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Very Special Framed Links for a Homotopy 3-Sphere

Ippei ISHII

Keio University

Introduction.

A *framed link* K in a closed 3-manifold M is a system of disjoint simple closed curves $K = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ equipped with another system of simple closed curves $\tilde{K} = (\tilde{\kappa}_1, \tilde{\kappa}_2, \cdots, \tilde{\kappa}_n)$ such that each component $\tilde{\kappa}_j$, called the *framing curve* of κ_j, lies on the boundary of a regular neighborhood V_j of κ_j in M and meets the meridian curve of V_j exactly once. Given a framed link K in M, a 3-manifold $\chi(M; K)$ obtained by a *Dehn surgery* along K is defined as follows:

$$\chi(M; K) = (M - (\mathring{V}_1 \cup \mathring{V}_2 \cup \cdots \cup \mathring{V}_n)) \cup (V_1' \cup V_2' \cup \cdots \cup V_n'),$$

where each V_j' is a solid torus glued back by a homeomorphism $h_j : \partial V_j' \to \partial V_j$ which takes a meridian curve of V_j' onto the framing curve $\tilde{\kappa}_j$. Furthermore we can define the *dual framed link* $K^* = (\kappa_1^*, \kappa_2^*, \cdots, \kappa_n^*)$ in $M^* \equiv \chi(M; K)$ so that each κ_j^* is a core of V_j' and its framing curve $\tilde{\kappa}_j^*$ is a meridian of V_j, and we get the dual surgery description $M = \chi(M^*; K^*)$.

In [1] it was shown that any closed 3-manifold M has a framed link K such that $\chi(M; K)$ is homeomorphic to a 3-sphere S^3 and the dual framed link K^* in S^3 enjoys some special properties, especially K^* forms a pure plat in S^3. It is pointed out in [1] that such a framed link of M is closely related to a Heegaard splitting (or diagram) of M. On the other hand, using a notion of a d-pseudo core, we proposed in [2] a condition for a Heegaard splitting to be reduced.

In this paper, to see how extent we can apply the conditions in [2] to a Heegaard splitting of a homotopy 3-sphere M induced by a surgery description, we will try to add further good properties to a special framed link given in [1]. In the case where $\chi(M; K) = S^3$ and the dual framed link K^* forms a pure plat, it was shown in [2] that we can take a link isotopic to K as a *generalized core* (see [2] for the definition) of the Heegaard splitting of M induced by the surgery description, and that a key for applying the reducibility condition is to find a *localizing arc system* (see [2] and §1.3 below) for a generalized core. In §1, adding some conditions on a localizing arc system to a special framed link defined in [1], we introduce a notion of a "very special framed link" of a closed 3-manifold, and in §§2, 3 we will prove that

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THEOREM 0.1. *A closed* 3*-manifold* M *has a very special framed link if and only if* M *is a homotopy* 3*-sphere.*

In §4, we will investigate on a relation between a very special framed link and a Heegaard splitting of a homotopy 3-sphere.

Throughout this paper, we work in the PL-category and use the following notation:

- $\mathrm{cl}(\cdot)$: the closure,
- $N(A, B)$: a regular neighborhood of A in B where $A \subset B$,
- $E(A, B)$: an exterior of A in B, namely $E(A, B) = \mathrm{cl}(B - N(A, B))$.

1. Definitions and notation.

1.1. Conventions. We will consider a framed link K in a general 3-manifold (mainly a homotopy 3-sphere) M with $\chi(M; K) = S^3$ and simultaneously consider its dual framed link K^* in S^3. To avoid confusion, we always mark an asterisk '*' for representing a link (or its component) in S^3, and do not mark it for a link in M.

In the case where $\chi(M; K) = S^3$, the exterior $E(K, M)$ coincides with the exterior $E(K^*, S^3)$ of the dual link K^* in S^3. In what follows, without any notice, we use this identification of the exteriors of K and K^*.

As usual, for representing the framing curve $\tilde{\kappa}_j$ of a component κ_j of a framed link in an oriented homology 3-sphere M, we use the linking number of κ_j and $\tilde{\kappa}_j$, which is called the *framing number* and denoted by $f.n.(\kappa_j)$.

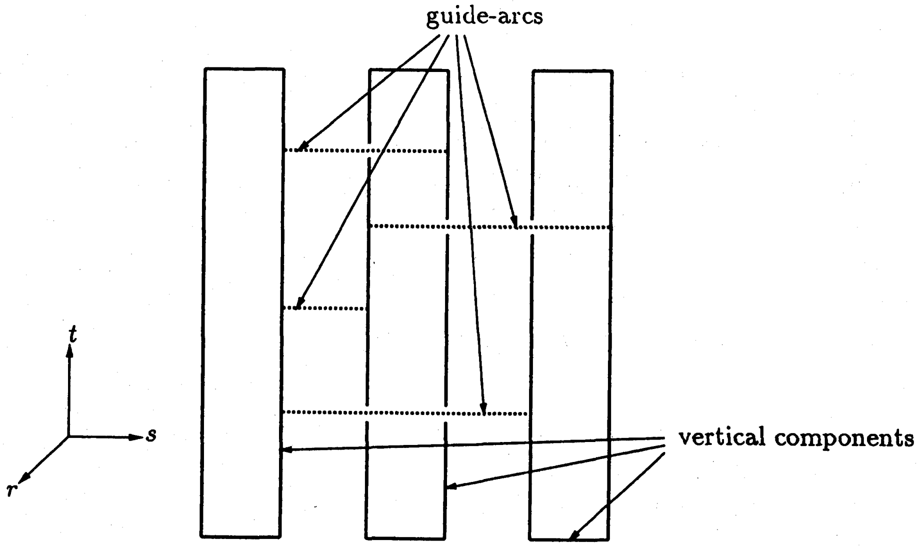
1.2. Pure plats and vh-plats in S^3. A link L^* in $S^3 = E^3 \cup \{\infty\}$ is said to be represented by a $2n$*-plat* (with respect to a height function on the t-axis) if it has n local maxima and n local minima with respect to the t-axis, when the Euclidian space E^3 is parametrized by rectangular coordinates (r, s, t). It is said to be a *pure plat* if it is a $2n$-plat with n components.

We will give a definition of a special class of pure plats in S^3. To define this class, we will fix a rectangular coordinate (r, s, t) on $E^3 = S^3 - \{\infty\}$ (of cause, we assume that all the objects which we consider do not meet with the point ∞). We say a subset X of E^3 to be *horizontal* if X is included in some plane $\{t = \text{const.}\} \subset E^3$.

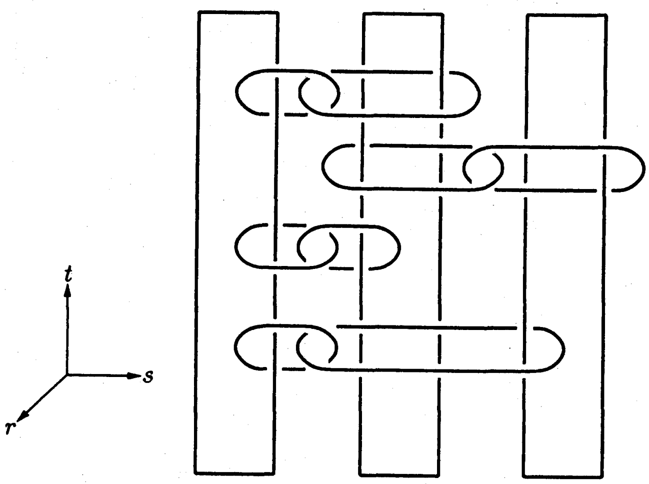
DEFINITION 1.1. A link Λ^* in S^3 is said to be a *vh-plat* ("vh" means "vertical and horizontal") if Λ^* is decomposed into two sublinks Λ_v^* (called the *vertical part*) and Λ_h^* (called the *horizontal part*), and these sublinks satisfy the following conditions (cf. Figure 1.1).

(1) Λ_v^* lies on the plane $\{r = 0\}$, is a pure plat with respect to the t-axis, and the components of Λ_v^* bound mutually disjoint 2-disks on $\{r = 0\}$. We may assume that the maximum points and the minimum points of the components of Λ_v^* are both on the same t-level.

(2) Λ_h^* consists of some Hopf links $x_{[1,j]}^* \cup x_{[2,j]}^*$ $(j = 1, 2, \cdots, m)$ with the following properties.



(a) vertical components and guide-arcs



(b) example of a vh-plat

FIGURE 1.1. vh-plat.

(i) For each $x^*_{[q,j]}$ there is a unique vertical component, say $\ell^*_{[q,j]}$, which is linking with $x^*_{[q,j]}$, and $\mathrm{lk}(x^*_{[q,j]}, \ell^*_{[q,j]}) = \pm 1$.

(ii) The j-th Hopf link $x^*_{[1,j]} \cup x^*_{[2,j]}$ is included in a regular neighborhood of a horizontal arc γ_j, called a *guide-arc*, which connects the two vertical components $\ell^*_{1,j}$ and $\ell^*_{2,j}$ within the half plane $\{t = t_j\} \cap \{r \geq 0\}$ ($t_j \neq t_{j'}$ for $j \neq j'$). We denote by $s_{q,j}$ ($q = 1, 2$) the s-coordinates of the end points of γ_j, and assume that $(0, s_{q,j}, t_j) \in \ell^*_{q,j}$ and $s_{1,j} < s_{2,j}$.

(iii) Each component $x^*_{[q,j]}$ is completely included in a plane

$$\Pi_{q,j} = \{t - t_j = a_{q,j} \cdot (r - b_q \cdot (s - s_{q,j}))\}$$

for some constants $a_{q,j}, b_q$ $(1 \le j \le m, q = 1, 2)$ with $a_{1,j} \cdot a_{2,j} < 0$, $b_1 > 0$ and $b_2 < 0$.

1.3. A localizing arc system for a link. Let $K = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ be a link in a 3-manifold M, and let δ be an embedded 2-disk such that $\delta \cap K$ consists of exactly one point. And let θ be an oriented arc going from a component κ_i to the boundary of δ. We assume that this θ does not intersect with $K \cup \delta$ except for its end points. The deformation κ_i into κ_i', which we denote by $\kappa_i[\theta; \delta]$, as in Figure 1.2 is called a *cross change along* θ. The result of this deformation is denoted by $K[\theta; \delta]$. Of cause $K[\theta; \delta]$ is determined up to twisting of the band centered at the arc θ. When we do not need to indicate the disk δ, considering θ to be an oriented arc going from a point on K to another point on K, we denote the cross changed link by $K[\theta]$. For a family $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_m)$ of mutually disjoint oriented arcs θ_k each of which connects two points on K, the link obtained by successive cross changes along $\theta_1, \cdots, \theta_m$ is denoted by $K[\vec{\theta}]$.

