TOKYO J. MATH.
VOL. 23, NO. 1, 2000

Completely Operator Semi-Selfdecomposable Distributions

Makoto MAEJIMA (*Keio University*), Ken-iti SATO
and Toshiro WATANABE (*The University of Aizu*)

Abstract. The class $L_\infty(b, Q)$ of completely operator semi-selfdecomposable distributions on $\mathbf{R}^d$ for b and Q is studied. Here $0 < b < 1$ and Q is a $d \times d$ matrix whose eigenvalues have positive real parts. This is the limiting class of the decreasing sequence of classes $L_m(b, Q)$, $m = -1, 0, 1, \cdots$, where $L_{-1}(b, Q)$ is the class of all infinitely divisible distributions on $\mathbf{R}^d$ and $L_m(b, Q)$ is defined inductively as the class of distributions μ with characteristic function $\hat{\mu}(z)$ satisfying $\hat{\mu}(z) = \hat{\mu}(b^{Q'}z)\hat{\rho}(z)$ for some $\rho \in L_{m-1}(b, Q)$. Q' is the transpose of Q. Distributions in $L_\infty(b, Q)$ are characterized in terms of Gaussian covariance matrices and Lévy measures. The connection with the class $OSS(b, Q)$ of operator semi-stable distributions on $\mathbf{R}^d$ for b and Q is established.

1. Introduction and a main result.

In our previous paper [MSW99], we have introduced the class of operator semi-selfdecomposable distributions and its decreasing subclasses. To explain those classes, we start with the necessary notation. $\mathcal{P}(\mathbf{R}^d)$ is the class of all probability distributions on $\mathbf{R}^d$, $I(\mathbf{R}^d)$ is the class of all infinitely divisible distributions on $\mathbf{R}^d$, $M_+(\mathbf{R}^d)$ is the class of all $d \times d$ matrices all of whose eigenvalues have positive real parts, Q' is the transpose of $Q \in M_+(\mathbf{R}^d)$, I is the identity matrix, $\hat{\mu}(z)$, $z \in \mathbf{R}^d$, is the characteristic function of $\mu \in \mathcal{P}(\mathbf{R}^d)$, μ^{*t}, $t \geq 0$, is the t-th convolution power of $\mu \in I(\mathbf{R}^d)$, $\mathcal{L}(X)$ is the law of X, $\langle\, ,\, \rangle$ is the Euclidean inner product in $\mathbf{R}^d$, and $|\cdot|$ is the norm induced by $\langle\, ,\, \rangle$ in $\mathbf{R}^d$. For $b > 0$, $b^Q = \sum_{n=0}^{\infty}(n!)^{-1}(\log b)^n Q^n$. Convergence of probability distributions is always weak convergence.

Let $0 < b < 1$, $Q \in M_+(\mathbf{R}^d)$, m a nonnegative integer, and $L_{-1}(b, Q) = I(\mathbf{R}^d)$. A distribution $\mu \in I(\mathbf{R}^d)$ is said to belong to the class $L_m(b, Q)$ if there exists $\rho \in L_{m-1}(b, Q)$ such that $\hat{\mu}(z) = \hat{\mu}(b^{Q'}z)\hat{\rho}(z)$. Actually the classes $L_m(b, Q)$ have been defined in [MSW99] in a different way and it has been shown there that the definition above is a necessary and sufficient condition for the definition in [MSW99]. Define $L_\infty(b, Q)$ by $L_\infty(b, Q) = \bigcap_{m\geq 0} L_m(b, Q)$. We have called distributions in $L_0(b, Q)$ operator semi-selfdecomposable in [MSW99].

On the other hand, Jurek [J83] and Sato and Yamazato [SY85] introduced and studied the classes $L_m(Q)$ for m a nonnegative integer or ∞. It has been proved in [MSW99] that $L_m(Q) = \bigcap_{0<b<1} L_m(b, Q)$, $0 \leq m \leq \infty$. Distributions in $L_\infty(Q)$ are called completely operator selfdecomposable and characterized in several ways in [SY85]. For this reason, we want to call distributions in $L_\infty(b, Q)$ completely operator semi-selfdecomposable. In

Received September 10, 1998

[SY85], they studied the relationship between the class $L_\infty(Q)$ and that of operator stable distributions. The purpose of this paper is to give characterization of distributions in $L_\infty(b, Q)$ and to investigate the relationship between the class $L_\infty(b, Q)$ and that of operator semi-stable distributions.

Let $Q \in M_+(\mathbf{R}^d)$. A class $H \subset I(\mathbf{R}^d)$ is said to be Q-completely closed in the strong sense if H is closed under convergence, convolution, and Q-type equivalence, and is closed under going to the t-th convolution power for any $t > 0$. Here H is said to be closed under Q-type equivalence if $\mathcal{L}(X) \in H$, $a > 0$, and $c \in \mathbf{R}^d$ imply $\mathcal{L}(a^{-Q}X + c) \in H$. We can easily see from the definition that $L_m(b, Q)$, $0 \le m \le \infty$, are Q-completely closed in the strong sense, because $L_{-1}(b, Q) = I(\mathbf{R}^d)$ is so. Furthermore let $OSS(b, Q)$ be the class of $\mu \in I(\mathbf{R}^d)$ such that $\hat{\mu}(z)^a = \hat{\mu}(b^{Q'}z)e^{i\langle c,z\rangle}$ for some $0 < a < 1$ and $c \in \mathbf{R}^d$. Distributions in $OSS(b, Q)$ are called operator semi-stable. They are studied by Jajte [J77], Krakowiak [K80], Laha and Rohatgi [LR80], Łuczak [Ł81, Ł91], and others.

One of our main theorems is the following.

THEOREM 1.1. *Let $0 < b < 1$ and $Q \in M_+(\mathbf{R}^d)$. Then the class $L_\infty(b, Q)$ is the smallest Q-completely closed class in the strong sense containing the class $OSS(b, Q)$.*

This theorem is a "semi"-version of Theorem 7.3 in [SY85]. In Section 2, we state some results we need in the subsequent sections. In Section 3, we characterize Gaussian distributions in $L_\infty(b, Q)$, and in Section 4, we treat purely non-Gaussian distributions in $L_\infty(b, Q)$. The proof of Theorem 1.1 is given in Section 5. As our results are new even in case $Q = I$, we make some remarks on this case in Section 6.

2. Preliminary results.

The following three propositions have recently been shown in [MSW99]. Since we need them in the subsequent sections, we state them below without proofs.

For a $d \times d$ matrix B we use the following notation: $BE = \{Bx : x \in E\}$ for $E \subset \mathbf{R}^d$ and $(T_B\nu)(E) = \nu(\{x : Bx \in E\})$ for a measure ν on $\mathbf{R}^d$. We use a mapping Ψ_B from the class of symmetric $d \times d$ matrices into itself defined by $\Psi_B(A) = A - BAB'$. Its iteration is $\Psi_B^l = \Psi_B \circ \Psi_B^{l-1}$ for $l = 2, 3, \cdots$ with $\Psi_B^1 = \Psi_B$. Also let $\mathcal{B}_0(\mathbf{R}^d)$ be the class of Borel sets E in $\mathbf{R}^d$ such that $E \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$.

In what follows, we fix $0 < b < 1$ and $Q \in M_+(\mathbf{R}^d)$. We use C_i, $i = 1, 2, \cdots$, for positive constants. Following (3.4.3) in [JM93], we introduce a norm $|\cdot|_Q$ in $\mathbf{R}^d$ depending on Q:

$$|x|_Q = \int_0^1 \frac{|u^Q x|}{u} du , \quad x \in \mathbf{R}^d .$$

Since $C_1 u^{C_2}|x| \le |u^Q x| \le C_3 u^{C_4}|x|$, $0 < u \le 1$, $|x|_Q$ is well defined. The norm $|\cdot|_Q$ is comparable with the Euclidean norm $|\cdot|$, and has an advantage that, for any $x \in \mathbf{R}^d \setminus \{0\}$, $t \to |t^Q x|_Q$ $(t > 0)$ is strictly increasing (Proposition 3.4.3 in [JM93]). Thus $\sup_{|x|_Q \le 1} |b^Q x|_Q <$

1. Define

$$B = b^Q,$$

$$S_B = \{x \in \mathbf{R}^d : |x|_Q \le 1 \text{ and } |B^{-1}x|_Q > 1\},$$

and $\mathcal{B}(S_B)$ as the class of Borel sets in S_B. It might be better to write $S_{Q,b}$ instead of S_B, because it depends on Q and b.

We note that all our results in this paper remain true if S_B is defined by the usual norm in place of the norm $|\cdot|_Q$, provided that $|B| = \sup_{|x|\le 1}|Bx|$, the operator norm of B, is less than 1. We also note that, since $|B^n x| \to 0$ for any $x \in \mathbf{R}^d$ as $n \to \infty$ and since the space is finite-dimensional, there is a positive integer n such that $|B^n| < 1$. Since $B^n = b^{nQ}$ and $L_m(b, Q) \subset L_m(b, nQ)$, $0 \le m \le \infty$, study of distributions in $L_m(b, Q)$ in the case $|B| < 1$ covers all cases in some sense. However, in characterization of the class $L_m(b, Q)$ itself, we cannot assume that $|B| < 1$. This is the reason that we use the norm $|\cdot|_Q$.

