) to 14°) correspond to nilpotent rotationals. In the cases 9°) and 10°) the nilpotence degree is 3, it is 2 for the cases 11°), 12°) and 13°), and for 14°) the rotational vanishes.

22. Star-product of a quadratic Poisson structure.

PROPOSITION 221 [7]. *Every quadratic Poisson structure whose structure constants* (a_{ij}^{kl}) *satisfy*

$$(*) \qquad \sum_{p=1}^{n} a_{ip}^{uv} a_{jk}^{pw} + a_{jp}^{uv} a_{ki}^{pw} + a_{kp}^{uv} a_{ij}^{pw} = 0 \qquad \text{for all} \quad i, j, k, u, v, w = 1, 2, \cdots, n$$

admits a star-product.

PROPOSITION 222. *Every quadratic Poisson structure P which has eigenvalues λ_i with no relation of the form*

$$\lambda_i + \lambda_j = \lambda_r + \lambda_s \qquad \text{with } r \neq s \text{ and } \{i, j\} \neq \{r, s\}$$

admits $\exp(tP)$ *as a star-product.*

PROOF. From Theorem 213, one has the existence of an automorphism g of F such that $g^{-1}(P) = \sum_{i,j} a_{ij} x_i x_j \partial_i \wedge \partial_j$. If $h_i = x_i \partial_i$ and $g_i = g \circ h_i \circ g^{-1}$ then $P = \sum_{i,j} a_{ij} g_i \wedge g_j$ and $[g_i, g_j] = g \circ [h_i, h_j] \circ g^{-1} = 0$ because $[h_i, h_j] = 0$ $\forall i, j \in \{1, \cdots, n\}$. Therefore, by Proposition 125, P admits a star-product and $\exp(\lambda P)$ is one. In [7], it is shown that every bracket $\sum_{i,j} a_{ij} x_i x_j \partial_i \wedge \partial_j$ admits a star-product.

PROPOSITION 223. *In dimension* 3, *every quadratic Poisson structure with non-vanishing rotational and which has eigenvalues λ_1, λ_2, λ_3 such that relations $\lambda_1 = 0$, $\lambda_2 = -\lambda_3$ and $\lambda_2 . \lambda_3 \neq 0$ are not verified admits a star-product.*

PROOF. The quadratic Poisson structure satisfying the hypotheses of the proposition are isomorphic to one of the models 1°), 2°), 3°), 5°), 6°), 7°), 9°), 10°), 11°), 12°) and 13°) of Theorem 214. We need the following lemmas to achieve the proof:

LEMMA 1. *The quadratic Poisson structure P of the form* 1°), 2°), 3°), 5°), 6°), 7°), 9°), 11°), 12°) *and* 13°) *can be written as*

$$P = D_1 \wedge D_2 \qquad \text{where } D_1, D_2 \in \mathrm{Der}(E) \text{ with } [D_1, D_2] = 0 .$$

PROOF OF LEMMA 1. For the model of the form 1°), one has:

$$D_1 = ax\partial_1 - bz\partial_3\ ;\quad D_2 = y\partial_2 - \frac{c}{a}z\partial_3 \qquad \text{if } a \neq 0\,,$$

$$D_1 = by\partial_2 - cx\partial_1\ ;\quad D_2 = x\partial_3 \qquad \text{if } a = 0\,.$$

For those of the form 2°), one has:

$$D_1 = (2bx - ay)\partial_2 - (ax + 2by)\partial_1\ ;$$
$$D_2 = -\tfrac{1}{4}a(2bx + ay)\partial_2 - \tfrac{1}{4}a(ax - 2by)\partial_1 + z\partial_3 \qquad \text{if } a \neq 0\,,$$
$$D_1 = bx\partial_2 - by\partial_1\ ;\quad D_2 = -y\partial_2 - x\partial_1 + 2z\partial_3 \qquad \text{if } a = 0\,.$$

For those of the form 3°), one has:

$$D_1 = (-ay + 2x)\partial_2 - ax\partial_1\ ;\qquad D_2 = -\tfrac{1}{2}y\partial_2 - \tfrac{1}{2}x\partial_1 + z\partial_3\,.$$

For those of the forms 5°) and 6°):

$$P = ax^2\partial_1 \wedge \partial_2 + (byz + (1 + 2a)xz)\partial_2 \wedge \partial_3 - bxz\partial_3 \wedge \partial_1\,,$$

one has:

$$D_1 = ax\partial_2 + bz\partial_3\ ;\quad D_2 = -y\partial_2 + \frac{1+2a}{a}z\partial_3 - x\partial_1 \qquad \text{if } a \neq 0\,,$$

$$D_1 = (by + x)\partial_2 + bx\partial_1\ ;\quad D_2 = z\partial_3 \qquad \text{if } a = 0\,.$$

For those of the form 7°), one has:

$$D_1 = (by + (2a + c)x)\partial_2 - ((2a + c)y - bx)\partial_1\ ;$$

$$D_2 = \frac{a}{b}x\partial_2 - \frac{a}{b}y\partial_1 + z\partial_3 \qquad \text{if } b \neq 0\,,$$

$$D_1 = y\partial_1 - x\partial_2\ ;\quad D_2 = ax\partial_1 + ay\partial_2 - (2a + c)z\partial_3 \qquad \text{if } b = 0\,.$$

For those of the form 9°), one has:

$$D_1 = x\partial_2 + y\partial_3\ ;\quad D_2 = \tfrac{1}{3}x\partial_1 + \tfrac{1}{3}y\partial_2 + (ax + \tfrac{1}{3}z)\partial_3\,.$$

For those of the forms 11°), 12°) and 13°):

$$P = (cx^2 + dz^2 + pxz)\partial_1 \wedge \partial_2 + (ax^2 + qz^2 + (2c + 1)xz)\partial_2 \wedge \partial_3$$
$$\text{with } (p, q) = (0, 0), (2, 1)\,,$$

one has

$$D_1 = (cx^2 + dz^2 + pxz)\partial_1 - (ax^2 + qz^2 + (2c + 1)xz)\partial_3\ ;\qquad D_2 = \partial_2\,.$$

LEMMA 2. *Every quadratic Poisson structure P of the form* 10°) *can be written as $P = D_1 \wedge D_2$ with $D_1, D_2 \in \mathrm{Der}(E)$ and $\bar{P} = D_1 \otimes D_2 - D_2 \otimes D_1$ satisfies the Yang-Baxter* *equation.*

PROOF. Put

$$D_1=-(2b+1)x\partial_1+by\partial_2+(1+4b)z\partial_3\ ; \qquad D_2=x\partial_2+y\partial_3\,.$$

Then $P=D_1\wedge D_2$ and one can verify $[D_1, D_2]=-(3b+1)D_2$. Moreover,

$$\begin{aligned}[\bar{P}_0, \bar{P}_1]&=[1\otimes(D_1\otimes D_2-D_2\otimes D_1), D_1\otimes 1\otimes D_2-D_2\otimes 1\otimes D_1]\\ &=-D_2\otimes D_1\otimes[D_2, D_1]-D_1\otimes D_2\otimes[D_1, D_2]\\ &=(3b+1)(-D_2\otimes D_1\otimes D_2+D_1\otimes D_2\otimes D_2)\,,\\ [\bar{P}_0, \bar{P}_2]&=(3b+1)(D_2\otimes D_2\otimes D_1-D_1\otimes D_2\otimes D_2)\,,\\ [\bar{P}_1, \bar{P}_2]&=(3b+1)(D_2\otimes D_1\otimes D_2-D_2\otimes D_2\otimes D_1)\,.\end{aligned}$$

Therefore $[\bar{P}_0, \bar{P}_1]+[\bar{P}_0, \bar{P}_2]+[\bar{P}_1, \bar{P}_2]=0$.

END OF THE PROOF OF THE PROPOSITION. Let P be a quadratic Poisson structure satisfying the hypotheses of the proposition. From Lemmas 1 and 2, one can write $P=g(D_1)\wedge g(D_2)$ with $g\in GL(F)$, $D_1, D_2\in \mathrm{Der}(F)$ and $D_1\otimes D_2-D_2\otimes D_1$ satisfies the Yang-Baxter equation. Therefore $g(D_1)\otimes g(D_2)-g(D_2)\otimes g(D_1)$ satisfies it too and P admits a star-product thanks to Theorem 123.