DEFINITION 1.2. A family $\vec{\theta} = (\theta_1, \cdots, \theta_m)$ of oriented arcs as above is said to be a *localizing arc system* for the link K if the cross changed link $K[\vec{\theta}]$ is local in M for a suitable choice of twisting of bands centered at θ_k $(k = 1, \cdots, m)$, where "local in M" means "completely included in a 3-ball in M".

Obviously any link in a homotopy 3-sphere has a localizing arc system.

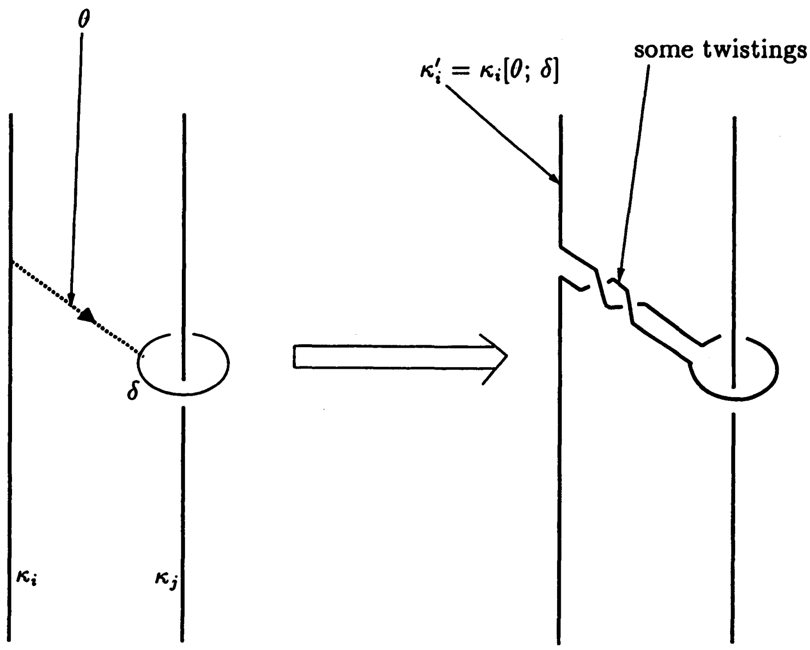


FIGURE 1.2. A cross change along θ.

1.4. A very special framed link. Let L be a framed link in a 3-manifold M such that $\chi(M; L) = S^3$ and the dual link L^* in $E^3 = S^3 - \{\infty\}$ forms a vh-plat as in Definition 1.1. Assume that L_1 is a sublink of L and that it has a localizing arc system $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_m)$. We may assume that $\tilde{\theta}_k \equiv \theta_k \cap E(L, M)$ is a connected arc for any k. The localizing arc system $\vec{\theta}$ is said to be *monotone* if any $\tilde{\theta}_k$ is monotone with respect to the height function h on the t-axis, and is said to be *separated* if $h(\tilde{\theta}_k) \cap h(\tilde{\theta}_{k'}) = \emptyset$ for any $k' \neq k$.

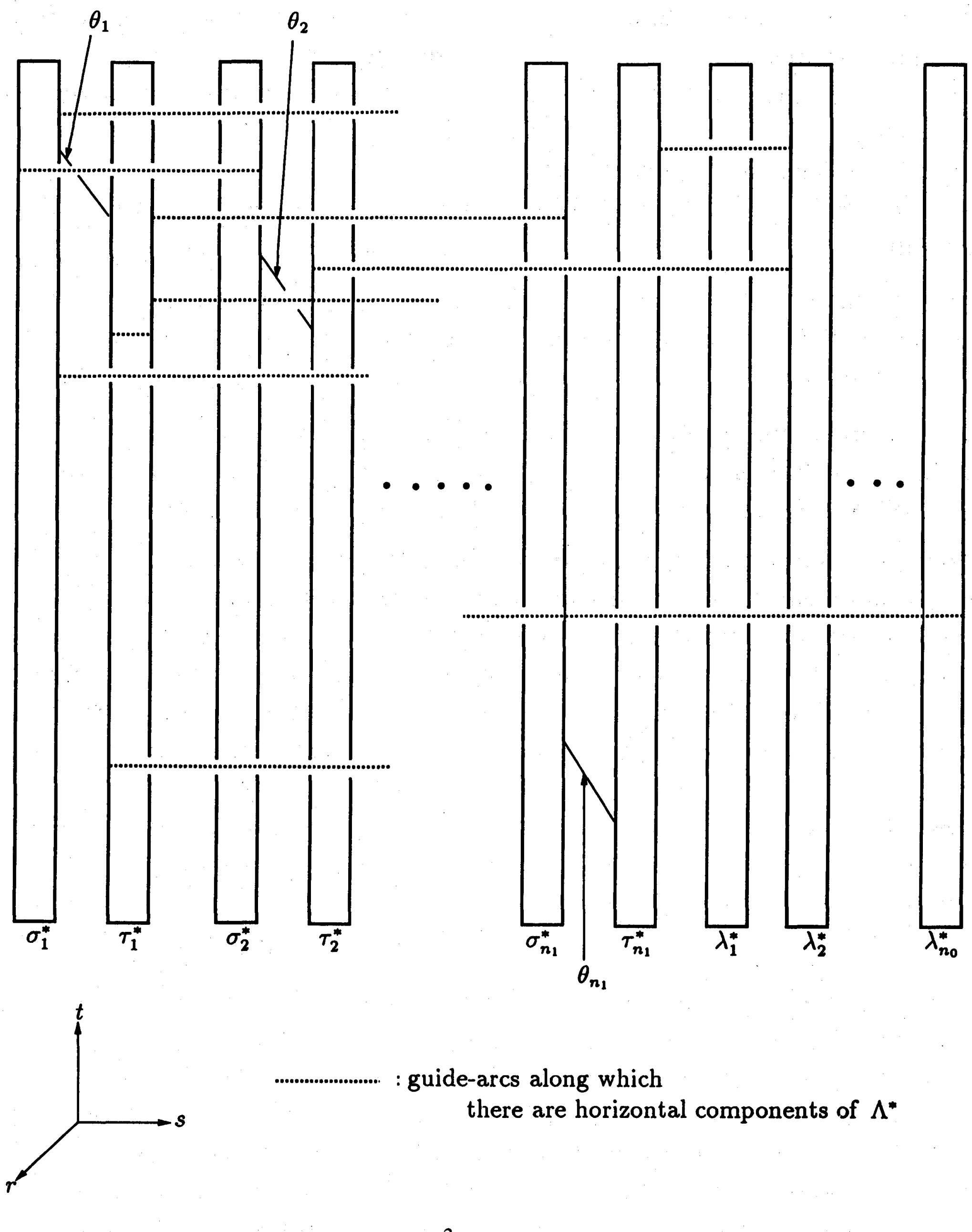


FIGURE 1.3. Dual link $\Lambda^* \subset S^3$ for a very special framed link $\Lambda \subset M$.

DEFINITION 1.3. A framed link Λ is a 3-manifold M is said to be *very special* if it satisfies the following conditions (a)–(d) (cf. Figure 1.3):

(a) the manifold $\chi(M; \Lambda)$ obtained by a surgery along Λ is homeomorphic to the 3-sphere S^3,

(b) the dual link $\Lambda^* \subset S^3$ forms a vh-plat, and satisfies the conditions (1) and (2) in Definition 1.1 with respect to some rectangular coordinate (r, s, t) on $E^3 = S^3 - \{\infty\}$,

(c) the framing numbers of the components of Λ^* are all 0,

(d) Λ_v, which is the sublink of Λ corresponding to the vertical part Λ_v^* of Λ^*, has a localizing arc system $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n_1})$ with the properties:

(i) the $2n_1$ end points of θ_k $(k = 1, \cdots, n_1)$ are all on different components of Λ_v from each other,

(ii) $\vec{\theta}$ is monotone and separated,

(iii) $\vec{\theta} \cap E(\Lambda^*, S^3)$ is included in $\{r < 0\}$, and its projection on $\{r = 0\}$ does not intersect with Λ_v^*.

The sublink Λ_v of a very special framed link Λ corresponding to the vertical part Λ_v^* of Λ^* is also called the *vertical part* of Λ, and the sublink Λ_h of Λ corresponding to Λ_h^* is called the *horizontal aprt* of Λ.

1.5. Kirby moves and dual Kirby moves. We now describe the well-known *Kirby moves* which can be made on a framed link $K \subset M$ and which do not alter the result of the surgery $\chi(M; K)$. There are three moves (cf. Figure 1.4):

Move (I). Introduce a new unknotted component with the framing number ± 1 lying in a 3-ball disjoint from K. It is well known that the effect of this move is to give the space a "full twist" as it passes through the new component.

Move (II). Introduce a Hopf link lying in a 3-ball disjoint from K, both of whose components have the framing number 0. This move can be generated by the above Move (I) and the next Move (III). However, because this move is frequently used, we list it as a basic move.

Move (III). Given components $\kappa_{j_1}, \kappa_{j_2}$ of K and a band β connecting κ_{j_1} to the framing curve $\tilde{\kappa}_{j_2}$ for κ_{j_2}, replace κ_{j_1} by $\kappa_{j_1} \# \tilde{\kappa}_{j_2}$, where # means the connected sum along the band β. The framing number of the new component $\kappa_{j_1} \# \tilde{\kappa}_{j_2}$ is defined to be

$$f.n.(\kappa_{j_1} \# \tilde{\kappa}_{j_2}) = f.n.(\kappa_{j_1}) + f.n.(\kappa_{j_2}) \pm 2 \cdot \mathrm{lk}(\kappa_{j_1}, \kappa_{j_2}), \tag{1.1}$$

and the framing numbers of the other components are not altered (cf. [3]).

This move is called a *band move of* κ_{j_1} *toward* κ_{j_2}.

Let K be a framed link and K_1 be one obtained by one of the Kirby moves from K. Then both dual links K^* and K_1^* are in $\chi(M; K)$. The change from K^* into K_1^* is said to be *dual moves*. As is observed by Rêgo and Rourke [4], the dual moves corresponding to the above Move (I)–(III) are given as in Figure 1.5. It is to be noticed that the dual move for a band move deforming κ_{j_1} into $\kappa_{j_1} \# \tilde{\kappa}_{j_2}$ is a band move which deforms $\kappa_{j_2}^*$ into $\kappa_{j_2}^* \# \tilde{\kappa}_{j_1}^*$.