PROPOSITION 2.1 (Proposition 3.2 of [MSW99]). (i) *If ν is the Lévy measure of $\mu \in I(\mathbf{R}^d)$, then there exist a finite measure ν_0 on S_B and a Borel measurable function $g_n : S_B \to \mathbf{R}_+$ for each $n \in \mathbf{Z}$ satisfying the following conditions*:

(a) *For $E \in \mathcal{B}(S_B)$, $\nu_0(E) = 0$ if and only if $\nu(B^n E) = 0$, $\forall n \in \mathbf{Z}$,*

(b) $\int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) g_n(x) < \infty$,

(c) $\sum_{n\in\mathbf{Z}} g_n(x) > 0$, *$\nu_0$-a.e.*,

(d) $\nu(E) = \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} g_n(x) 1_E(B^{-n}x)$, $\forall E \in \mathcal{B}(\mathbf{R}^d)$.

These $\{\nu_0, g_n, n \in \mathbf{Z}\}$ are uniquely determined in the following sense. If $\{\nu_0, g_n, n \in \mathbf{Z}\}$ and $\{\tilde{\nu}_0, \tilde{g}_n, n \in \mathbf{Z}\}$ satisfy the above conditions, then there exists a Borel measurable function $h(x)$ with $0 < h(x) < \infty$ such that

$$\tilde{\nu}_0(dx) = h(x)\nu_0(dx),$$

$$g_n(x) = h(x)\tilde{g}_n(x), \quad \nu_0\text{-a.e.}, \ \forall n \in \mathbf{Z}.$$

(ii) *Conversely, if ν_0, a finite measure on S_B, and g_n, $n \in \mathbf{Z}$, Borel measurable functions from S_B into $[0, \infty)$, are given, and satisfy* (b) *and* (c), *then ν defined by* (d) *is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$ and* (a) *is also satisfied.*

We call $\{\nu_0, g_n, n \in \mathbf{Z}\}$ determined uniquely from ν in (i) above the S_B-representation of ν. We may write $g(n, x)$ for $g_n(x)$ below. For $\{k(n), n \in \mathbf{Z}\}$, define $\Delta k(n) = k(n+1) - k(n)$ and $(\Delta^l k)(n) = \Delta(\Delta^{l-1}k)(n)$, $l = 2, 3, \cdots$. The sequence $\{k(n), n \in \mathbf{Z}\}$ is called completely monotone if

$$(-1)^l (\Delta^l k)(n) \ge 0 \quad \text{for } \forall l \ge 0, \ \forall n \in \mathbf{Z},$$

with $(\Delta^0 k)(n) = k(n)$.

PROPOSITION 2.2 (Lemma 4.3 and Corollary 4.1 of [MSW99]). (i) *If $\{k(n), n \in \mathbf{Z}\}$ is completely monotone, then*

(a) *there exists a unique measure ρ on $(0, 1]$ such that*

$$k(n) = \int_{(0,1]} x^n \rho(dx), \quad n \in \mathbf{Z}, \tag{2.1}$$

(b) *for each* $b \in (0, 1)$, *there exists a unique measure* Γ *on* $[0, \infty)$ *such that*

$$k(n) = \int_{[0,\infty)} b^{n\alpha} \Gamma(d\alpha) .$$

(ii) *Conversely,* $\{k(n), n \in \mathbf{Z}\}$ *having the representation* (2.1) *is completely monotone.*

PROPOSITION 2.3 (Theorem 3.1 of [MSW99]). *Let* $0 \le m \le \infty$, $\mu \in I(\mathbf{R}^d)$, A *its Gaussian covariance matrix,* ν *its Lévy measure, and let* $\{\nu_0, g(n, x), n \in \mathbf{Z}\}$ *be the* S_B*-representation of* ν. *Then the following three statements are equivalent*:

(i) $\mu \in L_m(b, Q)$.

(ii) $\Psi_B^l(A)$, $1 \le l \le m+1$, *are nonnegative definite, and* $(I - T_B)^l \nu \ge 0$, $1 \le l \le m+1$, *on* $\mathcal{B}_0(\mathbf{R}^d)$.

(iii) $\Psi_B^l(A)$, $1 \le l \le m+1$, *are nonnegative definite, and* $(-1)^l (\Delta^l g)(n, x) \ge 0$, $n \in \mathbf{Z}$, ν_0*-a.e.x for* $1 \le l \le m+1$.

(In the above, when $m = \infty$, $1 \le l \le m+1$ *should be read as* $1 \le l < \infty$.*)*

3. Gaussian distributions in $L_\infty(b, Q)$.

The following are generalizations of some results in [SY85] to "semi"-version.

Let $\{\beta_1, \cdots, \beta_p\}$ be the distinct eigenvalues of $B = b^Q$, and let $f(\zeta)$ be the minimal polynomial of B. Decompose it into linear factors

$$f(\zeta) = (\zeta - \beta_1)^{n_1} \cdots (\zeta - \beta_p)^{n_p} ,$$

where, for $1 \le j \le p$, n_j is a positive integer not exceeding the multiplicity of β_j. Let

$$V_j = \mathrm{Ker}(B - \beta_j I)^{n_j} \quad \text{in } \mathbf{C}^d \ (1 \le j \le p) .$$

Then

$$\mathbf{C}^d = V_1 \oplus \cdots \oplus V_p .$$

Let T_j be the projector of $\mathbf{C}^d$ onto V_j in this direct sum decomposition. Similarly we denote

$$V_j' = \mathrm{Ker}(B' - \bar{\beta}_j I)^{n_j} \quad \text{in } \mathbf{C}^d \ (1 \le j \le p) ,$$

and obtain

$$\mathbf{C}^d = V_1' \oplus \cdots \oplus V_p' .$$

Then the projector of $\mathbf{C}^d$ onto V_j' in this direct sum decomposition coincides with the adjoint operator T_j' of T_j. For $j \ne k$, V_j' and V_k are orthogonal, where we use the Hermitian inner product denoted also by $\langle\, ,\, \rangle$. The following is a characterization of Gaussian distribution in $L_\infty(b, Q)$.

THEOREM 3.1. *Let* μ *be a Gaussian distribution with covariance matrix A. Then the following three statements are equivalent*:

(i) $\mu \in L_\infty(b, Q)$.

(ii) $(B - \beta_j) A T_j' = 0$, *for* $1 \le j \le p$.

(iii) (a) $A(B' - \bar{\beta}_j) T_j' = 0$, *for* $1 \le j \le p$, *and* (b) $T_k A T_j' = 0$ *for* $j \ne k$.

To prove the theorem, we need a lemma.

LEMMA 3.1. *Let μ be a Gaussian distribution with covariance matrix A. Then $\mu \in L_\infty(b, Q)$ if and only if for any $z \in \mathbf{R}^d$, $\langle AB'^n z, B'^n z\rangle$, $n \in \mathbf{Z}$, is completely monotone.*

PROOF. Set $k_z(n) = \langle AB'^n z, B'^n z\rangle$, $n \in \mathbf{Z}$, $z \in \mathbf{R}^d$. Then observe that, for each $l \geq 1$, $(-1)^l(\Delta^l k_z)(n) \geq 0$, $\forall n \in \mathbf{Z}$, $\forall z \in \mathbf{R}^d$, if and only if $\Psi_B^l(A)$ is nonnegative definite. The nonnegative definiteness of $\Psi_B^l(A)$ for all $l \geq 0$ is a necessary and sufficient condition for that the Gaussian μ is in $L_\infty(b, Q)$ by Proposition 2.3. This concludes the lemma. □

PROOF OF THEOREM 3.1. We first show (i)⇒(iii). To show (iii)(a), it is enough to prove that, for any integer $k \geq 1$ and $z_0 \in \mathbf{C}^d$,

$$(B' - \bar\beta_j)^k z_0 = 0 \quad \text{implies} \quad A(B' - \bar\beta_j)z_0 = 0. \tag{3.1}$$

We prove this by induction in k. If $k = 1$, the assertion is trivial. Suppose that (3.1) is true for $k - 1$ in place of k, and assume $(B' - \bar\beta_j)^k z_0 = 0$. Since $B'^n(B' - \bar\beta_j) = (B' - \bar\beta_j)B'^n$ for any $n \in \mathbf{Z}$, we have $(B' - \bar\beta_j)^{k-2+l} B'^n z_0 = 0$ for any $l \geq 2$ and $n \in \mathbf{Z}$. Hence by the induction hypothesis,

$$A(B' - \bar\beta_j)^l B'^n z_0 = 0 \quad \text{for } l \geq 2 \text{ and } n \in \mathbf{Z}. \tag{3.2}$$