PROPOSITION 224. *Every quadratic Poisson structure isomorphic to models* 1°), 2°), 3°), 5°), 6°), 7°), 9°), 11°), 12°) *and* 13°) *admits* $\exp\lambda P$ *as a star-product.*

PROOF. Consequence of Lemma 1 and Proposition 125.

PROPOSITION 225. *Let* D_1, $D_2\in\mathrm{Der}(E)$ *and* $a\in K$ *such that* $[D_1, D_2]=aD_2$ ($[D_1, D_2]=D_2\circ D_1-D_1\circ D_2$), *then* $P=D_1\wedge D_2$ *is a quadratic Poisson structure which admits a star-product. The series* (**) *in Proposition* 124 *gives a star-product of* P, *where* $\varphi(\hat{C}_{\alpha\beta})$ *are*:

$$\varphi(\hat{C}_{\alpha\beta})=0 \qquad \text{if } \alpha_2+\beta_2\neq 1 \text{ or } \alpha_1=\beta_1=0 \text{ or } \alpha_1=\alpha_2=0 \text{ or } \beta_1=\beta_2=0\,.$$

$$\varphi(\hat{C}_{\alpha\beta})=\frac{(-a)^{|\alpha|+|\beta|-1}}{|\alpha|+|\beta|}\left(\sum_{k=1}^{|\alpha|+|\beta|+1}\frac{(-1)^{k+1}}{k}\right.$$
$$\left.\times\left(\sum_{(p_i,q_i)\in A^k_{\alpha\beta}}\frac{1}{p_1!q_1!\cdots p_k!q_k!}-\sum_{(p_i,q_i)\in B^k_{\alpha\beta}}\frac{1}{p_1!q_1!\cdots p_k!q_k!}\right)\right)$$

if $\alpha_2=1$, $\beta_2=0$ *and* $\beta_1\geq 1$.

$$\varphi(\hat{C}_{\alpha\beta})=\frac{(-a)^{|\alpha|+|\beta|}}{|\alpha|+|\beta|}\left(\sum_{k=1}^{|\alpha|+|\beta|+1}\frac{(-1)^{k+1}}{k}\right.$$
$$\left.\times\left(\sum_{(p_i,q_i)\in A^k_{\alpha\beta}}\frac{1}{p_1!q_1!\cdots p_k!q_k!}-\sum_{(p_i,q_i)\in B^k_{\alpha\beta}}\frac{1}{p_1!q_1!\cdots p_k!q_k!}\right)\right)$$

if $\alpha_2=0$, $\alpha_1\geq 1$ *and* $\beta_2=1$.

Here,

$$A^k_{\alpha\beta}=\left\{(p_i,q_i)_{1\le i\le k}\;\middle|\;\begin{array}{l}(p_i+q_i>0\ (\forall i=1,\cdots,k),\\ ((p_1=1,q_1>0)\ \textit{or}\ (p_1=1,q_1=p_2=0,q_2>0))\\ \textit{and}\left(\left(\sum_{i=1}^k p_i=\alpha_1+1,\ \sum_{i=1}^k q_i=\beta_1+1\ \textit{if}\ q_k>0\right)\right.\\ \qquad\left.\textit{or}\left(\sum_{i=1}^k p_i=\alpha_1+2,\ \sum_{i=1}^k q_i=\beta_1\ \textit{if}\ q_k=0\right)\right)\end{array}\right\},$$

$$B^k_{\alpha\beta}=\left\{(p_i,q_i)_{1\le i\le k}\;\middle|\;\begin{array}{l}(p_i+q_i>0\ (\forall i=1,\cdots,k),\ ((p_1=0,\ q_1=1)\ \textit{or}\ p_2>0)\\ \textit{and}\left(\left(\sum_{i=1}^k p_i=\alpha_1+1,\ \sum_{i=1}^k q_i=\beta_1+1\ \textit{if}\ q_k>0\right)\right.\\ \qquad\left.\textit{or}\left(\sum_{i=1}^k p_i=\alpha_1+2,\ \sum_{i=1}^k q_i=\beta_1\ \textit{if}\ q_k=0\right)\right)\end{array}\right\}.$$

Moreover, for all $p,q\in\mathbf{N}^*$, one has

i°) $H^p_1(e)=(-1)^pD^p_1$

ii°) $H^q_2(e)=\sum_{k=0}^{E(q/2)}\sum_{l=0}^{k}A^q_{k,l}a^{2k-l}D_2^{q-2k}D_1^l$

iii°) $H^p_1\circ H^q_2(e)=\sum_{k=0}^{E(q/2)}\sum_{l=0}^{p+k}A^{p,q}_{k,l}a^{p+2k-l}D_2^{q-2k}D_1^l$

where $A^q_{k,l}$ and $A^{p,q}_{k,l}$ are given by the following recurrency formulas:

ii°)′ $A^1_{0,0}=1,\ A^2_{0,0}=1,\ A^2_{1,0}=0,\ A^2_{1,1}=1.$

If q is even,

$$A^{q+1}_{k,0}=\sum_{s=0}^{k}A^q_{k,s}2^s\qquad \textit{for all } k=0,\cdots,q/2,$$

$$A^{q+1}_{k,l}=(q-2(k-1))A^q_{k-1,l-1}+\sum_{s=0}^{k-l}A^q_{k,s}2^sC^s_{l+s}$$

$$\textit{for all } k,l\ (1\le k\le q/2 \textit{ and } 1\le l\le k).$$

If q is odd,

$$A^{q+1}_{k,0}=\sum_{s=0}^{k}A^q_{k,s}2^s\quad \textit{for all } k=0,\cdots,(q-1)/2,\qquad A^{q+1}_{(q+1)/2,0}=0,$$

$$A^{q+1}_{(q+1)/2,l}=A^{q+1}_{(q-1)/2,l-1}\qquad \textit{for all } l=0,\cdots,(q+1)/2,$$

$$A^{q+1}_{k,l}=(q-2(k-1))A^q_{k-1,l-1}+\sum_{s=0}^{k-l}A^q_{k,s}2^sC^s_{l+s}$$

$$\textit{for all } k,l\ (1\le k\le (q-1)/2 \textit{ and } 1\le l\le k).$$

iii°)′ $A_{0,l}^{p,1}=(-1)^p 2^{p-l}C_p^l$, $A_{0,l}^{p,2}=(-1)^p 4^{p-l}C_p^l$, $A_{1,l+1}^{p,2}=(-1)^p 2^{p-l}C_p^l$ *for all* $l=0,\cdots,p$ *and* $A_{1,0}^{p,2}=0$.

If q is even,

$$A_{k,0}^{p,q+1}=\sum_{s=0}^{p+k}A_{k,s}^{p,q}2^s \qquad \textit{for all } k=0,\cdots,q/2\,,$$

$$A_{0,l}^{p,q+1}=\sum_{s=l}^{p}A_{0,s}^{p,q}2^{s-l}C_l^s \qquad \textit{for all } l=0,\cdots,p\,,$$

$$A_{k,l}^{p,q+1}=(q-2(k-1))A_{k-1,l-1}^{p,q}+\sum_{s=0}^{k-l}A_{k,s}^{p,q}2^sC_{l+s}^s$$

$$\textit{for all } k,l\ (1\le k\le q/2 \textit{ and } 1\le l\le k)\,.$$

If q is odd,

$$A_{k,0}^{p,q+1}=\sum_{s=0}^{p+k}A_{k,s}^{p,q}2^s \qquad \textit{for all } k=0,\cdots,(q-1)/2\,,$$

$$A_{0,l}^{p,q+1}=\sum_{s=1}^{p}A_{0,s}^{p,q}2^{s-l}C_s^l \qquad \textit{for all } l=0,\cdots,p\,,$$

$$A_{k,l}^{p,q+1}=(q-2(k-1))A_{k-1,l-1}^{p,q}+\sum_{s=l}^{p+k}A_{k,s}^{p,q}2^{s-l}C_l^s$$

$$\textit{for all } k,l\ (1\le k\le (q-1)/2 \textit{ and } 1\le l\le k)\,,$$

$$A_{(q+1)/2,0}^{p,q+1}=0\,,\qquad A_{(q+1)/2,l}^{p,q+1}=A_{(q-1)/2,l-1}^{p,q} \quad \textit{for all } l=0,\cdots,p+(q+1)/2\,,$$

$$A_{k,l}^{p,q+1}=(q-2(k-1))A_{k-1,l-1}^{p,q}+\sum_{s=l}^{p+k}A_{k,s}^{p,q}2^{s-l}C_l^s$$

$$\textit{for all } k,l\ (1\le k\le (q-1)/2 \textit{ and } 1\le l\le k)\,.$$

($E(q/2)$ is the integer part of $q/2$.)