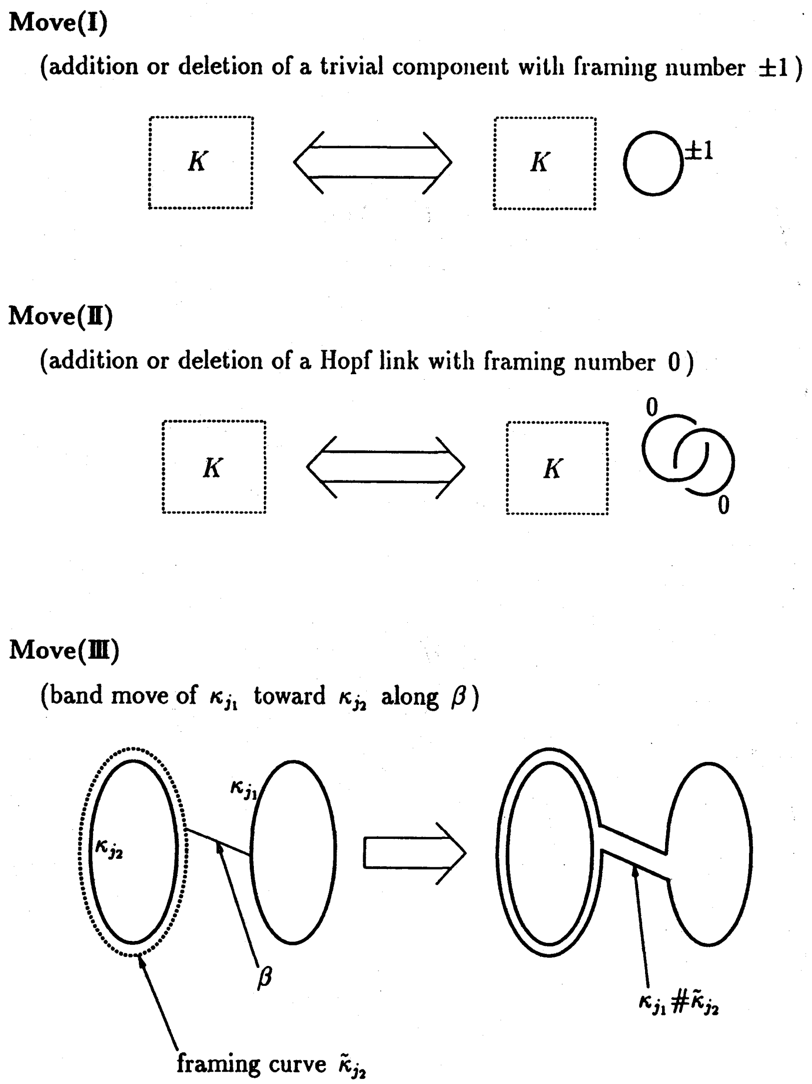


FIGURE 1.4. Kirby Moves.

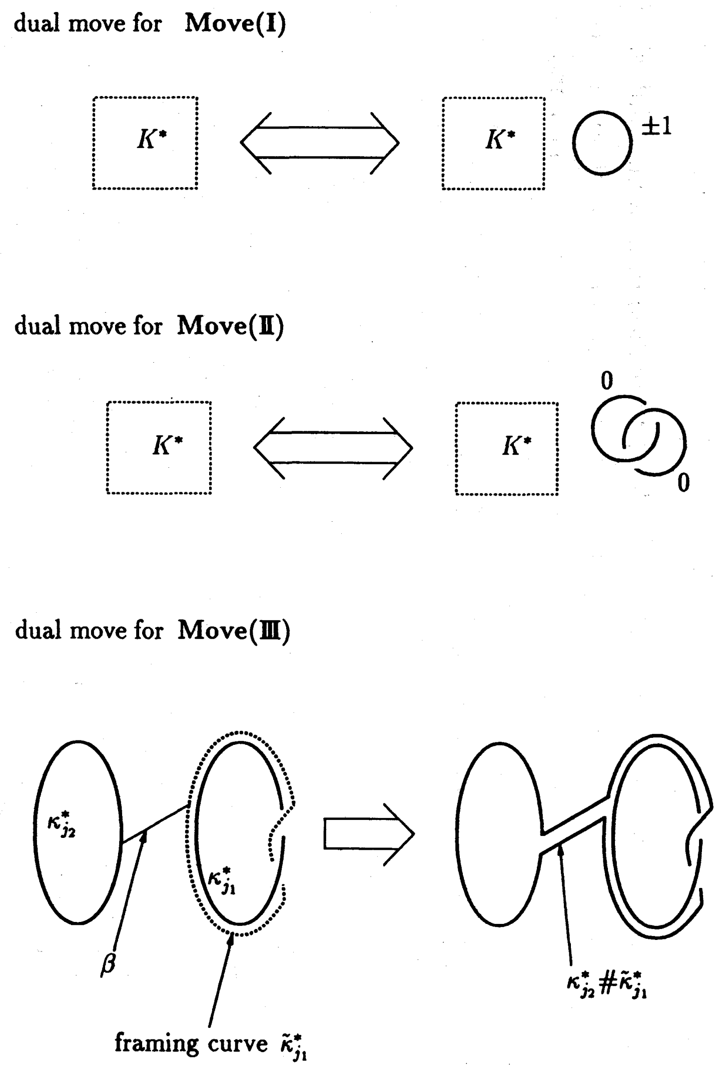


FIGURE 1.5. Dual moves for Kirby Moves in Figure 1.4.

Composing the above Kirby moves, we obtain a move defined below, which is often used in the following arguments.

DEFINITION 1.4 (Cross Change Move). The deformation of a framed link shown in Figure 1.6 is called a *cross change move*, which realizes a cross change of some two components (κ_1 and κ_2 in Figure 1.6). As is shown in Figure 1.6, this move is a composition of Kirby moves.

The dual of a cross change move is shown in Figure 1.7.

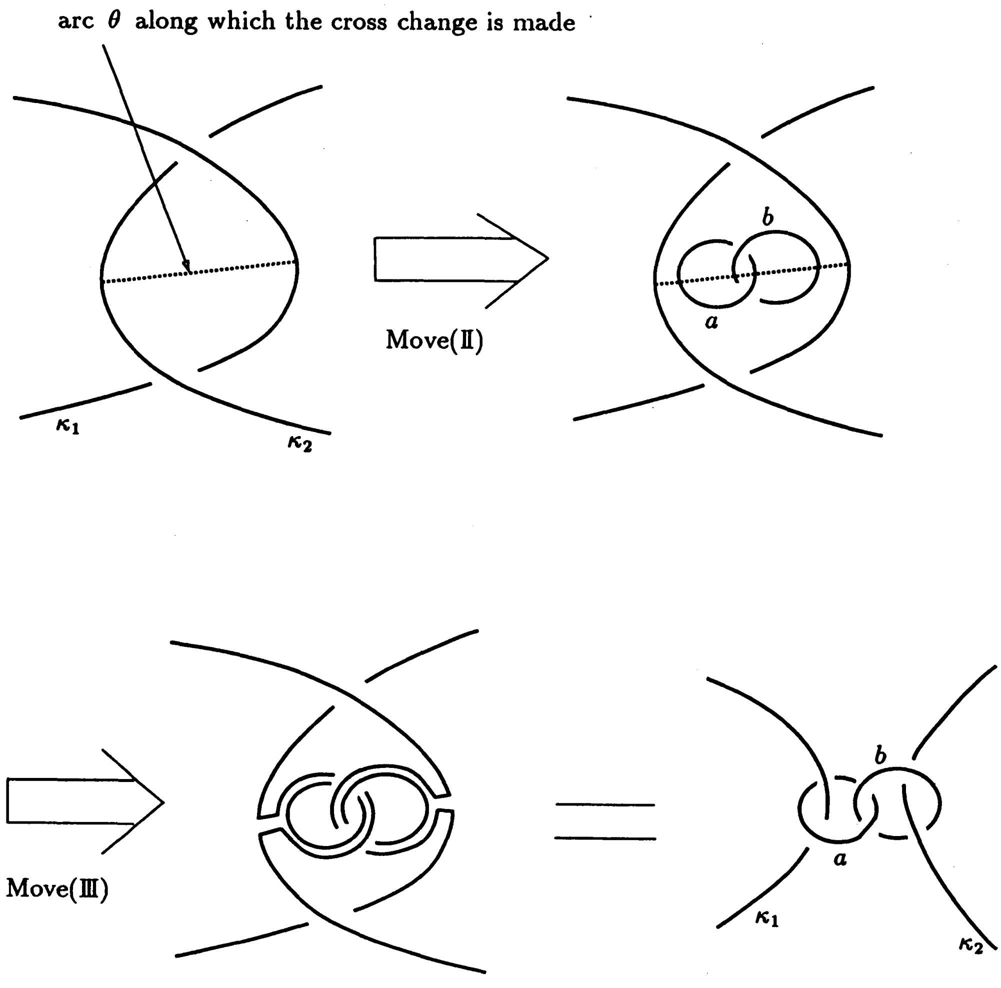


FIGURE 1.6. Cross change move.

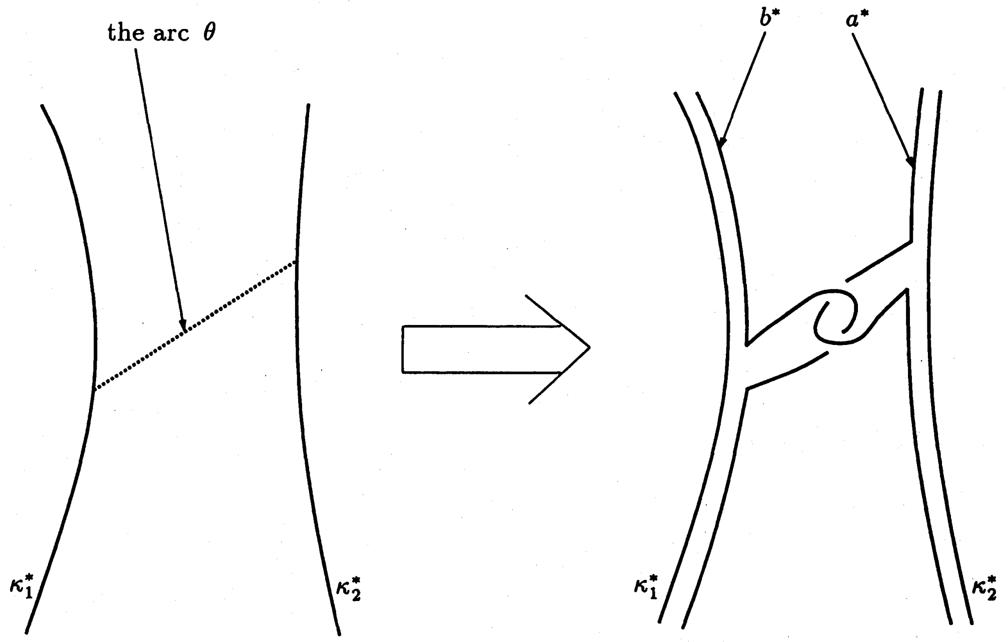


FIGURE 1.7. Dual move for the cross change move in Figure 1.6.