Let

$$L(n) = \langle AB'^n z_0, B'^n z_0\rangle \quad \text{for } n \in \mathbf{Z}.$$

We claim that

$$L(n) = |\beta_j|^{2n}\{\langle Az_0, z_0\rangle + 2n\Re\langle Az_0, z_1\rangle + n^2\langle Az_1, z_1\rangle\}, \qquad n \in \mathbf{Z}, \tag{3.3}$$

where $z_1 = \bar\beta_j^{-1}(B' - \bar\beta_j)z_0$. If $n = 0$, this is trivial. We write $z_l = \bar\beta_j^{-l}(B' - \bar\beta_j)^l z_0$. If $n \geq 1$, then

$$B'^n z_0 = (\bar\beta_j + (B' - \bar\beta_j))^n z_0 = \bar\beta_j^n \sum_{l=0}^{n} \binom{n}{l} z_l$$

and, by (3.2),

$$L(n) = \langle \bar\beta_j^n A(z_0 + nz_1), B'^n z_0\rangle = \langle \bar\beta_j^n (z_0 + nz_1), \bar\beta_j^n A(z_0 + nz_1)\rangle,$$

which is (3.3). Suppose $n \leq -1$ and write $n = -h$. Let $w = B'^n z_0 - \bar\beta_j^n(z_0 + nz_1)$. Then

$$\begin{aligned} w &= B'^n\{z_0 - \bar\beta_j^n B'^h(z_0 + nz_1)\} \\ &= B'^n\left\{z_0 - \sum_{l=0}^{h}\binom{h}{l}(z_l + nz_{l+1})\right\} \\ &= B'^n\left\{n^2 z_2 - \sum_{l=2}^{h}\binom{h}{l}(z_l + nz_{l+1})\right\}, \end{aligned}$$

where the sum over $2 \le l \le h$ is considered as zero if $h = 1$. Hence by (3.2), $Aw = 0$. Thus

$$\begin{aligned} L(n) &= \langle AB'^n z_0 - Aw, B'^n z_0 \rangle \\ &= \langle \bar{\beta}_j^n A(z_0 + nz_1), B'^n z_0 \rangle \\ &= \langle \bar{\beta}_j^n (z_0 + nz_1), AB'^n z_0 - Aw \rangle \\ &= \langle \bar{\beta}_j^n (z_0 + nz_1), \bar{\beta}_j^n A(z_0 + nz_1) \rangle . \end{aligned}$$

Hence (3.3) is true for all $n \in \mathbf{Z}$. Since $L(n)$ is completely monotone in n by Lemma 3.1 under the assumption that $\mu \in L_\infty(b, Q)$, we have

$$L(n) = \int_{(0,1]} \beta^n \rho(d\beta) \tag{3.4}$$

for some measure ρ by Proposition 2.2. If we let

$$E_1 = \{\beta \in (0, 1] : \beta > |\beta_j|^2\},$$
$$E_2 = \{\beta \in (0, 1] : \beta < |\beta_j|^2\},$$

then from (3.3) and (3.4)

$$\rho(\{|\beta_j|^2\}) + \int_{E_1} \left(\frac{\beta}{|\beta_j|^2}\right)^n \rho(d\beta) + \int_{E_2} \left(\frac{\beta}{|\beta_j|^2}\right)^n \rho(d\beta)$$
$$= \langle Az_0, z_0 \rangle + 2n\Re\langle Az_0, z_1 \rangle + n^2 \langle Az_1, z_1 \rangle = I ,$$

say. If $\rho(E_1) > 0$, then there exists $\varepsilon > 0$ such that $\rho((1+\varepsilon)|\beta_j|^2 \le \beta \le 1) > 0$, and hence

$$I \ge (1+\varepsilon)^n \rho((1+\varepsilon)|\beta_j|^2 \le \beta \le 1), \quad n > 0 .$$

Letting $n \to \infty$, we get a contradiction. Thus $\rho(E_1) = 0$. Similarly, if $\rho(E_2) > 0$, then there exists $\varepsilon > 0$ such that

$$I \ge (1-\varepsilon)^n \rho(0 < \beta \le (1-\varepsilon)|\beta_j|^2), \quad n < 0,$$

and letting $n \to -\infty$ yields a contradiction. Thus $\rho(E_2) = 0$. Consequently,

$$\int_{(0,1]} \beta^n \rho(d\beta) = |\beta_j|^{2n} \rho(\{|\beta_j|^2\}) = |\beta_j|^{2n} \langle Az_0, z_0 \rangle ,$$

and $\langle Az_1, z_1 \rangle = 0$. By Lemma 3.1 of [SY85], we conclude that $Az_1 = 0$. This proves (3.1).

Let us show (iii)(b). It is enough to show that

$$\langle Az_0, w_0 \rangle = 0 \quad \text{for any } z_0 \in V'_j \text{ and } w_0 \in V'_k \text{ with } j \neq k .$$

Since V'_j and V'_k are invariant under B'^h, $h \in \mathbf{Z}$, we have, by (iii)(a),

$$A(B' - \bar{\beta}_j)^l B'^h z_0 = A(B' - \bar{\beta}_k)^l B'^h w_0 = 0 \quad \text{for } l \ge 1 \text{ and } h \in \mathbf{Z} .$$

Hence, for $n \in \mathbf{Z}$,

$$\begin{aligned} L_\pm(n) \langle AB'^n(z_0 \pm w_0), B'^n(z_0 \pm w_0) \rangle \\ = |\beta_j|^{2n} \langle Az_0, z_0 \rangle \pm 2\Re \bar{\beta}_j^n \beta_k^n \langle Az_0, w_0 \rangle + |\beta_k|^{2n} \langle Aw_0, w_0 \rangle . \end{aligned} \tag{3.5}$$

We consider two cases.

Case I ($|\beta_j| \neq |\beta_k|$). As before, there exist measures ρ_+ and ρ_- on (0, 1] such that

$$L_\pm(n) = \int_{(0,1]} \beta^n \rho_\pm(d\beta)\,. \tag{3.6}$$

Let us show

$$\rho_\pm(\{|\beta_j|^2\}) = \langle Az_0, z_0\rangle\,, \tag{3.7}$$

$$\rho_\pm(\{|\beta_k|^2\}) = \langle Aw_0, w_0\rangle\,. \tag{3.8}$$

Without loss of generality, we assume that $|\beta_j| < |\beta_k|$. As in the case of $L(n)$, we observe

$$\rho_\pm(\beta < |\beta_j|^2) = \rho_\pm(\beta > |\beta_k|^2) = 0\,.$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}\langle Aw_0, w_0\rangle &= \lim_{n\to\infty} \frac{L_\pm(n)}{|\beta_k|^{2n}} \\ &= \lim_{n\to\infty} \int_{\beta<|\beta_k|^2} \left(\frac{\beta}{|\beta_k|^2}\right)^n \rho_\pm(d\beta) + \rho_\pm(\{|\beta_k|^2\}) \\ &= \rho_\pm(\{|\beta_k|^2\})\,.\end{aligned}$$

This shows (3.8). (3.7) can be shown similarly, by considering $L_\pm(n)/|\beta_j|^{2n}$ and letting $n \to -\infty$. Thus we have

$$\pm 2\Re \bar{\beta}_j^n \beta_k^n \langle Az_0, w_0\rangle = \int_{(|\beta_j|^2, |\beta_k|^2)} \beta^n \rho_\pm(d\beta) \geq 0\,,$$

concluding $\Re\langle Az_0, w_0\rangle = 0$. Since $iz_0 \in V_j$, we also have $\Im\langle Az_0, w_0\rangle = 0$.

Case II ($|\beta_j| = |\beta_k|$). We have

$$L_+(n) = |\beta_j|^{2n} \left\{ \langle Az_0, z_0\rangle + \langle Aw_0, w_0\rangle + 2\Re \left(\frac{\beta_k}{\beta_j}\right)^n \langle Az_0, w_0\rangle \right\}\,.$$

As in the case of $L(n)$, we see that

$$L_+(n) = \int_{(0,1]} \beta^n \rho_+(d\beta) = \text{const.} \times |\beta_j|^{2n}\,.$$

Therefore, $\Re(\beta_k/\beta_j)^n\langle Az_0, w_0\rangle$ is independent of n. Since $\beta_k/\beta_j \neq 1$, we have $\langle Az_0, w_0\rangle = 0$. This conclude (iii)(b).