PROOF. i) One can show that if $\bar{P}=D_2\circ D_1-D_1\circ D_2$ then

$$[\bar{P}_0,\bar{P}_1]=a(-D_2\otimes D_1\otimes D_2+D_1\otimes D_2\otimes D_2)\,,$$

$$[\bar{P}_0,\bar{P}_2]=a(D_2\otimes D_2\otimes D_1-D_1\otimes D_2\otimes D_2)\,,$$

$$[\bar{P}_1,\bar{P}_2]=a(D_2\otimes D_1\otimes D_2-D_2\otimes D_2\otimes D_1)\,.$$

So $[\bar{P}_0,\bar{P}_1]+[\bar{P}_0,\bar{P}_2]+[\bar{P}_1,\bar{P}_2]=0$ i.e. $\bar{P}$ satisfies the Yang-Baxter equation. Therefore P is a quadratic bracket thanks to Examples 212-2°). From Proposition 124, P admits a star-product and (∗∗) is the one.

ii) We use the same notations as in Proposition 124.

$$(a_{ij})=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\quad (b_{ij})=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix};$$

$$L = KD_1 \oplus KD_2\ ;\quad L' = KD_1 \oplus KD_2 \oplus Ku\ ;$$

$$[D_i, D_j]' = [D_i, D_j] - b_{ij}u\ ;\quad \varphi \in (L')^*\ ;\quad \varphi(L) = 0 \quad \text{and} \quad \varphi(u) = 1\ ;$$

$$H_1 = h_{12}D_2 - h_{11}D_1\ ;\quad H_2 = h_{22}D_2 - h_{21}D_1\ ;\quad h_{ij}(e) = \delta^i_j;$$

$$\begin{aligned} D_1(h_{11}) &= 0 &;\quad D_2(h_{11}) &= -ah_{22}\ ;\\ D_1(h_{12}) &= ah_{12}\ &;\quad D_2(h_{12}) &= 0 \quad ;\\ D_1(h_{21}) &= 0 &;\quad D_2(h_{21}) &= -ah_{22}\ ;\\ D_1(h_{22}) &= ah_{22}\ &;\quad D_2(h_{22}) &= 0 \quad . \end{aligned}$$

The values of $\varphi(\hat{C}_{\alpha\beta})$ come immediately from the Campbell-Hausdorff formula and from $[D_1, D_2]\hat{} = aD_2 - u$. The value of $H_1^p \circ H_2^q(e)$ is a consequence of the following lemma:

LEMMA. *For all $p, q \in N^*$ one has:*
1°) $D_1^p \circ D_2 = D_2 \circ (\sum_{k=0}^{p} C_p^k a^k D_1^{p-k})$
2°) $H_1^p(e) = (-1)^p D_1^p$
3°) $H_2^q(e) = \sum_{k=0}^{E(q/2)} \sum_{l=0}^{k} A_{k,l}^q a^{2k-l} D_2^{q-2k} (D_1)^l$
4°) $H_1^p \circ H_2^q(e) = \sum_{k=0}^{E(q/2)} \sum_{l=0}^{p+k} B_{k,l}^q a^{p+2k-l} D_2^{q-2k} \circ D_1^l$ *with $A_{k,l}^q$ and $B_{k,l}^q$ defined by the recurrency relations above.*

PROOF OF THE LEMMA. 1°) One has: $D_1 \circ D_2 = D_2 \circ D_1 + aD_2$; the formula is then satisfied for $p = 1$. Suppose it is true until order n, then

$$\begin{aligned} D_1^{n+1} \circ D_2 &= D_1^n \circ (D_1 \circ D_2) = D_1^n \circ (D_2 D_1 + aD_2) = (D_1^n \circ D_2) \circ (D_1 + aI)\\ &= D_2 \circ \left(\sum_{k=0}^{n} C_n^k a^k D_1^{n-k} \right) \circ (D_1 + aI) = D_2 \circ \left(\sum_{k=0}^{n} C_p^k a^k D_1^{n+1-k} + \sum_{k=0}^{n} C_n^k a^{k+1} D_1^{n-k} \right)\\ &= D_2 \circ \left(D_1^{n+1} + \sum_{k=1}^{n} (C_n^k + C_n^{k-1}) a^k D_1^{n+1-k} + a^{n+1} \right) = D_2 \circ \left(\sum_{k=0}^{n+1} C_n^k a^k D_1^{n-k} \right). \end{aligned}$$

2°) One has: $H_1(e) = (h_{12}D_2 - h_{11}D_1)(e) = -D_1$. If $H_1^n(e) = (-1)^n D_1^n$, then

$$\begin{aligned} H_1^{n+1}(e) &= (H_1^n(e) \circ H_1)(e) = (-1)^n (D_1^n \circ (h_{12}D_2 - h_{11}D_1))(e)\\ &= (-1)^n \left(-h_{11} D_1^{n+1} + \sum_{m \le n} C_n^m D_1^m(h_{12}) D_1^{n-m} \circ D_2 \right)(e)\\ &= (-1)^n \left(-h_{11} D_1^{n+1} + \sum_{m \le n} h_{12} a^m C_n^m D_1^{n-m} \circ D_2 \right)(e)\\ &= (-1)^{n+1} D_1^{n+1}\ . \end{aligned}$$

3°) One has:

$$H_2(e) = (h_{22}D_2 - h_{21}D_1)(e) = D_2\ ,$$

$$\begin{aligned}H_2^2(e)&=(D_2\circ(h_{22}D_2-h_{21}D_1)(e)\\&=(ah_{22}D_1-h_{21}D_2\circ D_1+h_{22}D_2^2)(e)=aD_1+D_2^2\,.\end{aligned}$$

Suppose that

$$H_2^n(e)=\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k.l}^n a^{2k-l}D_2^{n-2k}D_1^l\,,$$

then,

$$\begin{aligned}H_2^{n+1}(e)&=(H_2^n(e)\circ H_2)(e)=\left(\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k,l}^n a^{2k-l}D_2^{n-2k}\circ D_1^l\right)\circ H_2\right)(e)\\
&=\left(\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k,l}^n a^{2k-l}D_2^{n-2k}\circ D_1^l\right)\circ(h_{22}D_2-h_{21}D_1)\right)(e)\\
&=\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k,l}^n a^{2k-l}D_2^{n-2k}\circ\left(-h_{21}D_1^{l+1}+h_{22}\sum_{s\le l}C_l^s a^s D_1^{l-s}\circ D_2\right)\right)(e)\\
&=\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k,l}^n a^{2k-1}D_2^{n-2k}\circ\left(-h_{21}D_1^{l+1}+h_{22}\sum_{s\le l}C_l^s a^s D_2\right.\right.\\
&\qquad\left.\left.\circ\left(\sum_{r=0}^{l-s}C_{l-s}^r a^r D_1^{l-s-r}\right)\right)\right)(e)\\
&=\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k,l}^n a^{2k-l}D_2^{n-2k}\circ\left(-h_{21}D_1^{l+1}+h_{22}\sum_{s\le l}C_l^s a^s D_1^{l-a}\circ D_2\right)\right)(e)\\
&=\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k,l}^n a^{2k-l}D_2^{n-2k}\circ\left(-h_{21}D_1^{l+1}+h_{22}\sum_{s\le l}C_l^s a^s D_2\right.\right.\\
&\qquad\left.\left.\circ\left(\sum_{r=0}^{l-s}C_{l-s}^r a^r D_1^{l-s-r}\right)\right)\right)(e)\\
&=\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}A_{k,l}^n a^{2k-l}D_2^{n-2k}\circ\left(-h_{21}D_1^{l+1}+h_{22}\sum_{s\le l}2^s C_l^s a^s D_2\circ D_1^{l-s}\right)\right)(e)\\
&=\left(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}h_{22}A_{k,l}^n a^{2k-1}\left((n-2k)aD_2^{n-2k-1}D_1^{l+1}\right.\right.\\
&\qquad\left.\left.+\sum_{s\le l}2^s C_l^s a^s D_2^{n-2k+1}\circ D_1^{l-s}\right)\right)(e)\\
&=\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}(n-2k)A_{k,l}^n a^{2k-l+1}D_2^{n-2k-1}D_1^{l+1}\\
&\qquad+\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}\left(\sum_{s+r=l}A_{k,l}^n 2^s C_l^s\right)a^{2k-r}D_2^{n+1-2k}\circ D_1^r\\
&=\sum_{k=1}^{E(n/2)+1}\sum_{l=1}^{k}(n-2(k-1))A_{k-1,l-1}^n a^{2k-l}D_2^{n+1-2k}D_1^l\end{aligned}$$