2. Proof of the "only if" part in Theorem 0.1.

To show the "only if" part, we first prove that

LEMMA 2.1. *If M has a framed link K such that $\chi(M; K) = S^3$ and each component of K is null homotopic in M, then M is a homotopy 3-sphere.*

PROOF. Let γ be any loop in $E(K, M) \equiv E(K^*, S^*)$. For such a γ, we can take an immersed planar surface X in $E(K^*, S^3)$ such that the boundary ∂X consists of γ and some meridians of $N(\kappa_j^*, S^3)$, where κ_j^* is a component of the dual link $K^* \subset S^3$. Since a meridian of $N(\kappa_j^*, S^3)$ is parallel to its dual component κ_j of K and κ_j is null homotopic in M by the assumption, γ bounds an immersed disk within M. This shows that M has a trivial fundamental group, namely M is a homotopy 3-sphere. □

Using this lemma, we obtain the following lemma which completes the proof of the "only if" part of Theorem 0.1.

LEMMA 2.2. *M is a homotopy 3-sphere if M admits a very special framed link.*

PROOF. Let $\Lambda = \Lambda_v \cup \Lambda_h$ be a very special framed link in M. Since Λ_v has a localizing arc system, any component of Λ_v is null homotopic in M. Hence by Lemma 2.1 it is sufficient for the proof to show that each component $x_{[q,j]}$ of Λ_h is null homotopic.

Let $x^*_{[p,j]}$ $(p \neq q)$ be the horizontal component of Λ^* which together with $x^*_{[q,j]}$ forms a Hopf link, and let $\omega(x^*_{[p,j]})$ be a 2-disk in S^3 which is bounded by $x^*_{[p,j]}$ on the plane $\Pi_{p,j}$ (cf. Definition 1.1).

In order to see $x_{[q,j]}$ being null homotopic in M, we consider a planar surface $Y \equiv \omega(x^*_{[p,j]}) \cap E(\Lambda^*, S^3)$. The boundary ∂Y consists of three circles; one, which we denote by $\phi(x^*_{[p,j]})$, lies on $\partial N(x^*_{[p,j]}, S^3)$, one, which we denote by $\pi(x_{[q,j]})$, is a meridian of $N(x^*_{[q,j]}, S^3)$, and the other, which we denote by $\pi(\ell_{p,j})$ is a meridian of $N(\ell^*_{p,j}, S^3)$, where $\ell^*_{p,j}$ is the vertical component of Λ^* linking with $x^*_{[p,j]}$. Regard Y to be a subset of $E(\Lambda, M) \subset M$. Because the framing number of $x^*_{[p,j]}$ is 0, the boundary component $\phi(x^*_{[p,j]})$ bounds a 2-disk D within M. Hence the surface $Y \cup D$ is an annulus bounded by $\pi(x_{[q,j]})$ and $\pi(\ell_{p,j})$. Noticing that each $\pi(x_{[q,j]})$ and $\pi(\ell_{p,j})$ is parallel to $x_{[q,j]}$ and $\ell_{p,j}$ respectively and that a component $\ell_{p,j}$ belonging to Λ_v is null homotopic in M, we can conclude that $x_{[q,j]}$ is also null homotopic. This completes the proof. $\square$

3. Proof of the "if" part in Theorem 0.1.

Our starting point for constructing a very special framed link in a homotopy 3-sphere M is the following lemma established in [1].

LEMMA 3.1 ([1]). *Any closed 3-manifold M admits a framed link K such that*
(i) *$\chi(M; K) = S^3$ and the dual link K^* in S^3 forms a pure plat, and*
(ii) *the framing numbers of the components of K^* are all even.*

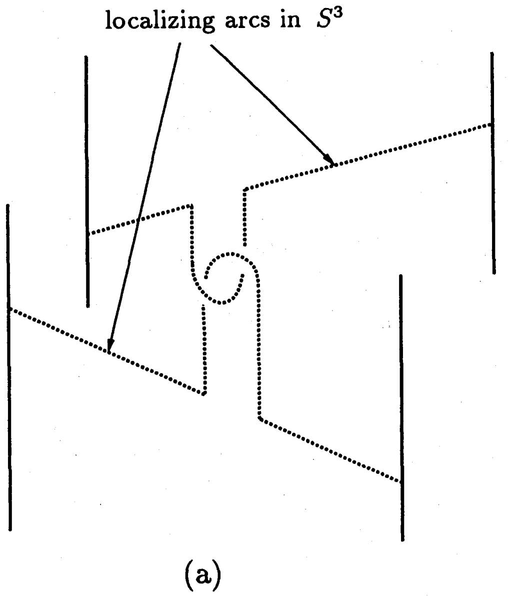
Using a framed link as in this lemma, we will show that a homotopy 3-sphere M admits a framed link with further properties as in the next lemma.

LEMMA 3.2. *For any homotopy 3-sphere M, there exists a framed link L in M such that*
(1) *$\chi(M; L) = S^3$, and the dual link L^* is a pure plat on the t-axis in $E^3 = S^3 - \{\infty\}$,*
(2) *the framing numbers of the dual framed link L^* are all even,*
(3) *L has a localizing arc system $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n_1})$ which is monotone and separated with respect to the height function on the t-axis.*

PROOF. By Lemma 3.1 there exists a framed link K in M which satisfies the conditions (1) and (2). For such a K we can take a localizing arc system $\vec{\theta}' = (\theta'_1, \theta'_2, \cdots \theta'_m)$ because M is a homotopy 3-sphere. However $\vec{\theta}'$ is neither monotone nor separated in general. So, applying some Kirby moves on K or K^*, we shall modify K into L with all the desired properties.

Applying an isotopic deformation if necessary, we may assume that the components of $\vec{\theta}'$ are monotone and separated apart from a finite number of crossings (as in Figure 3.1(a)). For each of these crossings, add to K^* new components with the framing number 0 (Move (II)) so as to cancel the crossings (see Figure 3.1(b)).

Denote by L^* the result of the above moves, and denote by L its dual. Obviously L satisfies the required conditions (1) and (2). It is also obvious that, after these moves, $\vec{\theta}'$ becomes monotone and separated. But, since the arcs θ'_j pass through the disks bounded by new components, this $\vec{\theta}'$ may not be a localizing arc system for L. To obtain the desired



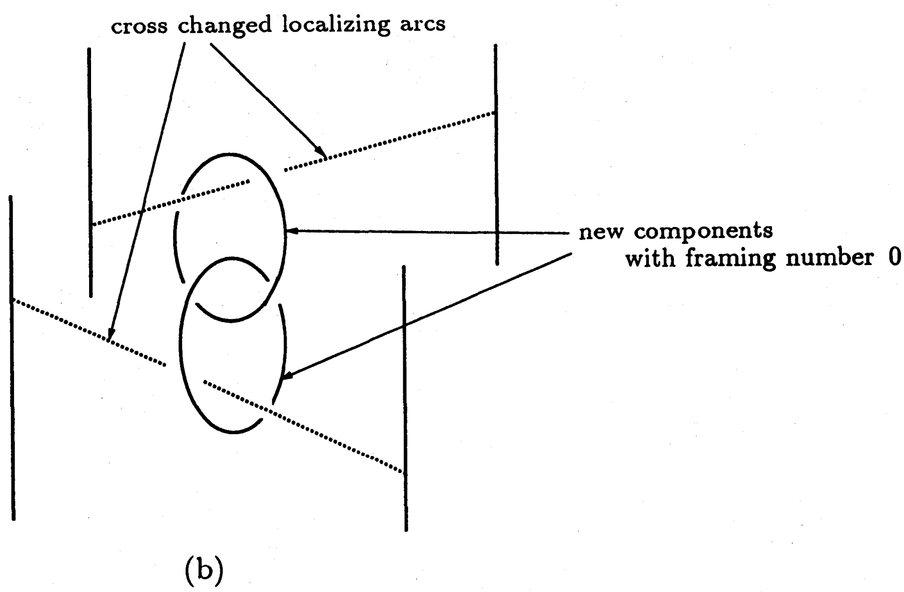


FIGURE 3.1. Cross change for localizing arcs.

localizing arc system $\vec{\theta}$ for L, we must add to $\vec{\theta}'$ two arcs indicated in Figure 3.2 for each new component of L. In fact, as is shown in Figure 3.3, we get a link having the same localizing arc system as the original K after applying cross changes along additional arcs.

As is shown in Figure 3.2(b), we can arrange L^* and $\vec{\theta}$ so that L^* is still a pure plat and $\vec{\theta}$ is monotone. Moreover, since the additional localizing arcs can be taken as short as we

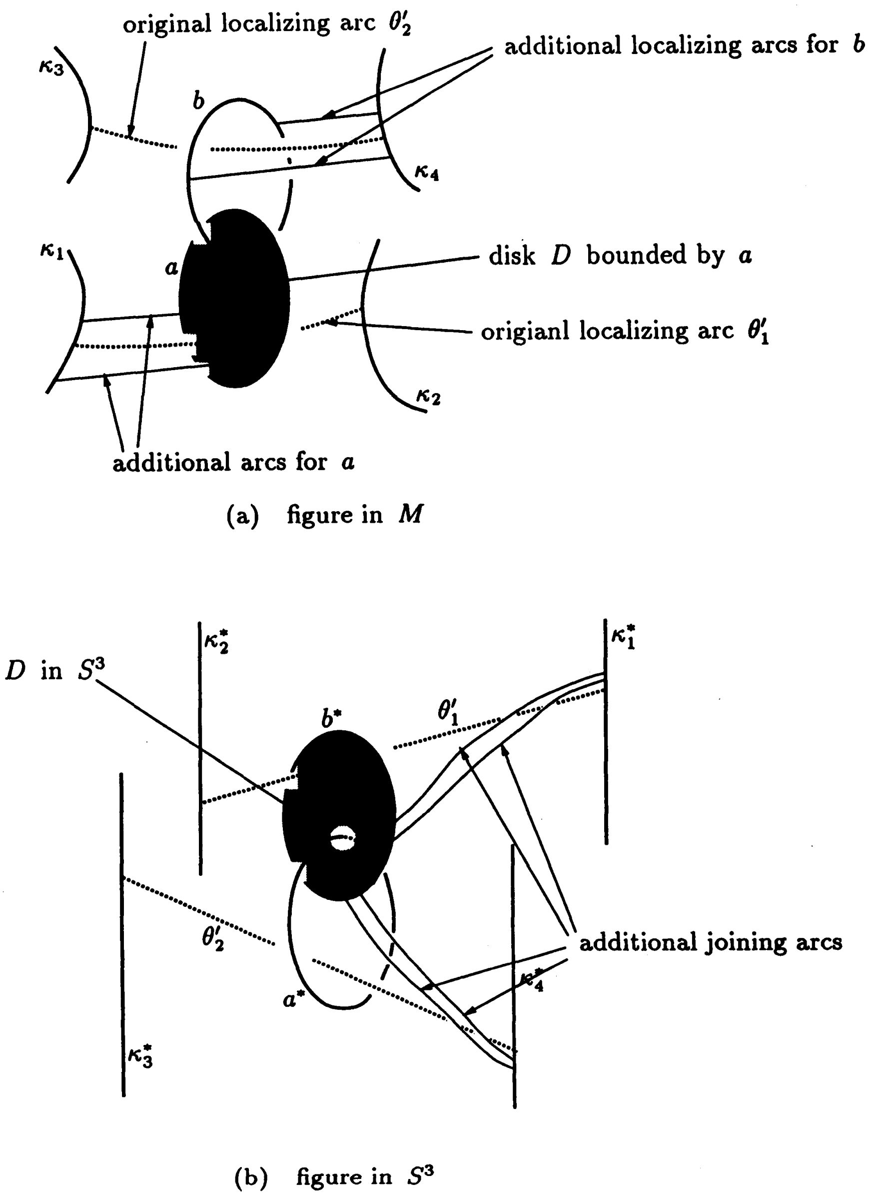


FIGURE 3.2. Additional localizing arcs arising from cross change of arcs.