We next show (iii)⇒(i). By the Jordan decomposition of B', we can find a basis $\{z_{jlk} = \bar{\beta}_j^{-k}(B' - \bar{\beta}_j)^k z_{jl},\ 1 \leq j \leq p,\ 1 \leq l \leq l_j,\ 0 \leq k \leq k(j,l)\}$ of $\mathbf{C}^d$ for some integers l_j and $k(j,l)$. Here $z_{jl} \in V_j'$ $(1 \leq l \leq l_j)$ and $(B' - \bar{\beta}_j)^{k(j,l)+1} z_{jl} = 0$. Thus, for any $z \in \mathbf{C}^d$,

$$z = \sum_{j,l,k} c_{jlk} z_{jlk} \quad \text{with some } c_{jlk} \in \mathbf{C}\,.$$

and, hence,

$$B'^n z = \sum_{j,l,k} \bar{\beta}_j^n c_{jlk} \sum_{m=0}^{(k(j,l)-k)\wedge n} \binom{n}{m} z_{j,l,k+m}$$

for all $n \in \mathbf{Z}_+$. Therefore, by (iii) for any $n \in \mathbf{Z}$,

$$A\left(B'^n z - \sum_{j,l} \bar{\beta}_j^n c_{jl0} z_{jl0}\right) = 0 .$$

Hence, for $n \in \mathbf{Z}$,

$$\begin{aligned}\langle AB'^n z, B'^n z\rangle &= \sum_{j=1}^{p} |\beta_j|^{2n} \left\langle A \sum_l c_{jl0} z_{jl0}, \sum_l c_{jl0} z_{jl0} \right\rangle \\ &= \int_{(0,1]} \beta^n \rho(d\beta) ,\end{aligned}$$

which is completely monotone, if we define ρ by $\rho(\{|\beta_j|^2\}) = \langle A \sum_l c_{jl0} z_{jl0}, \sum_l c_{jl0} z_{jl0}\rangle$ and $\rho((0, 1] \setminus \{|\beta_1|^2, \cdots, |\beta_p|^2\}) = 0$. It follows from Lemma 3.1 that $\mu \in L_\infty(b, Q)$.

We finally show (ii)⇒(iii). (iii)⇒(ii) is easily seen, because we have, for any $z, w \in \mathbf{C}^d$,

$$\langle (B - \beta_j) A T_j' z, w\rangle = \sum_{k=1}^{p} \langle T_j' z, A(B' - \bar{\beta}_j) T_k' w\rangle = 0 ,$$

using (iii)(a) for $k = j$ and (iii)(b) for $k \neq j$. As to (ii)⇒(iii), we have

$$\langle A(B' - \bar{\beta}_j) T_j' z, (B' - \bar{\beta}_j) T_j' z\rangle = \langle (B - \beta_j) A (B' - \bar{\beta}_j) T_j' z, T_j' z\rangle = 0 ,$$

which together with Lemma 3.1 of [SY85] implies that $A(B' - \bar{\beta}_j) T_j' z = 0$, namely (iii)(a). Also we have (iii)(b), since (ii) implies that AT_j' has its range in V_j. The proof of Theorem 3.1 is thus complete. □

Theorem 3.1 uses a direct sum decomposition of $\mathbf{C}^d$. Let us consider the corresponding decomposition of $\mathbf{R}^d$, and then prove a decomposition theorem of Gaussian distributions in $L_\infty(b, Q)$. For this purpose we arrange the distinct eigenvalues of $B = b^Q$ in such a way that $\beta_1, \cdots, \beta_q$ are real and $\beta_{q+1}, \cdots, \beta_p$ are not real, $\beta_j = \bar{\beta}_{j+r}$ $(q + 1 \leq j \leq q + r)$, and $q + 2r = p$. Here q or r may possibly be zero. Let γ_j and δ_j be the real and the imaginary part of β_j, respectively. The real factorization of the minimal polynomial $f(\zeta)$ of B' is

$$f(\zeta) = f_1(\zeta)^{n_1} \cdots f_{q+r}(\zeta)^{n_{q+r}} ,$$

where $f_j(\zeta) = \zeta - \beta_j = \zeta - \gamma_j$, $1 \leq j \leq q$, and $f_j(\zeta) = (\zeta - \gamma_j)^2 + \delta_j^2$, $q+1 \leq j \leq q+r$. Let

$$W_j' = \operatorname{Ker} f_j(B')^{n_j} \quad \text{in } \mathbf{R}^d , \quad 1 \leq j \leq q + r . \tag{3.9}$$

Then

$$\mathbf{R}^d = W_1' \oplus \cdots \oplus W_{q+r}' . \tag{3.10}$$

As in the proof (iii)⇒(i) of Theorem 3.1, let

$$\{z_{jlk} = \bar{\beta}_j^{-k}(B' - \bar{\beta}_j)^k z_{jl} : 1 \le j \le p, 1 \le l \le l_j, 0 \le k \le k(j,l)\}$$

be a basis of $\mathbf{C}^d$, where $z_{jl} \in V_j'$ and $(B' - \bar{\beta}_j)^{k(j,l)+1} z_{jl} = 0$. For $1 \le j \le q$, we can choose z_{jl} real so that $\{z_{jlk} : 1 \le l \le l_j, 0 \le k \le k(j,l)\}$ is a basis of W_j'. For $q+1 \le j \le q+r$, we have $l_j = l_{j+r}$ and $k(j,l) = k(j+r,l)$ and we can choose z_{jl} and $z_{j+r,l}$ in such a way that $z_{jl} = \bar{z}_{j+r,l}$. Let ξ_{jlk} and η_{jlk} be the real and the imaginary part of z_{jlk}, respectively, for $q+1 \le j \le q+r$. Here complex conjugates, real parts, and imaginary parts of vectors in $\mathbf{C}^d$ are taken component-wise. The system $\{\xi_{jlk}, \eta_{jlk} : 1 \le l \le l_j, 0 \le k \le k(j,l)\}$ is then a basis of W_j'. The following theorem gives a matrix representation when these bases are used.

THEOREM 3.2. *Let μ be Gaussian with covariance matrix A. Then $\mu \in L_\infty(b, Q)$ if and only if the following four conditions are satisfied*:

(i) $\langle Az_{jlk}, z_{jlk}\rangle = 0$ *for* $1 \le j \le q$, $1 \le l \le l_j$, $k \ge 1$,

(ii) $\langle A\xi_{jlk}, \xi_{jlk}\rangle = \langle A\eta_{jlk}, \eta_{jlk}\rangle = 0$ *for* $q+1 \le j \le q+r$, $1 \le l \le l_j$, $k \ge 1$,

(iii) $\langle A\xi_{jl0}, \xi_{jm0}\rangle = \langle A\eta_{jl0}, \eta_{jm0}\rangle$ *and* $\langle A\xi_{jl0}, \eta_{jm0}\rangle = -\langle A\eta_{jl0}, \xi_{jm0}\rangle$ *for* $q+1 \le j \le q+r$, $1 \le l \le l_j$, $1 \le m \le l_j$ *with* $l = m$ *inclusive*,

(iv) $\langle Az, w\rangle = 0$ *for* $z \in W_j'$, $w \in W_k'$, *for* $1 \le j \le q+r$, $1 \le k \le q+r$ *with* $j \ne k$.

PROOF. The proof is exactly the same as that of Theorem 4.1 of [SY85]. So we omit it here. □

Let us consider the direct sum decomposition of $\mathbf{R}^d$ associated with $B = b^Q$. Let

$$W_j = \operatorname{Ker} f_j(B)^{n_j} \quad \text{in } \mathbf{R}^d, \quad 1 \le j \le q+r.$$

Then

$$\mathbf{R}^d = W_1 \oplus \cdots \oplus W_{q+r}. \tag{3.11}$$

This is the decomposition dual to (3.10). Let U_j be the projector of $\mathbf{R}^d$ onto W_j in the decomposition (3.11). The transposed matrix U_j' of U_j is the projector onto W_j' in the decomposition (3.10). For $q+1 \le j \le q+r$, we have $V_j = \bar{V}_{j+r}$ and $T_j x = \overline{T_{j+r}x}$ for $x \in \mathbf{R}^d$. Thus $U_j x = T_j x + T_{j+r} x$ for $x \in \mathbf{R}^d$ for $q+1 \le j \le q+r$. For $1 \le j \le q$, we have $U_j x = T_j x$ for $x \in \mathbf{R}^d$. Let

$$N_j = \operatorname{Ker} f_j(B) \quad \text{in } \mathbf{R}^d, \quad 1 \le j \le q+r.$$

THEOREM 3.3. *Suppose that μ is a centered Gaussian distribution in $L_\infty(b, Q)$. Then, the support of μ is a B-invariant linear subspace of $\mathbf{R}^d$ and the minimal polynomial of the restriction of B to the support of μ does not have double roots. There exists a unique decomposition $\mu = \mu_1 * \cdots * \mu_{q+r}$, where each μ_j is a centered Gaussian distribution such that $\mu_j \in L_\infty(b, Q)$ and the support of μ_j is contained N_j and hence in W_j.*

PROOF. Again, the proof is exactly the same as that of Theorem 4.2 of [SY85]. So we omit it here. □

REMARK 3.1. The μ_j in Theorem 3.3 is centered Gaussian with covariance matrix $A_j = U_j A U_j'$. Thus, since $\mu_j \in L_\infty(b, Q)$, by Theorem 3.1,

$$\begin{aligned}\hat{\mu}_j(B'z) &= \exp\left\{-\frac{1}{2}\langle A_j B'z, B'z\rangle\right\}\\ &= \exp\left\{-\frac{1}{2}\langle A_j B'U_j'z, B'U_j'z\rangle\right\}\\ &= \exp\left\{-\frac{1}{2}|\beta_j|^2\langle A_j U_j'z, U_j'z\rangle\right\}\\ &= \exp\left\{-\frac{1}{2}|\beta_j|^2\langle A_j z, z\rangle\right\}\\ &= \hat{\mu}_j(z)^{|\beta_j|^2},\end{aligned}$$

which means that $\mu_j \in OSS(b, Q)$.