$$+\sum_{k=0}^{E(n/2)}\sum_{l=0}^{k}\left(\sum_{s+l\leq k}A_{k,s+l}^{n}2^{s}C_{s+l}^{s}\right)a^{2k-l}D_{2}^{n+1-2k}\circ D_{1}^{l}\,.$$

If n is even,

$$A_{k,0}^{n+1}=\sum_{s=0}^{k}A_{k,s}^{n}2^{s}\qquad\text{for all } k=0,\cdots,n/2\,,$$

$$A_{k,l}^{n+1}=(n-2(k-1))A_{k-1,l-1}^{n}+\sum_{s=0}^{k-l}A_{k,s}^{n}2^{s}C_{l+s}^{s}$$

$$\text{for all } k,l\ (1\leq k\leq n/2 \text{ and } 1\leq l\leq k)\,.$$

If n is odd,

$$A_{k,0}^{n+1}=\sum_{s=0}^{k}A_{k,s}^{n}2^{s}\quad\text{for all } k=0,\cdots,(n-1)/2\,,\qquad A_{(n+1)/2,0}^{n+1}=0\,,$$

$$A_{(n+1)/2,l}^{n+1}=A_{(n-1)/2,l-1}^{n+1}\qquad\text{for all } l=0,\cdots,(n+1)/2\,,$$

$$A_{k,l}^{n+1}=(n-2(k-1))A_{k-1,l-1}^{n}+\sum_{s=0}^{k-l}A_{k,s}^{n}2^{s}C_{l+s}^{s}$$

$$\text{for all } k,l\ (1\leq k\leq (n-1)/2 \text{ and } 1\leq l\leq k)\,.$$

4°) One has:

$$\begin{aligned}(H_1^p\circ H_2)(e)&=((H_1^p)(e)\circ H_2)(e)=(-1)^p(D_1^p\circ(h_{22}D_2-h_{21}D_1))(e)\\
&=(-1)^p\left(\sum_{k\leq p}C_p^kD_1^k(h_{22})D_1^{p-k}\circ D_2-h_{21}D_1^{p+1}\right)(e)\\
&=(-1)^p\left(\sum_{k\leq p}C_p^kh_{22}a^kD_1^{p-k}\circ D_2-h_{21}D_1^{p+1}\right)(e)\\
&=(-1)^p\sum_{k\leq p}C_p^kh_{22}a^kD_1^{p-k}\circ D_2\\
&=(-1)^p\sum_{k\leq p}C_p^kh_{22}a^kD_2\circ\left(\sum_{l\leq p-k}C_{p-k}^la^lD_1^{p-k-l}\right)\\
&=(-1)^p\sum_{q\leq p}2^qC_p^qa^qD_2\circ D_1^{p-q}=(-1)^p\sum_{l\leq p}2^{p-l}C_p^la^{p-l}D_2\circ D_1^l\,,\end{aligned}$$

$$\begin{aligned}(H_1^p\circ H_2^2)(e)&=((H_1^p\circ H_2)(e)\circ H_2)(e)\\
&=(-1)^p\left(\sum_{q\leq p}2^{p-q}C_p^qa^{p-q}D_2\circ D_1^q)\circ(h_{22}D_2-h_{21}D_1\right)(e)\\
&=(-1)^p\left(\sum_{q\leq p}2^{p-q}C_p^qa^{p-q}D_2\circ\left(\sum_{l\leq q}C_q^lh_{22}a^{q-l}D_1^l\circ D_2\right)\right.\end{aligned}$$

$$+\sum_{q\le p} 2^{p-q}C_p^q a^{p-q+1}h_{22}D_1^{q+1}\Bigg)(e)$$

$$=(-1)^p\Bigg(\sum_{q\le p} 2^{p-q}C_p^q a^{p-q}D_2^2\circ\Bigg(\sum_{l\le q} 2^{q-l}C_q^l h_{22}a^{q-1}D_1^l\Bigg)$$

$$+\sum_{q\le p} 2^{p-q}C_p^q a^{p-q+1}h_{22}D_1^{q+1}\Bigg)(e)$$

$$=(-1)^p\Bigg(\sum_{q\le p} 2^{p-q}C_p^q a^{p-q}D_2^2\circ\Bigg(\sum_{l\le q} 2^{q-l}C_q^l a^{q-l}D_1^l\Bigg)+\sum_{l\le p} 2^{p-l}C_p^l a^{p-l+1}D_1^{l+1}\Bigg)$$

$$=(-1)^p\Bigg(\sum_{l\le p}\Bigg(\sum_{l\le q} C_p^q C_q^l\Bigg)2^{p-1}a^{p-l}D_2^2\circ D_1^l+\sum_{l\le p} 2^{p-l}C_p^l a^{p-l+1}D_1^{l+1}\Bigg)$$

$$=(-1)^p\Bigg(\sum_{l\le p} 4^{p-l}C_p^l a^{p-l}D_2^2\circ D_1^l+\sum_{q\le p} 2^{p-l}C_p^l a^{p-l+1}D_1^{l+1}\Bigg).$$

Suppose that $(H_1^p\circ H_2^n)(e)=\sum_{k=0}^{E(n/2)}\sum_{l=0}^{p+k}A_{k,l}^{p,n}a^{p+2k-l}D_2^{n-2k}D_1^l$, then

$$(H_1^p\circ H_2^{n+1})(e)=((H_1^p\circ H_2^n)(e)\circ H_2)(e)$$

$$=\Bigg(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{p+k}A_{k,l}^{p,n}a^{p+2k-l}D_2^{n-2k}D_1^l\Bigg)\circ(h_{22}D_2-h_{21}D_1)(e)$$

$$=\Bigg(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{p+k}A_{k,l}^{p,n}a^{p+2k-l}D_2^{n-2k}\circ(D_1^l(h_{22}D_2)-h_{21}D_1^{l+1})\Bigg)(e)$$

$$=\Bigg(\sum_{k=0}^{E(n/2)}\sum_{l=0}^{p+k}A_{k,l}^{p,n}a^{p+2k-l}D_2^{n-2k}$$

$$\circ\Bigg(h_{22}D_2\circ\Bigg(\sum_{s=1}^{l}2^{l-s}a^{l-s}C_l^s\Bigg)-h_{21}D_1^{l+1}\Bigg)\Bigg)(e)$$

$$=\Bigg(\sum_{k=0}^{E(n/2)}\sum_{s=0}^{p+k}h_{22}\Bigg(\sum_{l=s}^{p+k}2^{l-s}C_l^s A_{k,l}^{p,n}\Bigg)a^{p+2k-s}D_2^{n-2k+1}\circ D_1^s$$

$$+\sum_{k=0}^{E(n/2)}\sum_{l=0}^{p+k}h_{22}(n-2k)A_{k,l}^{p,n}a^{p+2k-l+1}D_2^{n-2k-1}\circ D_1^{l+1}\Bigg)(e)$$

$$=\sum_{k=0}^{E(n/2)}\sum_{s=0}^{p+k}\Bigg(\sum_{l=s}^{p+k}2^{l-s}C_l^s A_{k,l}^{p,n}\Bigg)a^{p+2k-s}D_2^{n-2k+1}\circ D_1^s$$

$$+\sum_{k=0}^{E(n/2)}\sum_{l=0}^{p+k}(n-2k)A_{k,l}^{p,n}a^{p+2k-l+1}D_2^{n-2k-1}\circ D_1^{l+1}.$$