want (cf. Figure 3.4), we can choose L and $\vec{\theta}$ so that they satisfy also the condition (3). This completes the proof. □

Now we shall show that

LEMMA 3.3. *Any homotopy* 3*-sphere has a very special framed link.*

PROOF. Let L be a framed link in a homotopy 3-sphere M whose existence has been just proved in Lemma 3.2, and let $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n_1})$ be its localizing arc system which is

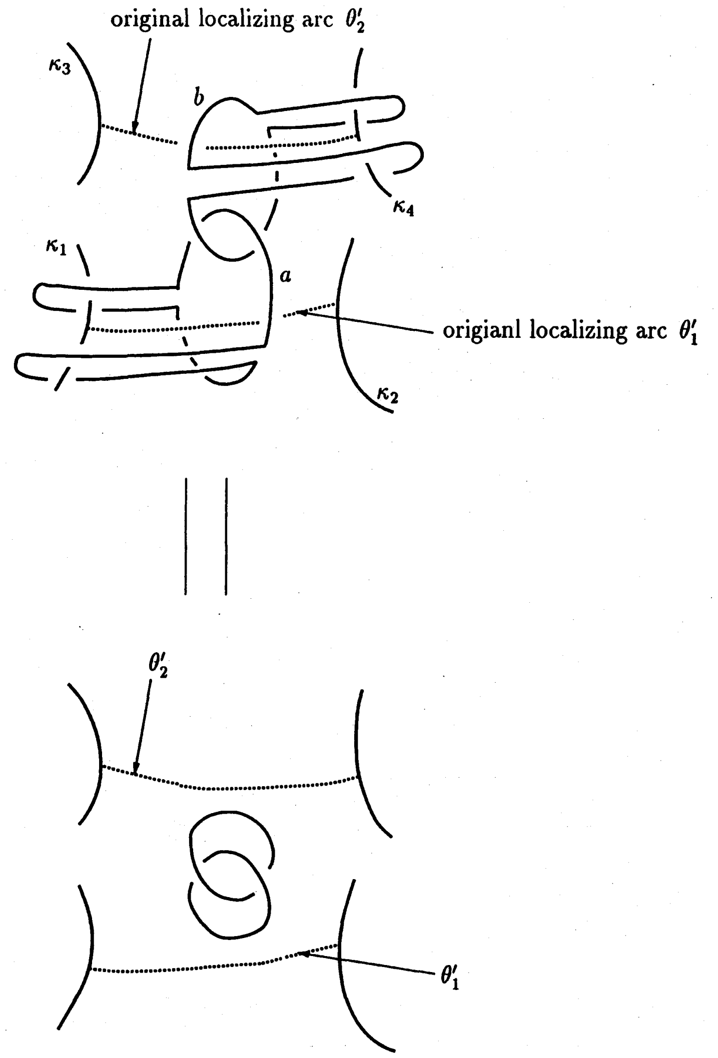


FIGURE 3.3. The link applied cross changes along additional localizing arcs.

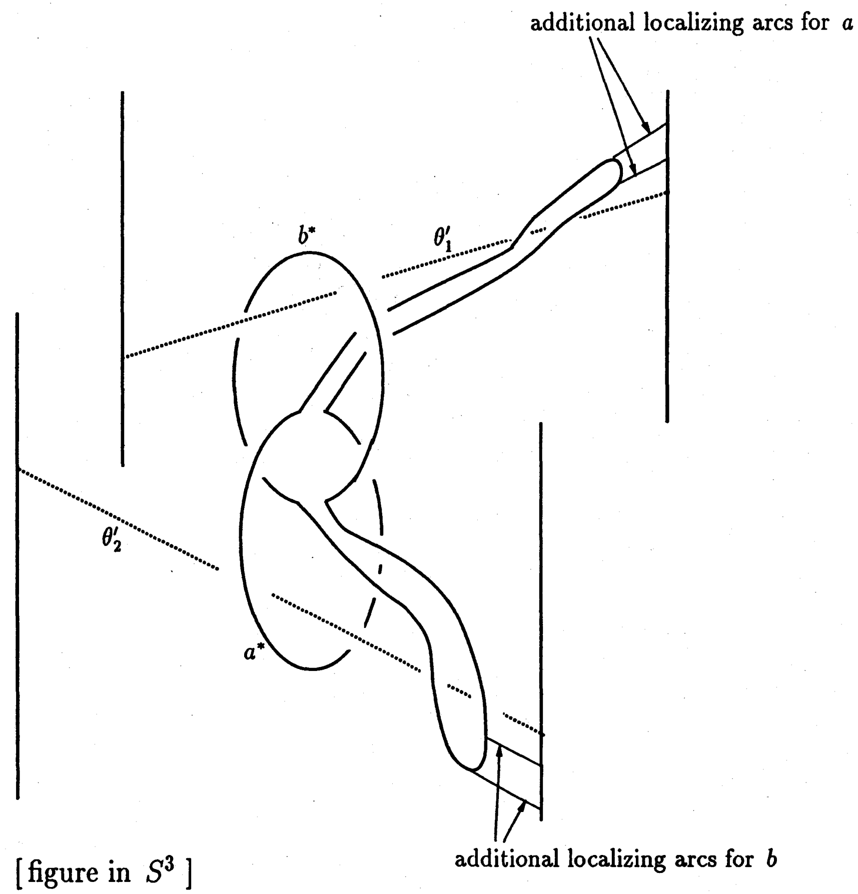
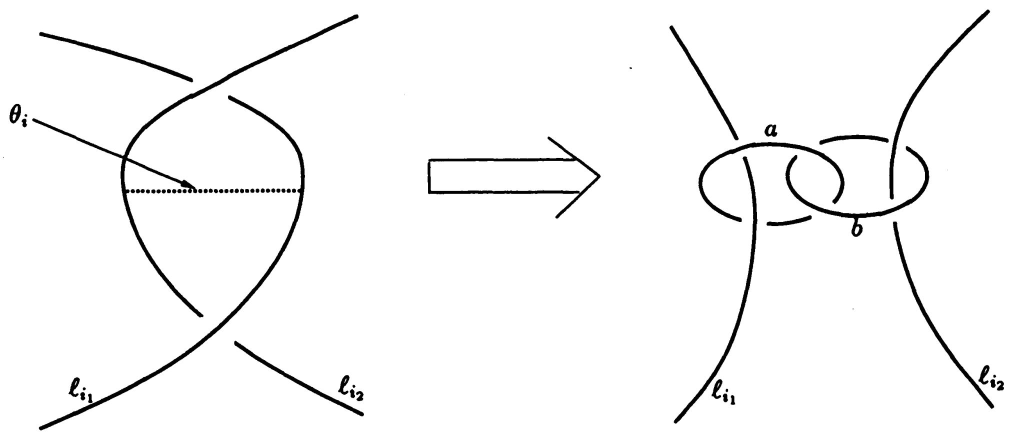


FIGURE 3.4. Additional localizing arcs in S^3.

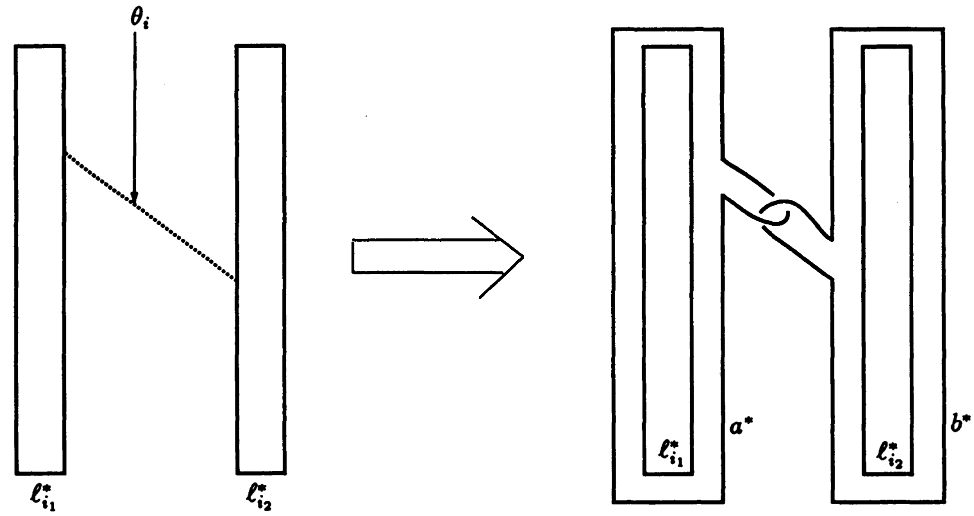
monotone and separated. First we shall show that we can modify L and $\vec{\theta}$ so that, besides the conditions (1)–(3) in Lemma 3.2, they satisfy the following additional condition (4):

(4) any component of Λ has at most one end point of $\vec{\theta}$ on it, that is, there are mutually different components σ_k and τ_k $(k = 1, 2, \cdots, n_1)$ such that the k-th arc θ_k goes from σ_k to τ_k.

In order to get a framed link satisfying (4), for each localizing arc θ_i $(i = 1, \cdots, m)$, which is assumed to go from ℓ_{i_1} to ℓ_{i_2}, we apply the following three steps (i)–(iii) of Kirby moves and dual Kirby moves (cf. Figures 3.5–3.7).