Combining Theorem 3.3 and Remark 3.1, we have

THEOREM 3.4. *Let s be the number of distinct absolute values of eigenvalues of $B = b^Q$. If μ is a Gaussian distribution in $L_\infty(b, Q)$, then μ can be expressed as the convolution of at most s Gaussian distributions in $OSS(b, Q)$.*

EXAMPLE. For $d = 2, 3, 4$, explicit forms of the covariance matrices of Gaussian distributions in $L_\infty(Q)$ are given in [SY85]. Let $d = 2$ and let I_G be the class of all Gaussian distributions on $\mathbf{R}^2$. Let $\mu \in I_G$ with covariance matrix A.

First consider the case $Q = \begin{pmatrix}\gamma & 1\\ 0 & \gamma\end{pmatrix}$ with $\gamma > 0$. For $0 < b < 1$, $b^Q = b^\gamma\begin{pmatrix}1 & \log b\\ 0 & 1\end{pmatrix}$ and hence b^Q has the Jordan form $\begin{pmatrix}b^\gamma & 1\\ 0 & b^\gamma\end{pmatrix}$. As is shown in [SY85], $\mu \in L_\infty(Q)$ if and only if $A = \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}$, $a \geq 0$. As our Theorem 3.2 is formally the same as Theorem 4.1 of [SY85], we see that $\mu \in L_\infty(b, Q)$ if and only if $A = \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}$, $a \geq 0$. Thus $L_\infty(b, Q) \cap I_G = L_\infty(Q) \cap I_G$ for any $0 < b < 1$ in this case.

Next consider the case $Q = \begin{pmatrix}\gamma & -\delta\\ \delta & \gamma\end{pmatrix}$ with $\gamma > 0$ and $\delta \in \mathbf{R} \setminus \{0\}$. Then, for $0 < b < 1$, $b^Q = b^\gamma\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ with $\theta = \delta \log b$. In this case, $\mu \in L_\infty(Q)$ if and only if $A = aI$, $a \geq 0$, as is shown [SY85]. If $b = e^{n\pi/\delta}$ with $n \in \mathbf{Z}$ and $n/\delta < 0$, then $b^Q = b^\gamma(-1)^n I$ and hence $L_\infty(b, Q) \cap I_G = I_G$. Otherwise, b^Q is of the same type as Q and $L_\infty(b, Q) \cap I_G = L_\infty(Q) \cap I_G$. Thus, $L_\infty(b, Q) \cap I_G$ truly depends on b and, for some b, $L_\infty(b, Q) \cap I_G$ is strictly larger than $L_\infty(Q) \cap I_G$.

4. Purely non-Gaussian distributions in $L_\infty(b, Q)$.

Now we give a representation of the Lévy measure of purely non-Gaussian $\mu \in L_\infty(b, Q)$. For each $x \in \mathbf{R}^d \setminus \{0\}$, let

$$\beta(x) = \max\{|\beta_j| : 1 \le j \le q + 2r,\ T_j x \ne 0\},$$
$$n(x, j) = \max\{n : n \ge 0,\ (B - \beta_j)^n T_j x \ne 0 \text{ for } T_j x \ne 0\},$$
$$n(x) = \max\{n(x, j) : 1 \le j \le q + 2r,\ T_j x \ne 0,\ |\beta_j| = \beta(x)\},$$
$$\gamma(x) = \frac{\log \beta(x)}{\log b}.$$

We show the following. Given two measurable spaces $(\Theta_1, \mathcal{B}_1)$ and $(\Theta_2, \mathcal{B}_2)$, we say that $\{\Gamma_{\theta_1}, \theta_1 \in \Theta_1\}$, a system of measures on $(\Theta_2, \mathcal{B}_2)$, is measurable in θ_1 if $\Gamma_{\theta_1}(E)$ is measurable in θ_1 for every $E \in \mathcal{B}_2$.

THEOREM 4.1. (i) *Suppose that μ is a purely non-Gaussian distribution in $I(\mathbf{R}^d)$ with nonzero Lévy measure ν. Then $\mu \in L_\infty(b, Q)$ if and only if ν is expressed as*

$$\nu(E) = \int_{S_B} \nu_0(dx) \int_{(0,2\gamma(x))} \Gamma_x(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha} 1_E(B^{-n}x), \quad E \in \mathcal{B}(\mathbf{R}^d), \tag{4.1}$$

where ν_0 is a nonzero finite measure on S_B and Γ_x, $x \in S_B$, are nonzero finite measures on $(0, \infty)$ measurable in x, each Γ_x is concentrated on $(0, 2\gamma(x))$ and

$$\int_{S_B} \nu_0(dx) \int_{(0,2\gamma(x))} \Gamma_x(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha} (|B^{-n}x|_Q^2 \wedge 1) < \infty. \tag{4.2}$$

(ii) *If a nonzero finite measure ν_0 on S_B and nonzero finite measures Γ_x, $x \in S_B$, on $(0, \infty)$ measurable in x are given and if each Γ_x is concentrated on $(0, 2\gamma(x))$ and (4.2) is satisfied, then the measure ν defined by (4.1) is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$.*

(iii) *If $\mu \in L_\infty(b, Q)$ has nonzero Lévy measure ν and if ν is expressed by ν_0* and *Γ_x as in (i), then ν_0 and Γ_x are unique in the following sense: if $\tilde{\nu}$ and $\tilde{\Gamma}_x$ give another expression of ν, then there exists a Borel measurable function $h(x)$ with $0 < h(x) < \infty$ such that $\tilde{\nu}_0(dx) = h(x)\nu_0(dx)$ and $\Gamma(d\alpha) = h(x)\tilde{\Gamma}_x(d\alpha)$ for ν_0-a.e. x. The measures Γ_x necessarily satisfy*

$$\int_{(0,2\gamma(x))} (\alpha^{-1} + (2\gamma(x) - \alpha)^{-2n(x)-1}) \Gamma_x(d\alpha) < \infty, \quad \nu_0\text{-a.e. } x. \tag{4.3}$$

LEMMA 4.1. *There exist positive constants C_j ($j = 5, 6, 7$) and $b_j(x)$ ($j = 1, 2, 3$) such that, for $x \in S_B$,*

$$|B^k x|_Q \le C_5 \beta(x)^k k^{n(x)} \quad \text{for } k \ge 1, \tag{4.4}$$

$$|B^k x|_Q \ge b_2(x) \beta(x)^k k^{n(x)} \quad \text{for } k \ge b_1(x), \tag{4.5}$$

$$\begin{aligned} C_6\alpha^{-1} + b_3(x)(2\gamma(x) - \alpha)^{-2n(x)-1} &\le \sum_{n\in\mathbf{Z}} (|B^n x|_Q^2 \wedge 1) b^{-n\alpha} \\ &\le C_7(\alpha^{-1} + (2\gamma(x) - \alpha)^{-2n(x)-1}) \quad \text{for } 0 < \alpha < 2\gamma(x). \end{aligned} \tag{4.6}$$

If $\alpha \geq 2\gamma(x)$, *then*

$$\sum_{n\in\mathbf{Z}}(|B^n x|_Q^2 \wedge 1)b^{-n\alpha} = \infty . \tag{4.7}$$

PROOF. Let $|x|_Q$ be defined by $|x|_Q = \int_0^1 (|n^Q x|/u)du$ also to $x \in \mathbf{C}^d$. We have

$$\begin{aligned} B^k T_j x &= (\beta_j + (B - \beta_j))^k T_j x \\ &= \beta_j^k \sum_{l=0}^{n(x,j)\wedge k} \binom{k}{l} \beta_j^{-l}(B - \beta_j)^l T_j x . \end{aligned} \tag{4.8}$$