If n is even,

$$A_{k,0}^{p,n+1} = \sum_{s=0}^{p+k} A_{k,s}^{p,n} 2^s \qquad \text{for all } k=0, \cdots, n/2,$$

$$A_{0,l}^{p,n+1} = \sum_{s=l}^{p} A_{0,s}^{p,n} 2^{s-l} C_l^s \qquad \text{for all } l=0, \cdots, p,$$

$$A_{k,l}^{p,n+1} = (n-2(k-1)) A_{k-1,l-1}^{p,n} + \sum_{s=0}^{k-l} A_{k,s}^{p,n} 2^s C_{l+s}^s$$

$$\text{for all } k, l \ (1 \le k \le n/2 \text{ and } 1 \le l \le k).$$

If n is odd,

$$A_{k,0}^{p,n+1} = \sum_{s=0}^{p+k} A_{k,s}^{p,n} 2^s \qquad \text{for all } k=0, \cdot\cdot, (n-1)/2,$$

$$A_{0,l}^{p,n+1} = \sum_{s=l}^{p} A_{0,s}^{p,n} 2^{s-l} C_s^l \qquad \text{for all } l=0, \cdots, p,$$

$$A_{k,l}^{p,n+1} = (n-2(k-1)) A_{k-1,l-1}^{p,n} + \sum_{s=l}^{p+k} A_{k,s}^{p,n} 2^{s-l} C_l^s$$

$$\text{for all } k, l \ (1 \le k \le (n-1)/2 \text{ and } 1 \le l \le k),$$

$$A_{(n+1)/2,0}^{p,n+1} = 0, \quad A_{(n+1)/2,l}^{p,n+1} = A_{(n-1)/2,l-1}^{p,n} \qquad \text{for all } l=0, \cdots, p+(n+1)/2,$$

$$A_{k,l}^{p,n+1} = (n-2(k-1)) A_{k-1,l-1}^{p,n} + \sum_{s=l}^{p+k} A_{k,s}^{p,n} 2^{s-l} C_l^s$$

$$\text{for all } k, l \ (1 \le k \le (n-1)/2 \text{ and } 1 \le l \le k).$$

COROLLARY 226. *Every Poisson quadratic bracket P with $P = Rp \wedge \omega$, $\omega \in \mathrm{Der}(E)$, admits a star-product.*

PROOF. Thanks to the formula (1) of [2]:

$$[u, v] = (-1)^q R(u \wedge v) - R(u) \wedge v - (-1)^q u \wedge R(v),$$

$$u \in A^q(E),\ v \in A(E) \text{ and } R^2 = 0.$$

We have $Rp = Rp \wedge R_\omega - [Rp, \omega]$ i.e. $[Rp, \omega] = Rp(R_\omega - 1)$. The result comes from Proposition 225.

PROPOSITION 227. 1°) *Every Poisson quadratic bracket isomorphic to the model #4 can be written: $P = D_1 \wedge D_2$ with $D_1, D_2 \in \mathrm{Der}(G)$ and $[D_1, D_2] = 0$.*

2°) *If $K = \mathbf{C}$, every Poisson quadratic bracket isomorphic to the model #8 can be written: $P = D_1 \wedge D_2$ with $D_1, D_2 \in \mathrm{Der}(G)$ and $[D_1, D_2] = 0$ where G is the algebra over K of fractions in x, y, z.*

PROOF. One can easily see that the ∂_i can be extended to derivations on G and that every isomorphism of F can be extended into an automorphism of G.

1. For a bracket P of the type #4, one has $P=D_1\wedge D_2$ with $D_1=-(x^2/z)\partial_2-cz\partial_3+ax\partial_1$ or $D_1=ax\partial_1-(cz+x^2/y)\partial_3$ and $D_2=y\partial_2-z\partial_3$. It's a matter of verification that $[D_1, D_2]=0$.

2. For a bracket P of the type #8, one has $P=D_1\wedge D_2$ with $D_1=x\partial_2-y\partial_1$ and

$$\begin{aligned} D_2= & -\{(a-b)/2(-yi+x)+\varepsilon z^2/(-iy+x)\}\partial_1 \\ & -\{(a-b)/2(y+xi)-i\varepsilon z^2/(-iy+x)\}\partial_2+az\partial_3\,, \qquad \text{or} \\ D_2= & -\{((a-b)(x^2+y^2)+\varepsilon z^2)/x+yi\}\partial_1 \\ & -i\{((a-b)(x^2+y^2)+\varepsilon z^2)/x+yi\}\partial_2+az\partial_3\,. \end{aligned}$$

We also have $[D_1, D_2]=0$.

Let now P be a Poisson quadratic bracket isomorphic to the model #4 or #8. From the above results, there exists $\psi\in GL(F)$ such that $\psi^{-1}(P)=D_1\wedge D_2$ with $D_1, D_2\in \mathrm{Der}(G)$ and $[D_1, D_2]=0$. Hence $P=\psi(D_1)\wedge\psi(D_2)$, $\psi(D_1)$, $\psi(D_2)\in\mathrm{Der}(G)$ and $[\psi(D_1, \psi(D_2)]=0$.

REMARKS. From Proposition 227, one has: each Poisson quadratic bracket P isomorphic to the model #4 or #8 admits $\exp tP$ as a star-product.

This star-product has its A_k in $\mathrm{Diff}^1(G)$ and is defined on $C^3\backslash S$ (S is a 2-plane in C^3). Nevertheless, for the structures of type #4 (resp. #8) verifying $a(c-a)(2a+c)=0$ (resp. $ab(3a+b)=0$), one can show the existence of a star-product such that $A_k\in\mathrm{Diff}^1(E)$. Indeed, a simple calculation shows that for the structures of type #4 (resp. type #8) verifying $a(c-a)=0$ (resp. $ab=0$), the structure constants satisfy $(*)$.

For the structures of type #4 verifying $2a+c=0$, one has $P=D_1\wedge D_2+D_3\wedge D_4$ with $D_1=ax\partial_1-cz\partial_3$; $D_2=y\partial_2-z\partial_3$; $D_3=x\partial_2$; $D_4=x\partial_3$ and

$$\begin{aligned}[\bar P_0, \bar P_1]+[\bar P_0, \bar P_2]+[\bar P_1, \bar P_2]= & (2a+c)(D_2\circ D_3\circ D_4-D_2\circ D_4\circ D_3-D_3\circ D_2\circ D_4 \\ & +D_3\circ D_4\circ D_2+D_4\circ D_2\circ D_3-D_4\circ D_3\circ D_2)=0\,.\end{aligned}$$

Hence $\bar P$ is a Poisson bracket and then admits a star-product and the series $(**)$ is the one.

For the structures of type #8 verifying $3a+b=0$, one has $P=D_1\wedge D_2+D_3\wedge D_4$ with $D_1=x\partial_2-y\partial_1$; $D_2=-(a+b)/2x\partial_1-(a+b)/2y\partial_2+az\partial_3$; $D_3=\varepsilon z\partial_1$; $D_4=z\partial_2$ and

$$\begin{aligned}[\bar P_0, \bar P_1]+[\bar P_0, \bar P_2]+[\bar P_1, \bar P_2]= & (3a+b)(D_1\circ D_3\circ D_4-D_1\circ D_4\circ D_3+D_4\circ D_3\circ D_1 \\ & -D_4\circ D_1\circ D_3+D_3\circ D_1\circ D_4-D_3\circ D_4\circ D_1)=0\,.\end{aligned}$$

Hence $\bar P$ is a Poisson bracket and then admits a star-product and the series $(**)$ is the one.