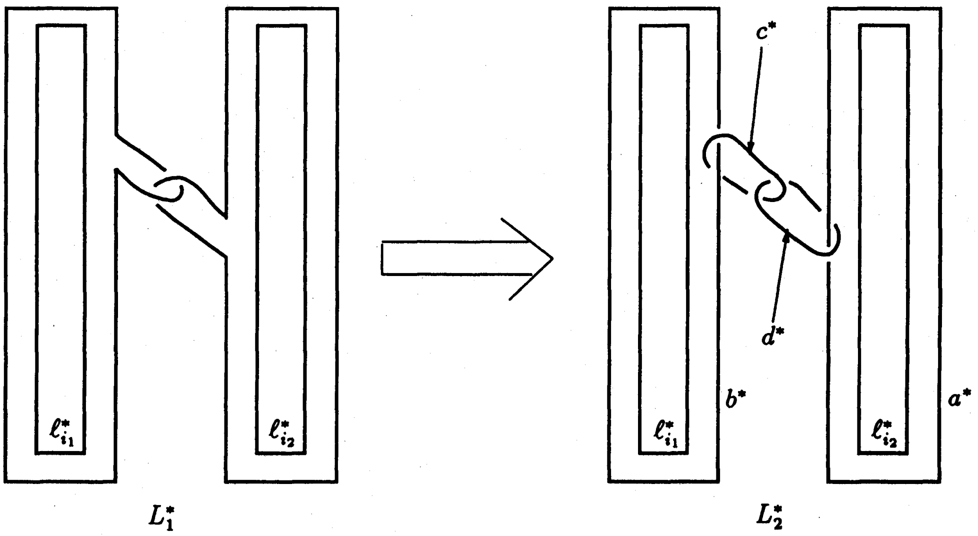


(i) cross change move in M making L local

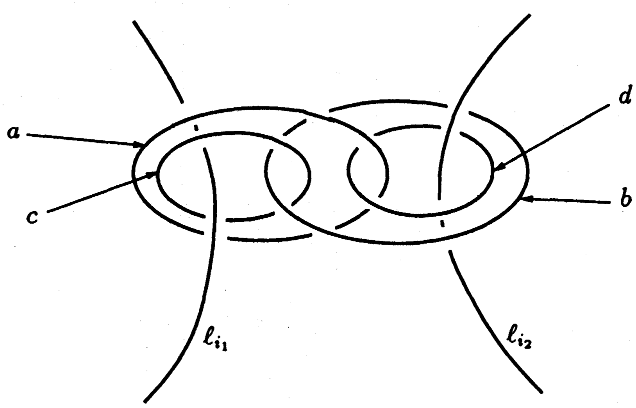


(ii) dual move in S^3

FIGURE 3.5. Moves and dual moves for the condition (4) (1-st step).



(a) cross change move in S^3

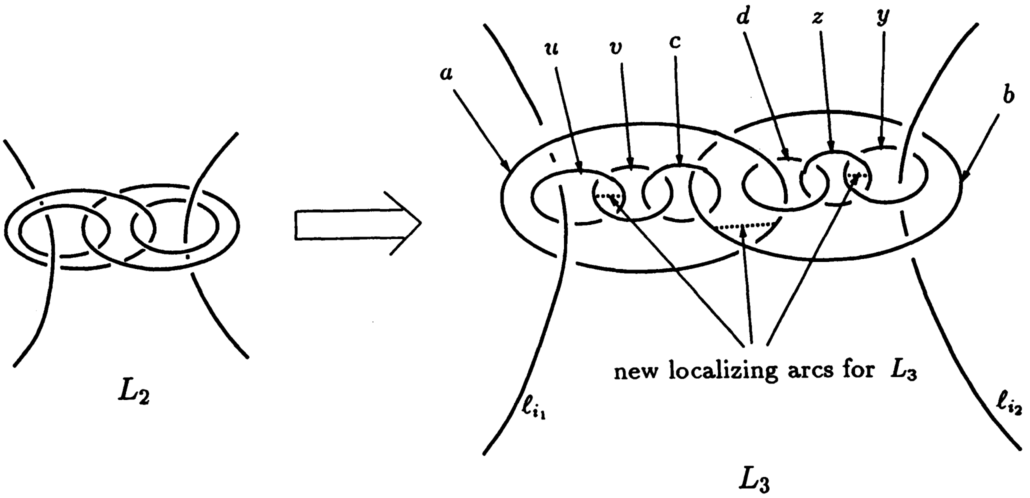


(b) dual link $L_2 \subset M$

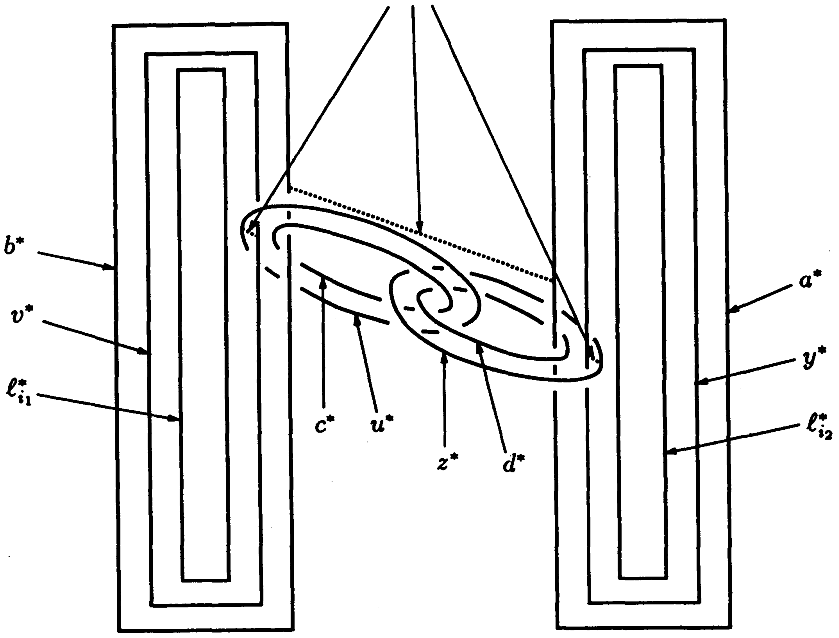
FIGURE 3.6. Moves and dual moves for the condition (4) (2-nd step).

(i) (Cross change moves in M) Along each arc θ_i apply a cross change move as in Figure 3.5(i) to make the link L local. We write L_1 for the new framed link, and a, b for the components of L_1 introduced by this move. As is shown in Figure 3.5(ii), the dual link $L_1^* \subset S^3$ is *not* a pure plat.

(ii) (Cross change move in S^3) To break the crossing between a^* and b^*, we make a cross change move on L_1^* shown in Figure 3.6. Let c^* and d^* be the components introduced for this cross change move. The new framed link L_2^* becomes a pure plat again. The dual moves lead us to the link $L_2 \subset M$ which is dual to L_2^* and illustrated in Figure 3.6.



(a) $L_3 \subset M$ and its localizing arcs



(b) dual link $L_3^* \subset S^3$ and localizing arcs for L_3 within S^3

FIGURE 3.7. Moves and dual moves for the condition (4) (3-rd step).

(iii) (Cross change move in M) To break the linking between c (or d) and the component of L_2, say ℓ_{i_ν} ($\nu = 1, 2$), we make a cross change move on L_2 as in Figure 3.7(a). New components introduced by this cross change move are denoted by u and v (or y and z). These moves result in a new framed link $L_3 \subset M$ as in Figure 3.7(a), and in its dual $L_3^* \subset S^3$ which is drawn in Figure 3.7(b).

Let $\tilde{L}$ be the sublink of L_3 which consists of the components corresponding to the original L. Because $\tilde{L}$ was made local by the above moves, the set of arcs shown in Figure 3.7(a) forms a localizing arc system $\vec{\theta}$ for L_3. In fact, by cross changes along $\vec{\theta}$ we obtain a link $L_3[\vec{\theta}]$ which can be shrunk into a regular neighborhood of $\tilde{L}$. Within S^3, the arcs in $\vec{\theta}$ are arranged as in Figure 3.7(b). Because two of three localizing arcs in Figure 3.7(b) can be taken as short as we want, making isotopic deformations if necessarily, we can see that the framed link L_3 and the localizing arc system $\vec{\theta}$ satisfy the required condition (4).

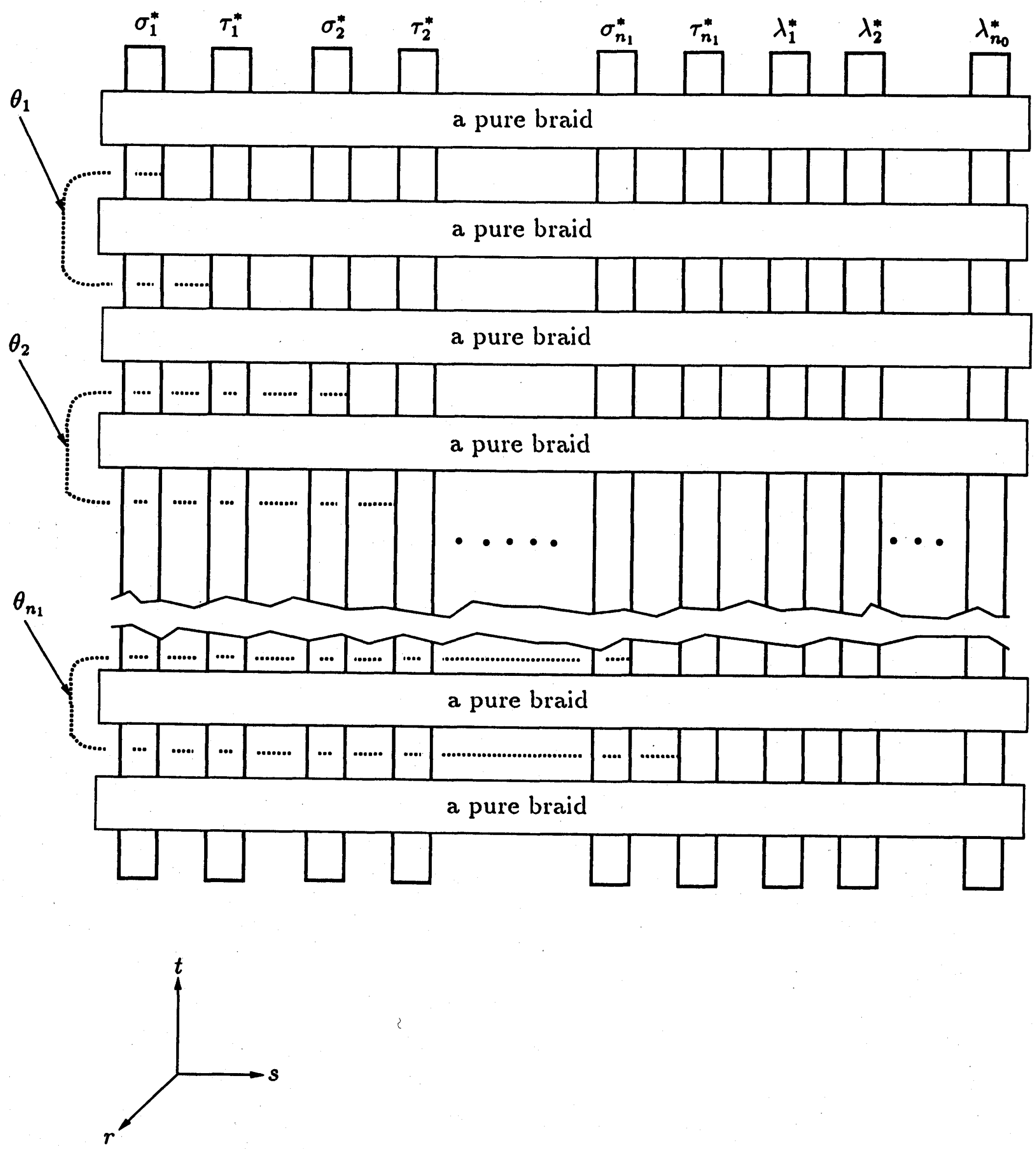


FIGURE 3.8. Well arranged $L^* \subset S^3$.