Thus

$$|B^k x|_Q \leq C_8 \beta(x)^k \sum_{j=1}^{p} k^{n(x,j)} \quad k \geq 1 .$$

Hence we have (4.4). It follows from (4.8) that there are $b_4(x)$ and $b_5(x)$ such that, for $k \geq b_4(x)$,

$$\begin{aligned} |B^k T_j x|_Q &\geq 2^{-1}|\beta_j|^k k^{n(x,j)}(n(x,j)!)^{-1}|(B - \beta_j)^{n(x,j)} T_j x|_Q \\ &\geq b_5(x)|\beta_j|^k k^{n(x,j)} \end{aligned}$$

for all j satisfying $T_j x \neq 0$. Choose a norm $\|\cdot\|$ in $\mathbf{C}^d$ as $\|x\| = \sum_{j=1}^p |T_j x|_Q$. Since arbitrary two norms are equivalent, we have $C_9|x|_Q \leq \|x\| \leq C_{10}|x|_Q$. Choosing j such that $\beta(x) = \beta_j$ and $n(x) = n(x, j)$, we obtain

$$|B^k x|_Q \geq C_{10}^{-1}\|B^k x\| \geq C_{10}^{-1}|B^k T_j x|_Q .$$

Hence (4.5) follows. Let $0 < \alpha < 2\gamma(x)$. Note that $\sum_{k=-\infty}^{0} b^{-k\alpha} = (1 - b^\alpha)^{-1}$ and $C_{11}\alpha^{-1} \leq (1 - b^\alpha)^{-1} \leq C_{12}\alpha^{-1}$. We see from (4.4) that

$$\begin{aligned} \sum_{k=1}^{\infty} |B^k x|_Q^2 b^{-k\alpha} &\leq C_5^2 \sum_{k=1}^{\infty} \beta(x)^{2k} k^{2n(x)} b^{-k\alpha} \\ &= C_5^2 \sum_{k=1}^{\infty} b^{k(2\gamma(x)-\alpha)} k^{2n(x)} \leq C_{13}(2\gamma(x) - \alpha)^{-2n(x)-1} . \end{aligned}$$

This proves the second inequality in (4.6). The first inequality is obtained from (4.5) as follows. We have

$$\begin{aligned} \sum_{k\geq b_1(x)} |B^k x|^2 b^{-k\alpha} &\geq b_2(x)^2 \sum_{k\geq b_1(x)} b^{k(2\gamma(x)-\alpha)} k^{2n(x)} \\ &\geq b_6(x)(2\gamma(x) - \alpha)^{-2n(x)-1} , \end{aligned}$$

for some $b_6(x) > 0$. Hence the first inequality in (4.6) is obtained. The proof of (4.7) for $\alpha \geq 2\gamma(x)$ is similar. □

PROOF OF THEOREM 4.1. In the following the conditions (a)–(d) refer to those in Proposition 2.1.

(i) Let $\{\nu_0, g_n, n \in \mathbf{Z}\}$ be the S_B-representation of ν. Suppose that $\mu \in L_\infty(b, Q)$. It follows from Propositions 2.2 and 2.3 that, for ν_0-a.e. x, there exists a measure Γ_x such that

$$g_n(x) = \int_{[0,\infty)} b^{n\alpha} \Gamma_x(d\alpha).$$

By (c), Γ_x is nonzero. By (d), for any $E \in \mathcal{B}(\mathbf{R}^d)$,

$$\nu(E) = \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} \int_{[0,\infty)} 1_E(B^{-n}x) b^{n\alpha} \Gamma_x(d\alpha).$$

Note that for any nonnegative measurable function $h(\alpha)$ on $[0, \infty)$, $\int h(\alpha)\Gamma_x(d\alpha)$ is measurable in x. By (b),

$$\infty > \sum_{n\geq 0} \int_{S_B} g_n(x)\nu_0(dx) = \int_{S_B} \nu_0(dx) \int_{[0,\infty)} \Gamma_x(d\alpha) \sum_{n\geq 0} b^{n\alpha}.$$

When $\alpha = 0$, $\sum_{n\geq 0} b^{n\alpha} = \infty$. Hence

$$\int_{S_B} \nu_0(dx)\Gamma_x(\{0\}) = 0.$$

Next we have, by (d),

$$\begin{aligned}
\infty &> \int_{\mathbf{R}^d} (|y|_Q^2 \wedge 1)\nu(dy) \\
&= \sum_{n\in\mathbf{Z}} \int_{S_B} (|B^{-n}x|_Q^2 \wedge 1)\nu_0(dx) \int_{(0,\infty)} b^{n\alpha}\Gamma_x(d\alpha) \\
&= \int_{S_B} \nu_0(dx) \int_{(0,\infty)} \Gamma_x(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha}(|B^{-n}x|_Q^2 \wedge 1).
\end{aligned}$$

Thus by Lemma 4.1, we have

$$\int_{S_B} \nu_0(dx) \int_{[2\gamma(x),\infty)} \Gamma_x(d\alpha) = 0,$$

concluding (4.1), and the integrability condition (4.2) is also proved.

Conversely suppose that ν has the representation (4.1) with ν_0 and Γ_x satisfying (4.2). Set

$$g_n(x) = \int_{(0,2\gamma(x))} b^{n\alpha}\Gamma_x(d\alpha), \quad n \in \mathbf{Z}. \tag{4.9}$$

Then (4.1) and (4.9) imply (d). We observe that $g_n(x)$ in (4.9) satisfies (a), (b), and (c). As to (b),

$$\begin{aligned}\int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) g_n(x) \\ &= \int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} (|B^{-n}x|_Q^2 \wedge 1) \int_{(0,2\gamma(x))} b^{n\alpha} \Gamma_x(d\alpha) \\ &= \int_{S_B} \nu_0(dx) \int_{(0,2\gamma(x))} \Gamma_x(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha} (|B^{-n}x|_Q^2 \wedge 1) < \infty\end{aligned}$$

by (4.2). (a) and (c) are obvious because $\Gamma_x(d\alpha)$ is nonzero for each x. Therefore $\{\nu_0, g_n, n \in \mathbf{Z}\}$ is the S_B-representation of ν. It follows from (4.9) and Proposition 2.2 that $g_n(x)$ is completely monotone in $n \in \mathbf{Z}$. Thus, by Proposition 2.3, $\mu \in L_\infty(b, Q)$.

(ii) In order to see that ν is the Lévy measure of some μ, it is enough to show that $\nu(\{0\}) = 0$ and that $\int_{\mathbf{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. The former is obvious from (4.1). The latter follows from (4.2) since $|x| \le \text{const.} \times |x|_Q$.

(iii) To show the uniqueness, suppose that both $\{\nu_0, \Gamma_x\}$ and $\{\tilde{\nu}_0, \tilde{\Gamma}_x\}$ represent ν. Let

$$g_n(x) = \int_{(0,2\gamma(x))} b^{n\alpha} \Gamma_x(d\alpha), \quad \tilde{g}_n(x) = \int_{(0,2\gamma(x))} b^{n\alpha} \tilde{\Gamma}_x(d\alpha). \tag{4.10}$$

By the proof above, $\{\nu_0, g_n\}$ and $\{\tilde{\nu}_0, \tilde{g}_n\}$ are S_B-representations of ν. Thus by the uniqueness of them in Proposition 2.1, there exists a Borel measurable function $h(x)$ with $0 < h(x) < \infty$ such that $\tilde{\nu}_0(dx) = h(x)\nu_0(dx)$ and $g_n(x) = h(x)\tilde{g}_n(x)$, ν_0-a.e. x for any $n \in \mathbf{Z}$. Thus by the uniqueness assertion in Proposition 2.2(i)(b) and by (4.10), we conclude that $\Gamma_x(d\alpha) = h(x)\tilde{\Gamma}_x(d\alpha)$. The assertion (4.3) for Γ_x follows from (4.2) and (4.6). □

5. Proof of Theorem 1.1.

We first show that $OSS(b, Q) \subset L_\infty(b, Q)$. Let $\mu \in OSS(b, Q)$. That is, for some $0 < a < 1$ and $c \in \mathbf{R}^d$,

$$\hat{\mu}(z)^a = \hat{\mu}(b^{Q'}z)e^{i\langle c,z\rangle}. \tag{5.1}$$

Then

$$\hat{\mu}(z) = \hat{\mu}(b^{Q'}z)\hat{\rho}(z), \tag{5.2}$$

with

$$\hat{\rho}(z) = \hat{\mu}(z)^{1-a}e^{i\langle c,z\rangle}. \tag{5.3}$$

To show that $\mu \in L_\infty(b, Q)$, by the definition, it is enough to show that $\rho \in L_\infty(b, Q)$. Since $\mu \in I(\mathbf{R}^d)$, $\rho \in I(\mathbf{R}^d)$. Hence by (5.2), $\mu \in L_0(b, Q)$. Since $L_0(b, Q)$ is Q-completely closed in the strong sense as mentioned in Section 1, ρ in (5.3) is in $L_0(b, Q)$. Thus by the definition, (5.2) implies that $\mu \in L_1(b, Q)$. Repeating this argument, we conclude that

$\mu \in L_m(b, Q)$ for any $1 \le m < \infty$ and therefore $\mu \in L_\infty(b, Q)$. Hence $OSS(b, Q) \subset L_\infty(b, Q)$.