PROPOSITION 228. *Every quadratic Poisson structure of the type* #14 *admits a star-product.*

PROOF. Let $P=\sum_{i,j,u,v} a_{ij}^{uv} x_u x_v \partial_i \wedge \partial_j$ be a quadratic Poisson structure of the type #14 and $Q \in S^3(E)$ such that $P=\partial_3(Q)\partial_1 \wedge \partial_2+\partial_1(Q)\partial_2 \wedge \partial_3+\partial_2(Q)\partial_3 \wedge \partial_1$. One can write:

$$\begin{aligned} Q= & Ax_1^3+Bx_2^3+Cx_3^3+Dx_1^2x_2+Ex_1^2x_3+Fx_2^2x_1+Gx_2^2x_3 \\ & +Hx_3^2x_1+Kx_3^2x_2+Lx_1x_2x_3 , \\ P= & (3Cx_3^2+Ex_1^2+Gx_2^2+2Hx_3x_1+2Kx_3x_2+Lx_1x_2)\partial_1 \wedge \partial_2 \\ & +(3Ax_1^2+2Dx_1x_2+2Ex_1x_3+Fx_2^2+Hx_3^2+Lx_2x_3)\partial_2 \wedge \partial_3 \\ & +(3Bx_2^2+Dx_1^2+2Fx_2x_1+2Gx_2x_3+Kx_3^2+Lx_1x_3)\partial_3 \wedge \partial_1 . \end{aligned}$$

For all $u, v, w=1, 2, 3$,

$$\begin{aligned} \sum_{p=1}^{3} a_{1p}^{uv}a_{23}^{pw}+a_{2p}^{uv}a_{31}^{pw}+a_{3p}^{uv}a_{12}^{pw} &= a_{12}^{uv}a_{23}^{2w}+a_{13}^{uv}a_{23}^{3w}+a_{21}^{uv}a_{31}^{1w}+a_{23}^{uv}a_{31}^{3w}+a_{31}^{uv}a_{12}^{1w}+a_{32}^{uv}a_{12}^{2w} \\ &= a_{12}^{uv}(a_{23}^{2w}-a_{31}^{1w})+a_{23}^{uv}(a_{31}^{3w}-a_{12}^{2w})+a_{31}^{uv}(a_{12}^{1w}-a_{23}^{3w}) \end{aligned}$$

and as $(a_{23}^{2w}-a_{31}^{1w})=(a_{31}^{3w}-a_{12}^{2w})=(a_{12}^{1w}-a_{23}^{3w})=0$ for all $w=1, 2, 3$, then

$$\sum_{p=1}^{3} a_{1p}^{uv}a_{23}^{pw}+a_{2p}^{uv}a_{31}^{pw}+a_{3p}^{uv}a_{12}^{pw}=0 .$$

By using Proposition 221, P admits a star-product.

In order to give examples of star-product for quadratic Poisson structures of the type #14 associated to the polynomials $Q=Q_1Q_2$ (Q_i ($i=1,2$) is a nonconstant polynomial of degree i), we shall prove a necessary and sufficient condition showing when such structures are isomorphic:

PROPOSITION 229. *Two quadratic Poisson structures P_1 and P_2 of type* 14: *$P_i=\partial_3(Q_i)\partial_1 \wedge \partial_2+\partial_1(Q_i)\partial_2 \wedge \partial_3+\partial_2(Q_i)\partial_3 \wedge \partial_1$, $i=1, 2$, are isomorphic if and only if there exists an automorphism ψ of E such that*: $Q_2=(\det \psi^{-1})\psi(Q_1)$. ($\det \psi^{-1}=(\det \psi^{-1})|F$).

PROOF. Let ψ be an automorphism of E and (b_{ij}) the matrix of ψ^{-1} restricted to F in the basis (x, y, z), then $\psi \circ \partial_i \circ \psi^{-1}=\sum_{j=1}^{3} b_{ij}\partial_j$ ($i=1, 2, 3$) and

$$\begin{aligned} \psi(P_1) &= \psi(\partial_3 Q_1)(\psi \circ \partial_1 \circ \psi^{-1}) \wedge (\psi \circ \partial_2 \circ \psi^{-1})+\psi(\partial_1 Q_1)(\psi \circ \partial_2 \circ \psi^{-1}) \wedge (\psi \circ \partial_3 \circ \psi^{-1}) \\ &\quad +\psi(\partial_2 Q_1)(\psi \circ \partial_3 \circ \psi^{-1}) \wedge (\psi \circ \partial_1 \circ \psi^{-1}) \\ &= \psi(\partial_3 Q_1)\left(\sum_{j=1}^{3} b_{1j}\partial_j\right) \wedge \left(\sum_{j=1}^{3} b_{2j}\partial_j\right)+\psi(\partial_1 Q_1)\left(\sum_{j=1}^{3} b_{2j}\partial_j\right) \wedge \left(\sum_{j=1}^{3} b_{3j}\partial_j\right) \\ &\quad +\psi(\partial_2 Q_1)\left(\sum_{j=1}^{3} b_{3j}\partial_j\right) \wedge \left(\sum_{j=1}^{3} b_{1j}\partial_j\right) \\ &= \psi(\partial_3 Q_1)((b_{11}b_{22}-b_{12}b_{21})\partial_1 \wedge \partial_2+(b_{12}b_{23}-b_{13}b_{22})\partial_2 \wedge \partial_3 \\ &\qquad +(b_{13}b_{21}-b_{11}b_{23})\partial_3 \wedge \partial_1) \end{aligned}$$

$$
\begin{aligned}
&+\psi(\partial_1 Q_1)((b_{21}b_{32}-b_{22}b_{31})\partial_1\wedge\partial_2+(b_{22}b_{33}-b_{23}b_{32})\partial_2\wedge\partial_3\\
&\qquad\qquad+(b_{23}b_{31}-b_{21}b_{33})\partial_3\wedge\partial_1)\\
&+\psi(\partial_2 Q_1)((b_{31}b_{12}-b_{32}b_{11})\partial_1\wedge\partial_2+(b_{32}b_{13}-b_{33}b_{12})\partial_2\wedge\partial_3\\
&\qquad\qquad+(b_{33}b_{11}-b_{31}b_{13})\partial_3\wedge\partial_1)\\
=&(\psi(\partial_3 Q_1)(b_{11}b_{22}-b_{12}b_{21})+\psi(\partial_1 Q_1)(b_{21}b_{32}-b_{22}b_{31})\\
&\qquad+\psi(\partial_2 Q_1)(b_{31}b_{12}-b_{32}b_{11}))\partial_1\wedge\partial_2\\
&+(\psi(\partial_3 Q_1)(b_{12}b_{23}-b_{13}b_{22})+\psi(\partial_1 Q_1)(b_{22}b_{33}-b_{23}b_{32})\\
&\qquad+\psi(\partial_2 Q_1)(b_{32}b_{13}-b_{33}b_{12}))\partial_2\wedge\partial_3\\
&+(\psi(\partial_3 Q_1)(b_{13}b_{21}-b_{11}b_{23})+\psi(\partial_1 Q_1)(b_{23}b_{31}-b_{21}b_{33})\\
&\qquad+\psi(\partial_2 Q_1)(b_{33}b_{11}-b_{31}b_{13}))\partial_3\wedge\partial_1\\
=&\left(\sum_{j=1}^{3} b_{3j}\partial_j\psi(Q_1)(b_{11}b_{22}-b_{12}b_{21})+\sum_{j=1}^{3} b_{1j}\partial_j\psi(Q_1)(b_{21}b_{32}-b_{22}b_{31})\right.\\
&\qquad\left.+\sum_{j=1}^{3} b_{2j}\partial_j\psi(Q_1)(b_{31}b_{12}-b_{32}b_{11})\right)\partial_1\wedge\partial_2\\
&+\left(\sum_{j=1}^{3} b_{3j}\partial_j\psi(Q_1)(b_{12}b_{23}-b_{13}b_{22})+\sum_{j=1}^{3} b_{1j}\partial_j\psi(Q_1)(b_{22}b_{33}-b_{23}b_{32})\right.\\
&\qquad\left.+\sum_{j=1}^{3} b_{2j}\partial_j\psi(Q_1)(b_{32}b_{13}-b_{33}b_{12})\right)\partial_2\wedge\partial_3\\
&+\left(\sum_{j=1}^{3} b_{3j}\partial_j\psi(Q_1)(b_{13}b_{21}-b_{11}b_{23})+\sum_{j=1}^{3} b_{1j}\partial_j\psi(Q_1)(b_{23}b_{31}-b_{21}b_{33})\right.\\
&\qquad\left.+\sum_{j=1}^{3} b_{2j}\partial_j\psi(Q_1)(b_{33}b_{11}-b_{31}b_{13})\right)\partial_3\wedge\partial_1\\
=&\sum_{j=1}^{3}(b_{3j}(b_{11}b_{22}-b_{12}b_{21})+b_{1j}(b_{21}b_{32}-b_{22}b_{31})\\
&\qquad+b_{2j}(b_{31}b_{12}-b_{32}b_{11}))\partial_j\psi(Q_1)\partial_1\wedge\partial_2\\
&+\sum_{j=1}^{3}(b_{3j}(b_{12}b_{23}-b_{13}b_{22})+b_{1j}(b_{22}b_{33}-b_{23}b_{32})\\
&\qquad+b_{2j}(b_{32}b_{13}-b_{33}b_{12}))\partial_j\psi(Q_1)\partial_2\wedge\partial_3\\
&+\sum_{j=1}^{3}(b_{3j}(b_{13}b_{21}-b_{11}b_{23})+b_{1j}(b_{23}b_{31}-b_{21}b_{33})\\
&\qquad+b_{2j}(b_{33}b_{11}-b_{31}b_{13}))\partial_j\psi(Q_1)\partial_3\wedge\partial_1\,,
\end{aligned}
$$