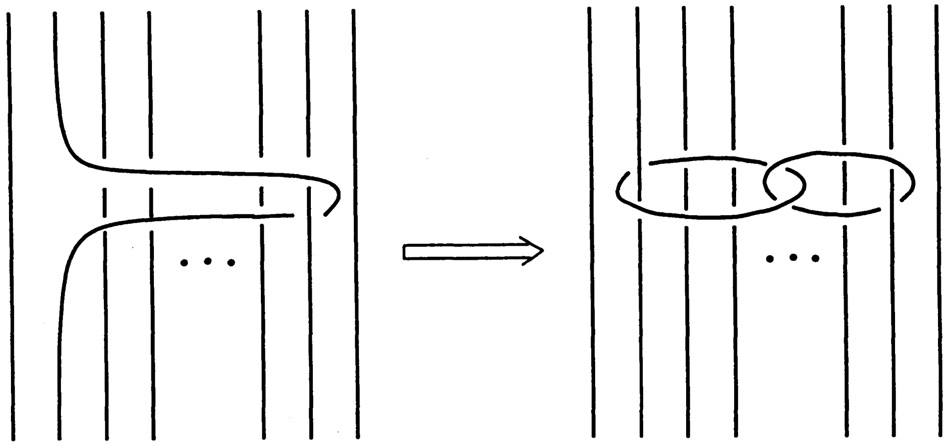


FIGURE 3.9. An elementary pure braid and the cross change move for making it a vh-plat.

Notice that the framing numbers of L_3^* are still all even, because all the new components introduced by the above moves have 0 framing and the other framing numbers are altered by the formula (1.1).

Rewriting L_3 by L, we may assume that L and $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n_1})$ satisfy the conditions (1)–(3) in Lemma 3.2 and the additional condition (4), namely there are $2n_1$ mutually different components σ_k and τ_k ($k = 1, 2, \cdots, n_1$) such that the k-th arc θ_k goes from σ_k down to τ_k. Then, by a horizontal isotopic deformation, we can arrange the dual link $L^* \subset S^3$ and $\vec{\theta}$ (correctly speaking $\vec{\theta} \cap E(L^*, S^3)$) as in Figure 3.8.

In order to make L^* a vh-plat, we employ the same method as in [1]. Since a pure braid is generated by an elementary pure braid (cf. Figure 3.9), breaking crossing of elementary pure braid by a cross change move in the manner illustrated in Figure 3.9, we can get a new framed link Λ^* which forms a vh-plat and still gives a surgery description of M. The components introduced by these cross change moves form the horizontal part Λ_h^*. The framing numbers of the horizontal part Λ_h^* are all 0, and those of the vertical part Λ_v^* are all even.

In order to adjust the framing numbers of the vertical part to 0, we make final moves indicated in Figure 3.10, which is a composition of isotopic deformations of vertical components and cross change moves. Indeed, since each vertical component has an even framing number, the formula (1.1) shows that, taking an adequate number of left or right full twists in the first move in Figure 3.10, we get a new framed link, which we denote by the same letter Λ^*, whose dual framed link $\Lambda \subset M$ satisfies the conditions (a), (b), (c) in Definition 1.3.

Because the dual moves for getting Λ from L do not alter the link L, the vertical part Λ_v is the same link as the original L. This implies that $\vec{\theta}$ is a localizing arc system also for Λ_v. Hence Λ satisfies also the condition (d) in Definition 1.3, that is, Λ is a very special framed link in M.

This completes the proof of Lemma 3.3, and so of Theorem 0.1. $\square$

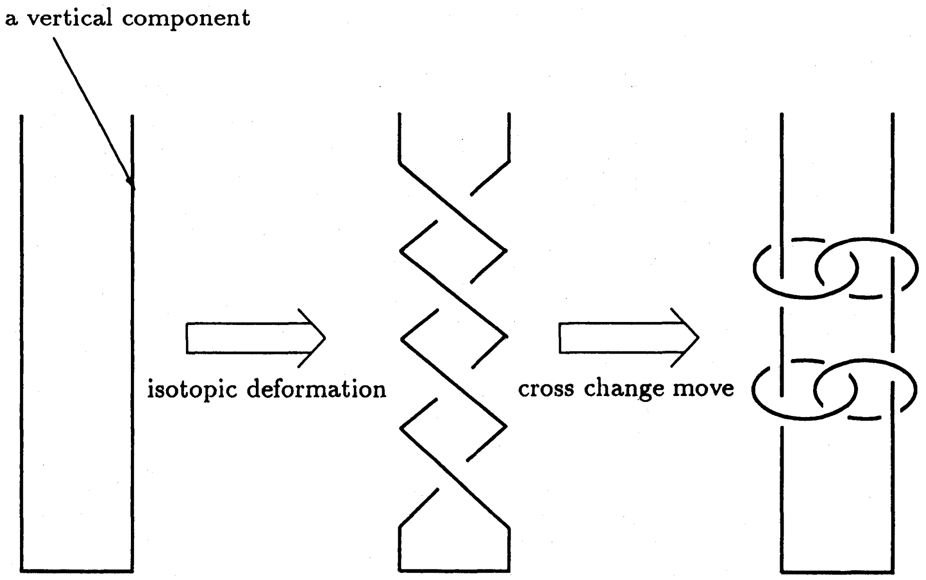


FIGURE 3.10. Moves for adjusting the framing number of a vertical component to 0.

4. A very special framed link and a Heegaard splitting.

In this section, we investigate on a Heegaard splitting of a homotopy 3-sphere M which is induced by a very special framed link Λ in M, whose dual link Λ^* forms a very special framed link. For definiteness of our argument, we recall the notation used in this section.

• We assume that the vertical part Λ_v has $(n_0 + 2n_1)$ components and a localizing arc system consisting of n_1 arcs, and that the horizontal part Λ_h consists of $2n_2$ components.

• The components of the vertical part Λ_v are denoted by

$$\lambda_i \quad (i = 1, 2, \cdots, n_0), \quad \sigma_k, \ \tau_k \quad (k = 1, 2, \cdots, n_1),$$

and the k-th arc θ_k of the localizing arc system $\vec{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n_1})$ of Λ_v is assumed to go from σ_k down to τ_k.

• The components of Λ_h are denoted by

$$x_{[1,j]}, \ x_{[2,j]} \quad (j = 1, 2, \cdots, n_2),$$

where, for each j, their dual components $x^*_{[1,j]}$ and $x^*_{[2,j]}$ are those which form a Hopf link along the j-th guide-arc.

4.1. A Heegaard splitting and its generalized core. We will define a Heegaard splitting of M by a 1-complex Γ^* in S^3 such that Γ^* includes the vh-plat Λ^* and the exterior $E(\Gamma^*, S^3)$ is a handle body. Because $H_1 \equiv E(\Gamma^*, S^3)$ can be viewed as a handle body in M and $H_2 \equiv \mathrm{cl}(M - H_1)$, which is a regular neighborhood of some 1-complex in M, is also a handle body, such a 1-complex Γ^* determines a Heegaard splitting $\mathcal{H} = (H_1, H_2)$ of M. Such a 1-complex Γ^* can be defined by adding some arcs to Λ^*. The added arcs are given as the following (i)–(iii).

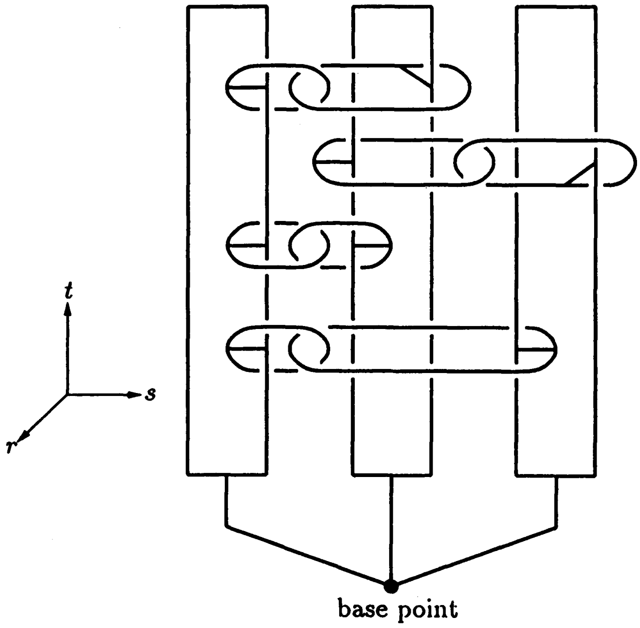


FIGURE 4.1. 1-complex Γ^*.

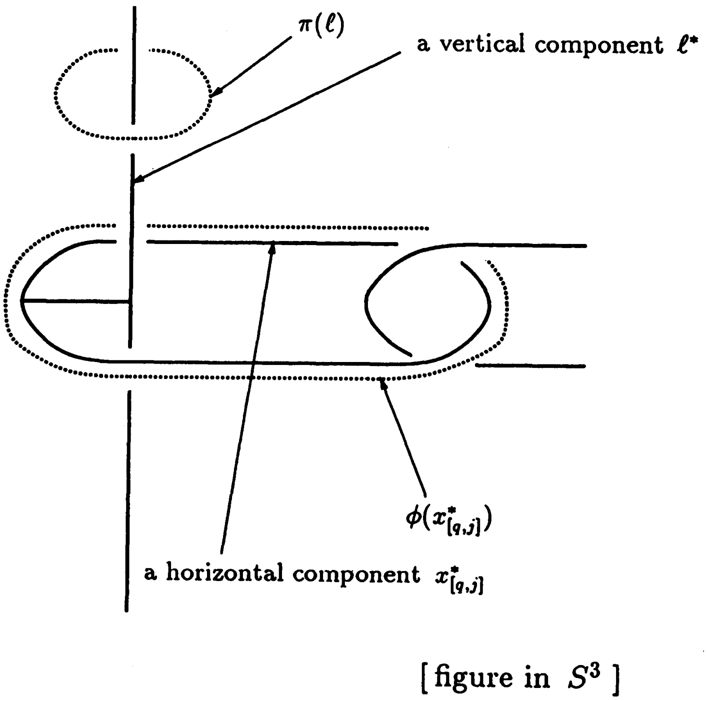


FIGURE 4.2. $\phi(\ell^*)$ and $\pi(\ell)$.