Since $L_\infty(b, Q)$ is Q-completely closed in the strong sense, it only remains to prove "the smallest". Let K be any Q-completely closed class in the strong sense containing $OSS(b, Q)$. First, notice the following fact. Let $\alpha > 0$ and $r(z, x) = e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle/(1 + |x|^2)$. If ν_0 is a finite measure concentrated on $S_B \cap \{x : 2\gamma(x) > \alpha\}$ satisfying

$$\int_{S_B} \nu_0(dx) \sum_{n<0} b^{n\alpha} |b^{-nQ}x|^2 < \infty$$

and if μ is a distribution with

$$\hat{\mu}(z) = \exp\left\{\int_{S_B} \nu_0(dx) \sum_{n\in\mathbf{Z}} b^{n\alpha} r(z, b^{-nQ}x)\right\},$$

then $\mu \in OSS(b, Q)$, since (5.1) holds with $a = b^\alpha$. Now let μ be a purely non-Gaussian distribution in $L_\infty(b, Q)$. Then its Lévy measure is represented as in Theorem 4.1, and we have

$$\hat{\mu}(z) = \exp\left\{i\langle\gamma, z\rangle + \int_{S_B} \nu_0(dx) \int_{(0,2\gamma(x))} \Gamma(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha} r(z, b^{-nQ}x)\right\}$$

with some $\gamma \in \mathbf{R}^d$. For $\varepsilon > 0$, define μ_ε by

$$\hat{\mu}_\varepsilon(z) = \exp\left\{i\langle\gamma, z\rangle + \int_{S_B} \nu_0(dx) \int_{(\varepsilon,2\gamma(x)-\varepsilon)} \Gamma_x(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha} r(z, b^{-nQ}x)\right\}.$$

Since the number of the possible values of $\gamma(x)$ is finite, $2\gamma(x) - \varepsilon > \varepsilon$ for all x, if ε is small enough. Then, for fixed $\varepsilon > 0$, we can choose measures $\Gamma_x^{(n)}(d\alpha)$ satisfying the following conditions: $\Gamma_x^{(n)}(d\alpha)$ is concentrated on the points $\{k2^{-n}, k = 1, 2, \cdots\} \cap (\varepsilon, 2\gamma(x) - \varepsilon)$, $\Gamma_x^{(n)}(d\alpha)$ converges to $\Gamma_x(d\alpha)$ for each $x \in S_B$ as $n \to \infty$, the total mass of $\Gamma_x^{(n)}(d\alpha)$ does not exceed that of $\Gamma_x(d\alpha)$ for each $x \in S_B$, and $\{\Gamma_x^{(n)}\}$ is measurable in x. Define $\mu_\varepsilon^{(n)}$ by

$$\hat{\mu}_\varepsilon^{(n)}(z) = \exp\left\{i\langle\gamma, z\rangle + \int_{S_B} \nu_0(dx) \int_{(\varepsilon,2\gamma(x)-\varepsilon)} \Gamma_x^{(n)}(d\alpha) \sum_{j\in\mathbf{Z}} b^{j\alpha} r(z, b^{-jQ}x)\right\}.$$

We see that $\mu_\varepsilon^{(n)}$ is a convolution of a finite number of purely non-Gaussian distributions in $OSS(b, Q)$. Hence $\mu_\varepsilon^{(n)} \in K$. We see from (4.6) that, for any fixed $\varepsilon > 0$,

$$C_{14} \le \sum_{n\in\mathbf{Z}} b^{n\alpha}(|b^{-nQ}x|_Q^2 \wedge 1) \le C_{15} \quad \text{for } \alpha \in (\varepsilon, 2\gamma(x) - \varepsilon) \text{ and } x \in S_B. \tag{5.4}$$

Hence, by (4.2),

$$\int_{S_B} \nu_0(dx)\Gamma_x((\varepsilon, 2\gamma(x) - \varepsilon)) < \infty. \tag{5.5}$$

We show that, for fixed $z \in \mathbf{R}^d$, $\sum_{n\in\mathbf{Z}} b^{n\alpha} r(z, b^{-nQ}x)$ is bounded in $\alpha \in (\varepsilon, 2\gamma(x) - \varepsilon)$ and $x \in S_B$, and continuous in α. Since $|r(z, x)| \le C_{16}(|x|_Q^2 \wedge 1)$, we see the boundedness from (5.4). The continuity is obvious. Thus by (5.5) and Lebesgue's dominated convergence theorem, we have that as $n \to \infty$ $\hat{\mu}_\varepsilon^{(n)} \to \hat{\mu}_\varepsilon(z)$, and that $\mu_\varepsilon \in K$. Finally, $\hat{\mu}_\varepsilon \to \hat{\mu}$ as $\varepsilon \downarrow 0$. Thus $\mu \in K$. This proves that if $\mu \in L_\infty(b, Q)$ is purely non-Gaussian, then $\mu \in K$.

If $\mu \in L_\infty(b, Q)$ is Gaussian, then by Theorem 3.4, μ is a convolution of finite number of Gaussian distributions in $OSS(b, Q) \subset K$, and thus $\mu \in K$. As Proposition 2.3 shows, any $\mu \in L_\infty(b, Q)$ is decomposable in $L_\infty(b, Q)$ as the convolution of a Gaussian and a purely non-Gaussian. Hence $L_\infty(b, Q) \subset K$ and the proof is complete. □

6. Remarks on the case $Q = I$.

Let us consider the case $Q = I$. Let $0 < b < 1$. The classes $L_m(b, I)$, $0 \leq m \leq \infty$, were introduced by Maejima and Naito [MN98], of which the paper [MSW99] was a matrix generalization on $\mathbf{R}^d$. Distributions in $L_0(b, I)$ are called semi-selfdecomposable. Distributions in $L_\infty(b, I)$ should be called completely semi-selfdecomposable. The class $OSS(b, I)$ consists of $\mu \in \mathcal{P}(\mathbf{R}^d)$ that satisfies $\hat{\mu}(z)^a = \hat{\mu}(bz)e^{i\langle c,z\rangle}$ for some $0 < a < 1$ and $c \in \mathbf{R}^d$; namely it is the class of $\mu \in \mathcal{P}(\mathbf{R}^d)$ that satisfies $\hat{\mu}(z)^{b^\alpha} = \hat{\mu}(bz)e^{i\langle c,z\rangle}$ for some $0 < \alpha \leq 2$ and $c \in \mathbf{R}^d$. Thus distributions in $OSS(b, I)$ are exactly semi-stable distributions studied by many authors beginning with Lévy [L37]. Now we have $B = bI$, $|x|_Q = |x|$, and $S_B = \{x \in \mathbf{R}^d : b < |x| \leq 1\}$. We write S_B as S_b. Further we have $p = q = 1$, $b_1 = b$, $n_1 = 1$, and $f(\zeta) = \zeta - b$. Since $\Psi_B(A) = (1 - b^2)A$, Proposition 2.3 shows that all Gaussian distributions are in $L_\infty(b, I)$. Since $\beta(x) = b$ and $\gamma(x) = 1$ for all $x \in \mathbf{R}^d$, the following result is obtained from Theorem 4.1.

THEOREM 6.1. (i) *Suppose that μ is in $I(\mathbf{R}^d)$ with nonzero Lévy measure ν. Then $\mu \in L_\infty(b, I)$ if and only if ν is expresed as*

$$\nu(E) = \int_{S_b} \nu_0(dx) \int_{(0,2)} \Gamma_x(d\alpha) \sum_{n \in \mathbf{Z}} b^{n\alpha} 1_E(b^{-n}x), \quad E \in \mathcal{B}(\mathbf{R}^d), \tag{6.1}$$

where ν_0 is a nonzero finite measure on S_b and Γ_x, $x \in S_b$, are measures on $(0, 2)$ *measurable in x satisfying*

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha}\right) \Gamma_x(d\alpha) = 1. \tag{6.2}$$

(ii) *If a nonzero finite measure ν_0 on S_b and measures Γ_x, $x \in S_b$, on* $(0, 2)$ *measurable in x satisfying* (6.2) *are given, then the measure ν defined by* (6.1) *is the Lévy measure of some $\mu \in I(\mathbf{R}^d)$.*

(iii) *If $\mu \in L_\infty(b, I)$ with nonzero Lévy measure ν, then the measure ν_0 in* (i) *is uniquely determined by ν and the measures Γ_x, $x \in S_b$, are unique in the sense that $\tilde{\Gamma}_x = \Gamma_x$ for ν_0-a.e. x for any $\tilde{\Gamma}_x$ which expresses ν by* (6.1) *in place of Γ_x.*

PROOF. (i) We apply Theorem 4.1. Note that we do not assume that μ is purely non-Gaussian, since all Gaussians are in $L_\infty(b, I)$. Suppose that $\mu \in L_\infty(b, I)$. Then we get

(6.1) with some ν_0 and Γ_x satisfying

$$\int_{S_b} \nu_0(dx) \int_{(0,2)} \Gamma_x(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha}((b^{-2n}|x|^2) \wedge 1) < \infty .$$

This is equivalent to

$$\int_{S_b} \nu_0(dx) \int_{(0,2)} \Gamma_x(d\alpha) \left(\sum_{n\geq 0} b^{n\alpha} + \sum_{n<0} b^{-n(2-\alpha)} \right) < \infty .$$

Since $\sum_{n\geq 0} b^{n\alpha} \sim C_{17}/\alpha$ as $\alpha \downarrow 0$ and $\sum_{n<0} b^{-n(2-\alpha)} \sim C_{18}/(2-\alpha)$ as $\alpha \uparrow 2$, the condition is equivalent to

$$\int_{S_B} \nu_0(dx) \int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma_x(d\alpha) < \infty .$$

Let $h(x) = \int_{(0,2)} (1/\alpha + 1/(2-\alpha)) \Gamma_x(d\alpha)$ and use $h(x)\nu_0(dx)$ and $(1/h(x))\Gamma_x(d\alpha)$ in place of ν_0 and Γ_x to obtain (6.2). The converse is similarly proved. (ii) and (iii) follow from Theorem 4.1(ii) and (iii), respectively. □

Another form of the theorem above is as follows.