$$b_{31}(b_{11}b_{22}-b_{12}b_{21})+b_{11}(b_{21}b_{32}-b_{22}b_{31})+b_{21}(b_{31}b_{12}-b_{32}b_{11})=0\,,$$
$$b_{32}(b_{11}b_{22}-b_{12}b_{21})+b_{12}(b_{21}b_{32}-b_{22}b_{31})+b_{22}(b_{31}b_{12}-b_{32}b_{11})=0\,,$$
$$b_{33}(b_{11}b_{22}-b_{12}b_{21})+b_{13}(b_{21}b_{32}-b_{22}b_{31})+b_{23}(b_{31}b_{12}-b_{32}b_{11})=\det\psi^{-1}\,,$$
$$b_{31}(b_{12}b_{23}-b_{13}b_{22})+b_{11}(b_{22}b_{33}-b_{23}b_{32})+b_{21}(b_{32}b_{13}-b_{33}b_{12})=\det\psi^{-1}\,,$$
$$b_{32}(b_{12}b_{23}-b_{13}b_{22})+b_{12}(b_{22}b_{33}-b_{23}b_{32})+b_{22}(b_{32}b_{13}-b_{33}b_{12})=0\,,$$
$$b_{33}(b_{12}b_{23}-b_{13}b_{22})+b_{13}(b_{22}b_{33}-b_{23}b_{32})+b_{23}(b_{32}b_{13}-b_{33}b_{12})=0\,,$$
$$b_{31}(b_{13}b_{21}-b_{11}b_{23})+b_{11}(b_{23}b_{31}-b_{21}b_{33})+b_{21}(b_{33}b_{11}-b_{31}b_{13})=0\,,$$
$$b_{32}(b_{13}b_{21}-b_{11}b_{23})+b_{12}(b_{23}b_{31}-b_{21}b_{33})+b_{22}(b_{33}b_{11}-b_{31}b_{13})=\det\psi^{-1}\,,$$
$$b_{33}(b_{13}b_{21}-b_{11}b_{23})+b_{13}(b_{23}b_{31}-b_{21}b_{33})+b_{23}(b_{33}b_{11}-b_{31}b_{13})=0.$$

Therefore

$$\psi(P_1)=P_2 \Leftrightarrow \begin{cases} \partial_3 Q_2=\det\psi^{-1}\partial_3\psi(Q_1) \\ \partial_2 Q_2=\det\psi^{-1}\partial_2\psi(Q_1) \\ \partial_1 Q_2=\det\psi^{-1}\partial_1\psi(Q_1) \end{cases} \Leftrightarrow Q_2=\det\psi^{-1}\psi(Q_1)\,.$$

REMARK 2210. 1°) The previous proposition gives a classification (by isomorphisms) of the quadratic Poisson structures which have a vanishing rotational.

2°) One can show that the homogenous Poisson structures of degree $p\geq 2$ having a vanishing rotational can be written as

$$P=\partial_3(Q)\partial_1\wedge\partial_2+\partial_1(Q)\partial_2\wedge\partial_3+\partial_2(Q)\partial_3\wedge\partial_1 \qquad \text{where } Q\in S^{p+1}(F)\,.$$

And the same proof as before is still available for two such structures

$$P_i=\partial_3(Q_i)\partial_1\wedge\partial_2+\partial_1(Q_i)\partial_2\wedge\partial_3+\partial_2(Q_i)\partial_3\wedge\partial_1 \qquad i=1,2\,.$$

3°) If Q is a homogeneous polynomial of degree p which depends on two variables, then the associated Poisson structure admits a star-product of the form $\exp\lambda P$. Indeed, according to 2°), one can suppose that Q does not depend on z; P can be written as

$$P=\partial_1(Q)\partial_2\wedge\partial_3+\partial_2(Q)\partial_3\wedge\partial_1$$

and if $D_1=\partial_1(Q)\partial_2-\partial_2(Q)\partial_1$; $D_2=\partial_3$, then $P=D_1\wedge D_2$ and $[D_1, D_2]=0$. Hence the result is given by Proposition 125.

PROPOSITION 2211. *If the field $K=\mathbf{C}$, every quadratic Poisson structure of type #14 such that $Q=Q_1Q_2$ (where Q_i is a homogeneous polynomial of degree i) has a star-product and an example is given by the Proposition* 225.

PROOF. First of all, we remark that if a quadratic Poisson structure verifies the Yang-Baxter equation, so does its image by isomorphisms or by multiplication by a constant. Because of the propositions before, one can suppose that

$$Q = xQ' \quad \text{where} \quad Q' = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz\,,$$
$$A, B, C, D, E \in K\,, \quad F \in \mathbf{R}\,.$$

If $F^2 - 4BC \neq 0$, one can reduce Q to the form $Q = x(Ax^2 + Dxy + Exz + Fyz)$ with $F \neq 0$ thanks to the automorphism ψ defined by

$$\psi(x) = x\,; \qquad \psi(y) = \frac{-F + \sqrt{F^2 - 4BC}}{2B}\,ay - \frac{F + \sqrt{F^2 - 4BC}}{2B}\,bz \quad (a, b \in \mathbf{R}_+^*)$$
$$\text{and} \quad \psi(z) = ay + bz \qquad \text{if } B \neq 0\,,$$

or

$$\psi(x) = x\,; \qquad \psi(z) = \frac{-F + \sqrt{F^2 - 4BC}}{2C}\,ay - \frac{F + \sqrt{F^2 - 4BC}}{2C}\,bz$$
$$\text{and} \quad \psi(y) = ay + bz \qquad \text{if } C \neq 0\,,$$

and $Q = x(Ax^2 + Fyz)$ thanks to the automorphism φ defined by

$$\varphi(x) = x\,; \quad \varphi(y) = y - \frac{E}{F}x\,; \quad \varphi(z) = z - \frac{D}{F}x\,.$$

If $F^2 - 4BC = 0$, Q can be reduced to $Q = x(Ax^2 + By^2 + Dxy + Exz)$ because $By^2 + Cz^2 + Fyz = (\sqrt{B}y + \sqrt{C}z)^2$, and to $Q = x(By^2 + Exz)$ if $E \neq 0$.

Finally, one has three cases to study:
- i°) $Q = x(Ax^2 + By^2 + Cxy)$,
- ii°) $Q = x(Ay^2 + Bxz)$,
- iii°) $Q = x(Ax^2 + Byz)$.