(i) Take a base point P_0 on the plane $\{r = 0\}$ so that it has t-coordinate less than any point on Λ^*.

(ii) For each vertical component ℓ^* ($= \lambda_i^*$ or σ_k^* or τ_k^*), take an arc $\rho(\ell^*)$ on $\{r = 0\}$ which connects the bottom point of ℓ^* down to the base point P_0.

(iii) For each horizontal component $x^*_{[q,j]}$, there is a unique vertical component ℓ^* linking with $x^*_{[q,j]}$. We take an arc $\rho(x^*_{[q,j]})$ on $\{r = 0\}$ which connects $x^*_{[q,j]}$ horizontally to ℓ^*.

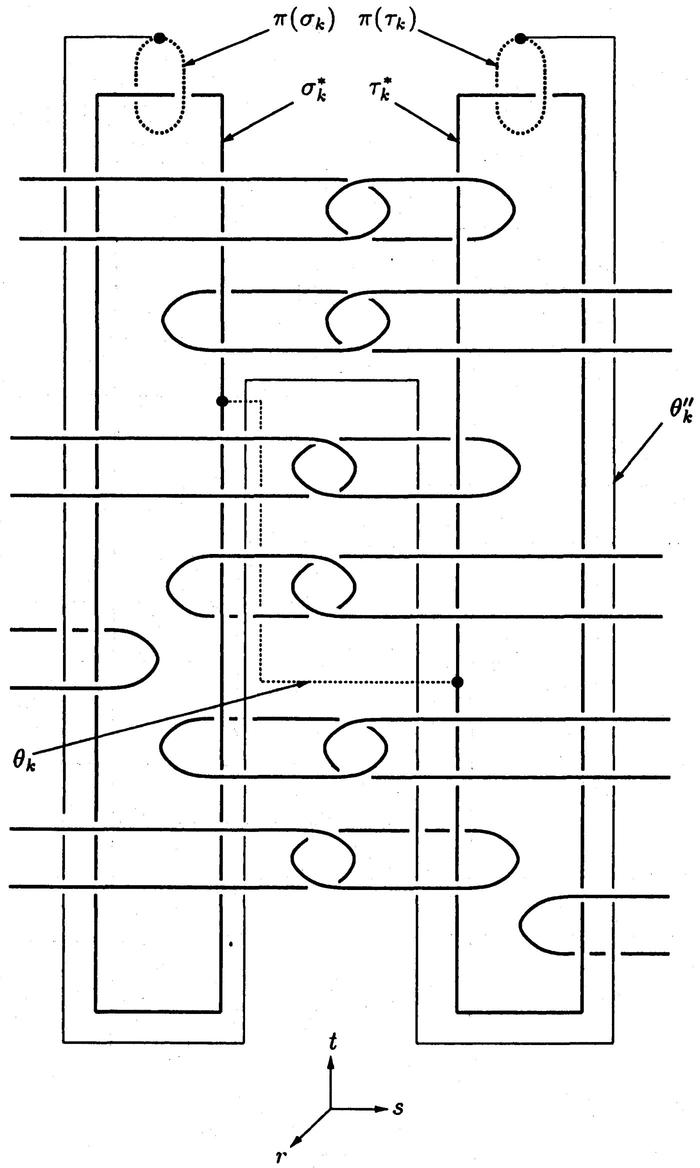


FIGURE 4.3. A localizing arc for $\mathcal{L}$.

Adding all the above defined arcs $\rho(\kappa^*)$, we obtain a 1-complex Γ^* as in Figure 4.1. It can be easily seen that the exterior $H_1 \equiv E(\Gamma^*, S^3)$ is a handle body. Hence, as is noticed above, this 1-complex Γ^* determines a Heegaard splitting of M with genus $g(\mathcal{H}) = n_0 + 2n_1 + 2n_2$.

Now we shall give a *generalized core* of H_1, namely a link $\mathcal{L}$ in the handle body H_1 for which there exists a 1-complex $T \subset H_1$ such that $\mathcal{L} \subset T$ and $H_1 \searrow T$ (see [2] for the precise definition). For a component κ of a very special framed link Λ, we denote by $\pi(\kappa)$ a meridian of $N(\kappa^*, S^3)$, and by $\phi(\kappa^*)$ a framing curve of the dual component κ^* (cf. Figure 4.2). We may assume that $\pi(\kappa)$ is included in the interior of H_1, and that $\phi(\kappa^*)$ lies on the boundary ∂H_1. Define a link $\mathcal{L}$ in H_1 so that it consists of $g(\mathcal{H})$ components

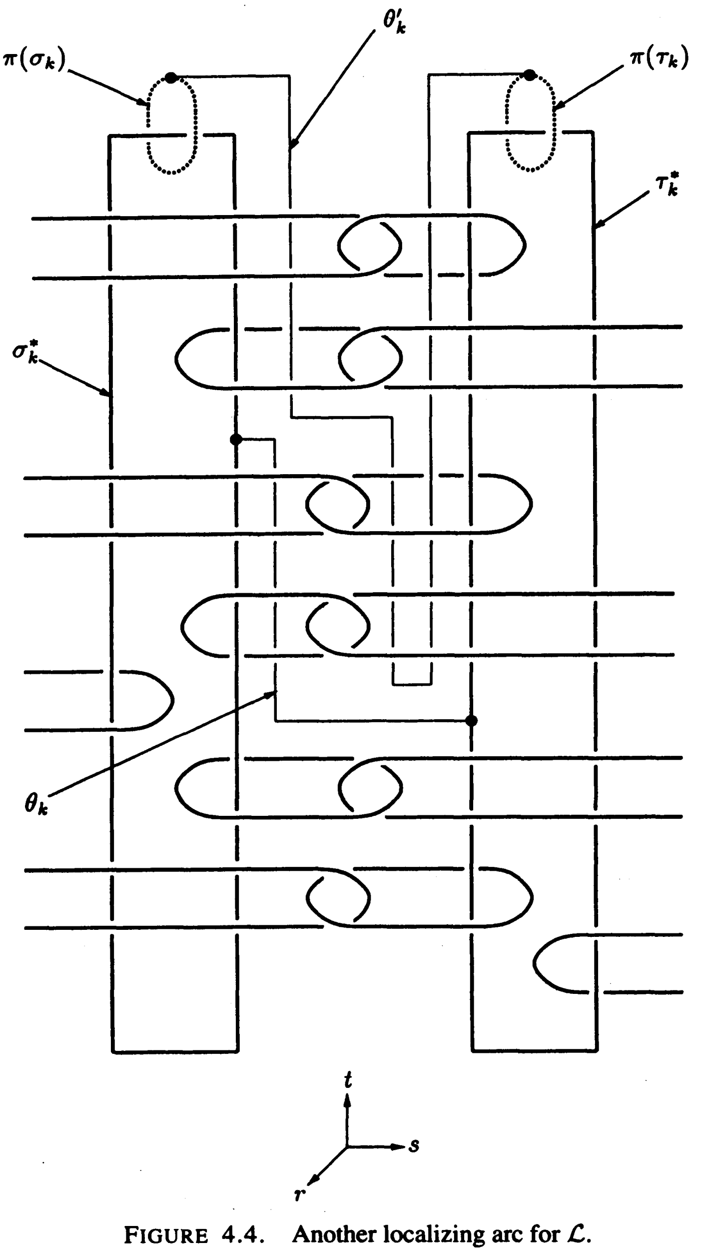


FIGURE 4.4. Another localizing arc for $\mathcal{L}$.

$$\pi(\lambda_i) \quad (i = 1, 2, \cdots, n_0), \quad \pi(\sigma_k), \ \pi(\tau_k) \quad (k = 1, 2, \cdots, n_1),$$
$$\phi(x^*_{[q,j]}) \quad (q = 1, 2, \ j = 1, 2, \cdots, n_2),$$

where $\pi(\ell)$ ($\ell = \lambda_i$ or σ_k or τ_k) is taken at the top of the vertical component ℓ^*.

Using the results in §1 of [2], we can see that the above defined $\mathcal{L}$ is a generalized core of H_1. It is shown in [2] that a localizing arc system for $\mathcal{L}$ plays an important role to see how extent the Heegaard splitting can be reduced. Hence we investigate on a localizing arc system for $\mathcal{L}$ in the next subsection.

4.2. A localizing arc system for the generalized core $\mathcal{L}$. Let $\mathcal{L}_0$ be the sublink of $\mathcal{L}$ consisting of the components

$$\pi(\lambda_i) \quad (i = 1, 2, \cdots, n_0), \quad \pi(\sigma_k), \ \pi(\tau_k) \quad (k = 1, 2, \cdots, n_1).$$

Since $\pi(\kappa)$ is parallel to κ within M for any component κ of Λ, $\mathcal{L}_0$ is isotopic to the vertical part Λ_v within M. On the other hand, Λ_v has a localizing arc system $\vec{\theta}$ whose k-th localizing arc θ_k runs within S^3 as in Figure 4.3. Hence $\mathcal{L}_0$ has a localizing arc system $\vec{\theta}'$ with the k-th arc θ'_k as in Figure 4.3. Because the framing curve $\phi(x^*_{[q,j]})$ bounds a 2-disk in M which is disjoint from the other components of $\mathcal{L}$, also $\mathcal{L}$ has $\vec{\theta}'$ as its localizing arc system.

An isotopic deformation of $\vec{\theta}'$ within $M - \mathcal{L}$ yields again a localizing arc system of $\mathcal{L}$. For example, the arcs θ''_k as in Figure 4.4 form a localizing arc system of $\mathcal{L}$. This example shows that there is some flexibility in a choice of a localizing arc system for $\mathcal{L}$. An adequate choice of a localizing system might lead us to a Heegaard splitting of a homotopy 3-sphere which can be sufficiently reduced.

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Present Address:
DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY,
HIYOSHI, YOKOHAMA, 223–8522 JAPAN.
e-mail: ishii@math.keio.ac.jp