THEOREM 6.2. (i) *Suppose that* $\mu \in I(\mathbf{R}^d)$ *with nonzero Lévy measure* ν. *Then* $\mu \in L_\infty(b, I)$ *if and only if*

$$\nu(E) = \int_{(0,2)} \Gamma(d\alpha) \sum_{n\in\mathbf{Z}} b^{n\alpha} \nu_\alpha((b^n E) \cap S_b) , \quad E \in \mathcal{B}(\mathbf{R}^d) , \tag{6.3}$$

where Γ *is a nonzero measure on* $(0, 2)$ *with*

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma(d\alpha) < \infty \tag{6.4}$$

and ν_α, $\alpha \in (0, 2)$, *are probability measures on* S_b, *measurable in* α.

(ii) *If a nonzero measure* Γ *on* $(0, 2)$ *satisfying* (6.4) *and probability measures* ν_α, $\alpha \in (0, 2)$, *on* S_b *measurable in* α *are given, then the measure* ν *defined by* (6.3) *is the Lévy measure of some* $\mu \in I(\mathbf{R}^d)$.

(iii) *If* $\mu \in L_\infty(b, I)$ *with nonzero Lévy measure* ν, *then the measure* Γ *in* (i) *is uniquely determined by* ν *and the probability measures* ν_α, $\alpha \in (0, 2)$, *on* S_b *are unique in the sense that* $\tilde{\nu}_\alpha = \nu_\alpha$ *for* Γ*-a.e.* α *for any* $\tilde{\nu}_\alpha$ *that expresses* ν *by* (6.3) *in place of* ν_α.

PROOF. In order to go to the representation (6.3) from (6.1), consider the probability measure $(1/\nu_0(S_b))\nu_0(dx)((1/\alpha) + 1/(2-\alpha))\Gamma_x(d\alpha)$ on $S_b \times (0, 2)$ and apply the existence theorem for conditional distributions. Transfer in the reverse direction is similar. □

Finiteness and infiniteness of the moments of distributions in $L_\infty(b, I)$ are determined only by the measure Γ. This is an application of Theorem 6.2.

THEOREM 6.3. *Let* μ *be a distribution in* $L_\infty(b, I)$ *with nonzero Lévy measure* ν. *Let* $\Gamma(d\alpha)$ *be the nonzero measure on* $(0, 2)$ *uniquely determined by* ν *in Theorem* 6.2. *Let*

$\alpha_0 \in [0,2)$ *be the infimum of the support of* Γ. *Then, finiteness and infiniteness of* $M_\eta = \int_{\mathbf{R}^d} |x|^\eta \mu(dx)$ *are as follows.*

(i) *If* $\eta > \alpha_0$, *then* $M_\eta = \infty$.

(ii) *If* $\alpha > 0$ *and* $0 < \eta < \alpha_0$, *then* $M_\eta < \infty$.

(iii) *If* $\alpha > 0$ *and* $\Gamma(\{\alpha_0\}) > 0$, *then* $M_{\alpha_0} = \infty$.

(iv) *If* $\alpha_0 > 0$ *and* $\Gamma(\{\alpha_0\}) = 0$, *then* M_{α_0} *is finite or infinite according as* $\int_{(\alpha_0,2)} (1/(\alpha - \alpha_0))\Gamma(d\alpha)$ *is finite or infinite.*

PROOF. It is known that $M_\eta < \infty$ if and only if $\int_{|x|>1} |x|^\eta \nu(dx) < \infty$ (Kruglov [K70]). We have, from (6.3),

$$\int_{|x|>1} |x|^\eta \nu(dx) = \int_{(0,2)} \Gamma(d\alpha) \int_{S_b} \nu_\alpha(dx) \sum_{n \in \mathbf{Z}} b^{n\alpha} |b^{-n}x|^\eta 1_E(b^{-n}x)$$
$$= \int_{(0,2)} \Gamma(d\alpha) \int_{S_b} \nu_\alpha(dx) |x|^\eta \sum_{n \geq 1} b^{n(\alpha - \eta)},$$

where $E = \{x : |x| > 1\}$. If $\eta > \alpha_0$, then $\sum_{n\geq 1} b^{n(\alpha-\eta)} = \infty$ for $\alpha \in [\alpha_0, \eta)$ and $\int_{|x|>1} |x|^\eta \nu(dx) = \infty$. If $\alpha_0 > 0$ and $0 < \eta < \alpha_0$, then

$$\int_{|x|>1} |x|^\eta \nu(dx) \leq \frac{1}{1 - b^{\alpha_0 - \eta}} \int_{[\alpha_0,2)} \Gamma(d\alpha) \int_{S_b} |x|^\eta \nu_\alpha(dx) < \infty.$$

If $\alpha_0 > 0$ and $\Gamma(\{\alpha_0\}) > 0$, then

$$\int_{|x|>1} |x|^{\alpha_0} \nu(dx) \geq \Gamma(\{\alpha_0\}) \int_{S_b} \nu_{\alpha_0}(dx) |x|^{\alpha_0} \sum_{n\geq 1} 1 = \infty.$$

Hence we obtain (i), (ii), and (iii). Consider the final case, $\alpha_0 > 0$ and $\Gamma(\{\alpha_0\}) = 0$. Since $\sum_{n\geq 1} b^{n(\alpha - \alpha_0)} \sim C_{19}/(\alpha - \alpha_0)$ as $\alpha \downarrow \alpha_0$, we have

$$C_{20} \int_{(\alpha_0,2)} \frac{1}{\alpha - \alpha_0} \Gamma(d\alpha) \leq \int_{|x|>1} |x|^{\alpha_0} \nu(dx) \leq C_{21} \int_{(\alpha_0,2)} \frac{1}{\alpha - \alpha_0} \Gamma(d\alpha).$$

Hence the assertion (iv) follows. □

References

[J77] R. JAJTE, Semi-stable probability measures on $\mathbf{R}^N$, Studia Math. **61** (1977), 29–39.

[J83] Z. J. JUREK, Limit distributions and one-parameter groups of linear operators on Banach spaces. J. Multivariate Anal. **13** (1983), 578–604.

[JM93] Z. J. JUREK and J. D. MASON, *Operator-Limit Distributions in Probability Theory*, John Wiley (1993).

[K80] W. KRAKOWIAK, Operator semi-stable probability measures on Banach spaces, Colloq. Math. **43** (1980), 351–363.

[K70] V. M. KRUGLOV, A note on infinitely divisible distributions, Theory Probab. Appl. **15** (1970), 319–324.

[LR80] R. G. LAHA and V. K. ROHATGI, Operator semi-stable probability measures on Hilbert space, Bull. Austral. Math. Soc. **22** (1980), 397–406. (Corrigenda, **22**, 479–480.)

[L37] P. LÉVY, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars (1937).

[Ł81] A. ŁUCZAK, Operator semi-stable probability measures on $\mathbf{R}^N$, Colloq. Math. **45** (1981), 287–300. (Corrigenda, **52** (1987), 167–169.)

[Ł91] A. ŁUCZAK, Operator-semi-stable operator Lévy's measures on finite dimensional vector spaces, Probab. Theory Related Fields **90** (1991), 317–340.

[MN98] M. MAEJIMA and Y. NAITO, Semi-selfdecomposable distributions and a new class of limit theorems, Probab. Theory Related Fields **112** (1998), 13–31.

[MSW99] M. MAEJIMA, K. SATO and T. WATANABE, Operator semi-selfdecomposability, (C, Q)-decomposability and related nested classes, Tokyo J. Math. **22** (1999), 473–509.

[SY85] K. SATO and M. YAMAZATO, Completely operator-selfdecomposable distributions and operator-stable distributions, Nagoya Math. J. **97** (1985), 71–94.

Present Addresses:
MAKOTO MAEJIMA
DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY,
HIYOSHI, YOKOHAMA, 223–8522 JAPAN.
e-mail: maejima@math.keio.ac.jp

KEN-ITI SATO
HACHIMAN-YAMA 1101–5–103, TENPAKU-KU, NAGOYA, 468–0074 JAPAN.
e-mail: ken-iti.sato@nifty.ne.jp

TOSHIRO WATANABE
CENTER FOR MATHEMATICAL SCIENCES, THE UNIVERSITY OF AIZU,
AIZU-WAKAMATSU, FUKUSHIMA, 965–8580 JAPAN.
e-mail: t-watanb@u-aizu.ac.jp