For the case i°) P admits a star-product because Q does not depend on z. For the case ii°) $P = Bxy\partial_1 \wedge \partial_2 + (3Ax^2 + Byz)\partial_2 \wedge \partial_3 + Bxz\partial_3 \wedge \partial_1$, so we are brought to structures of type 1) if $A = 0$ or type 4) if $A \neq 0$ (see remarks following Proposition 227). For the case iii°), $P = Bx^2\partial_1 \wedge \partial_2 + (Ay^2 + 2Bxz)\partial_2 \wedge \partial_3 + 2Axy\partial_3 \wedge \partial_1$. Putting $D_1 = x\partial_1 - (y/2)\partial_2 - 2z\partial_3$; $D_2 = -2Ay\partial_3 + Bx\partial_2$, we have $P = D_1 \wedge D_2$ and $[D_1, D_2] = 3/2D_2$, then thanks to Proposition 225, P admits a star-product.

The following result generalises the previous proposition:

PROPOSITION 2212. *Let $a, b \in K$, P_1, P_2 and P_3 three homogeneous polynomials of SF of degree 1 and $Q = P_1^p P_2^q P_3^r (aP_1^s + bP_2^t P_3^k)$ with $s = t + k$ and $qk - rt \neq 0$. If $abs \neq 0$, then the structure $P = \partial_3(Q)\partial_1 \wedge \partial_2 + \partial_1(Q)\partial_2 \wedge \partial_3 + \partial_2(Q)\partial_3 \wedge \partial_1$ admits a star-product.*

PROOF. Let E_1 be the vector subspace of F spanned by P_1, P_2 and P_3.

1st case. $\dim E_1 < 3$. In this case, there is an automorphism ψ of SF such that

$(\det\psi^{-1})\psi(Q)$ is a polynomial in the variables x and y, then P admits a star-product thanks to Remark 2210.

2^{nd} case. $\dim E_1 = 3$. In this case, we can find an automorphism ψ of SF such that

$$Q_1 = (\det\psi^{-1})\psi(Q) = cx^p y^q z^r(ax^s + by^t z^k) \qquad \text{where } c\in K^* .$$

Because of Proposition 224, it's sufficient to show that $P_1 = \partial_3(Q_1)\partial_1 \wedge \partial_2 + \partial_1(Q_1)\partial_2 \wedge \partial_3 + \partial_2(Q_1)\partial_3 \wedge \partial_1$admits a star-product.

i) $ab=0$ or $s=0$. Q_1 is of the form $dx^\alpha y^\beta z^\gamma$, $d\in K$. We can suppose that $\alpha\beta\gamma\neq 0$ (otherwise Q_1 will depend on two variables so P will admit a star-product thanks to Remark 2210). Then

$$P_1 = d\gamma x^\alpha y^\beta z^{\gamma-1}\partial_1 \wedge \partial_2 + d\alpha x^{\alpha-1} y^\beta z^\gamma \partial_2 \wedge \partial_3 + d\beta x^\alpha y^{\beta-1} z^\gamma \partial_3 \wedge \partial_1 .$$

Put $D_1 = x\partial_1 + \alpha/\beta y\partial_2$ and $D_2 = d\gamma x^{\alpha-1} y^\beta z^{\gamma-1}\partial_2 - d\beta x^{\alpha-1} y^{\beta-1} z^\gamma \partial_3$. One verifies easily that $P_1 = D_1 \wedge D_2$ and $[D_1, D_2] = (\alpha - 1 + (\alpha/\beta)(\beta-1))D_2$. Therefore a star-product of P_1 is given by Proposition 225.

ii) $abs\neq 0$. One has

$$\partial_1 Q_1 = cx^{p-1}y^q z^r(a(p+s)x^s + pby^t z^k) ,$$
$$\partial_2 Q_1 = cx^p y^{q-1} z^r(aqx^s + b(q+t)y^t z^k) ,$$
$$\partial_3 Q_1 = cx^p y^q z^{r-1}(arx^s + b(r+k)y^t z^k) ,$$

and

$$\begin{aligned} P_1 = {} & cx^p y^q z^{r-1}(arx^s + b(r+k)y^t z^k)\partial_1 \wedge \partial_2 + cx^{p-1} y^q z^r(a(p+s)x^s \\ & + pby^t z^k)\partial_2 \wedge \partial_3 + cx^p y^{q-1} z^r(aqx^s + b(q+t)y^t z^k)\partial_3 \wedge \partial_1 . \end{aligned}$$

Put

$$D_1 = x\partial_1 + uy\partial_2 + vz\partial_3,$$
$$D_2 = cx^{p-1}y^q z^{r-1}(arx^s + b(r+k)y^t z^k)\partial_2 - cx^{p-1}y^{q-1}z^r(aqx^s + b(q+t)y^t z^k)\partial_3$$

with $u, v\in K$. Let's seek u, v such that $P_1 = D_1 \wedge D_2$.

$$\begin{aligned} P_1 = D_1 \wedge D_2 \Leftrightarrow {} & -vx^{p-1}y^q z^r(arx^s + b(r+k)y^t z^k) \\ & -ux^{p-1}y^q z^r(aqx^s + b(q+t)y^t z^k) \\ & = x^{p-1}y^q z^r(a(p+s)x^s + pby^t z^k) \\ \Leftrightarrow {} & qau + rav = -a(p+s), \ (q+t)bu + (r+k)bv = -pb \\ \Leftrightarrow {} & u = (rp - (p+s)(r+k))/(qk - rt) , \\ & v = (-pq + (p+s)(q+t))/(qk - rt) . \end{aligned}$$

On the other hand, one has

$$
\begin{aligned}
[D_1, D_2] = c\{&(p-1)x^{p-1}y^q z^{r-1}(arx^s + b(r+k)y^t z^k) \\
&+ rsax^{p-1}y^q z^{r-1}x^s + qux^{p-1}y^q z^{r-1}(arx^s + b(r+k)y^t z^k) \\
&+ ub(r+k)x^{p-1}y^q z^{r-1}y^t z^k - ux^{p-1}y^q z^{r-1}(arx^s + b(r+k)y^t z^k) \\
&+ v(r-1)x^{p-1}y^q z^{r-1}(arx^s + b(r+k)y^t z^k) + bkv(r+k)x^{p-1}y^q z^{r-1}y^t z^k\}\partial_2 \\
- c\{&(p-1)x^{p-1}y^{q-1}z^r(aqx^s + b(q+t)y^t z^k) + qasx^{p-1}y^{q-1}z^r x^s \\
&+ u(q-1)x^{p-1}y^{q-1}z^r(aqx^s + b(q+t)y^t z^k) + ubt(q+t)x^{p-1}y^{q-1}z^r y^t z^k \\
&+ vrx^{p-1}y^{q-1}z^r(aqx^s + b(q+t)y^t z^k) + vbk(q+t)x^{p-1}y^{q-1}z^r y^t z^k \\
&- vx^{p-1}y^{q-1}z^r(aqx^s + b(q+t)y^t z^k)\}\partial_3 \\
= c\{&((p-1) + (q-1)u + (r-1)v)x^{p-1}y^q z^{r-1}(arx^s + b(r+k)y^t z^k) \\
&+ sx^{p-1}y^q z^{r-1}(arx^s + (b(r+k)/s)(ut+kv)y^t z^k)\}\partial_2 \\
- c\{&((p-1) + (q-1)u + (r-1)v)x^{p-1}y^{q-1}z^r(aqx^s + b(q+t)y^t z^k) \\
&+ sx^{p-1}y^{q-1}z^r(aqx^s + b(q+t)(1/s)(ut+kv)y^t z^k)\}\partial_3 .
\end{aligned}
$$

If

$$u = (rp - (p+s)(r+k))/(qk - rt), \qquad v = (-pq + (p+s)(q+t))/(qk - rt),$$

then

$$(ut + vk) = s, \qquad [D_1, D_2] = (p - 1 + (q-1)u + (r-1)v + s)D_2 .$$

Hence P_1 admits a star-product, and an example is given by Proposition 225.

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