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Between Lie Norm and Dual Lie Norm

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Introduction.

In August 1999, we discussed the double series expansion of holomorphic functions on the dual Lie ball ([2]). Looking at our results we conjectured that there was a series of norms between the Lie norm and the dual Lie norm.

The Lie norm $L(z)$ on $\mathbf{C}^n$ is defined by

$$L(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}}\,, \tag{1}$$

where $\|z\|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$, $\quad z^2 = z_1^2 + z_2^2 + \cdots + z_n^2$ for $z = (z_1, z_2, \cdots, z_n)$.

The dual Lie norm $L^*(z)$ is defined as follows: $L^*(z) = \sup\{|z \cdot \zeta|;\ L(\zeta) \le 1\}$, where $z \cdot \zeta = z_1\zeta_1 + z_2\zeta_2 + \cdots + z_n\zeta_n$ for $z = (z_1, z_2, \cdots, z_n)$ and $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$. $L^*(z)$ has the following expression:

$$L^*(z) = \sqrt{(\|z\|^2 + |z^2|)/2} = \frac{1}{2}\left(L(z) + \frac{|z^2|}{L(z)}\right).$$

Noting $|z^2|/L(z) = \sqrt{\|z\|^2 - \sqrt{\|z\|^4 - |z^2|^2}}$, we can write

$$L^*(z) = \frac{1}{2}\left(\sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}} + \sqrt{\|z\|^2 - \sqrt{\|z\|^4 - |z^2|^2}}\right) \tag{2}$$

(see [1] and [5]).

For $p \ge 1$, we define the function $N_p(z)$ on $\mathbf{C}^n$ as follows:

$$N_p(z) = \left\{\frac{1}{2}\left((\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2})^{p/2} + (\|z\|^2 - \sqrt{\|z\|^4 - |z^2|^2})^{p/2}\right)\right\}^{1/p}.$$

It is clear that $N_2(z)$ is equal to the Euclidean norm $\|z\|$. We have $N_1(z) = L^*(z)$ by (2) and $\lim_{p\to\infty} N_p(z) = L(z)$ by (1). If $n = 2$, then $N_p(z)$ is equivalent to the Lebesgue L^p norm (see (5)).

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We shall prove in this note that $N_p(z)$ are norms on $\mathbf{C}^n$ for $p \geq 1$ and that $N_q(z)$ is the dual norm of $N_p(z)$, where $1/p + 1/q = 1$ (see Theorem 13 and Corollary 14).

In Section 1, we shall prove that $L(z)$ and $N_1(z)$ are norms on $\mathbf{C}^n$ and dual to each other. Our new proof relies on the rotation invariance and is reduced to the 2 dimensional case. In Section 2, using the same idea we shall prove our main theorems. T. Kimura ([4]) proved that $N_p(z)$ are norms on $\mathbf{C}^n$. Our proof is different from his.

1. Lie norm and dual Lie norm.

For $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$ and $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{C}^n$ we denote the canonical bilinear form by

$$z \cdot \zeta = z_1\zeta_1 + \cdots + z_n\zeta_n ,$$

$z^2 = z \cdot z$ and $\|z\|^2 = z \cdot \bar{z}$, where $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_n)$ is the complex conjugate of z. Further we define

$$\begin{aligned} L(z) &= \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}} , \\ M(z) &= \sqrt{\|z\|^2 - \sqrt{\|z\|^4 - |z^2|^2}} , \\ N_1(z) &= \frac{1}{2}(L(z) + M(z)) . \end{aligned} \tag{3}$$

It is clear that $L(x) = M(x) = N_1(x) = \|x\|$ for $x \in \mathbf{R}^n$.

It is known that $L(z)$ and $N_1(z)$ are norms on the complex vector space $\mathbf{C}^n$ and dual to each other. $L(z)$ is called the Lie norm and $N_1(z)$ is equal to the dual Lie norm $L^*(z)$. In this section, we give a new proof of these facts relying only on (3).

LEMMA 1. *For $z \in \mathbf{C}^n$ we have*

(a) $L(z) \geq M(z)$ *and* $L(z)M(z) = |z^2|$,

(b) $N_1(z) = \sqrt{\frac{1}{2}(\|z\|^2 + |z^2|)}$,

(c) $L(z) \geq \|z\| \geq N_1(z) \geq \frac{1}{2}L(z)$.

PROOF. (a) is clear. By (3) we have

$$N_1(z)^2 = \frac{1}{4}(L(z)^2 + 2L(z)M(z) + M(z)^2) = \frac{1}{2}(\|z\|^2 + |z^2|) ,$$

which implies (b). (c) results from (3) and (b). □

The following lemma asserts the complex homogeneity of $L(z)$ and $N_1(z)$.

LEMMA 2. *For $\lambda \in \mathbf{C}$ and $z \in \mathbf{C}^n$ we have*

$$L(\lambda z) = |\lambda| L(z) , \quad M(\lambda z) = |\lambda| M(z) , \quad N_1(\lambda z) = |\lambda| N_1(z) .$$

PROOF. Lemma results from $\|\lambda z\| = |\lambda|\|z\|$ and $(\lambda z)^2 = \lambda^2 z^2$. □

In the sequel we identify $\mathbf{C}^2$ with the subspace

$$\{z = (z_1, \cdots, z_n) \in \mathbf{C}^n;\ z_3 = z_4 = \cdots = z_n = 0\}.$$

For $z = (z_1, z_2, z_3, \cdots, z_n) \in \mathbf{C}^n$ we denote by $\tilde{z}$ the projection of z to $\mathbf{C}^2$:

$$\tilde{z} = (z_1, z_2, 0, \cdots, 0).$$

We put $\hat{z} = z - \tilde{z} = (0, 0, z_3, \cdots, z_n)$. Then we have

$$z^2 = \tilde{z}^2 + \hat{z}^2 \quad \text{and} \quad \|z\|^2 = \|\tilde{z}\|^2 + \|\hat{z}\|^2.$$

LEMMA 3. *For $z \in \mathbf{C}^n$ we have*
(a) $\|\tilde{z}\|^2 - |\tilde{z}^2| \leq \|z\|^2 - |z^2|$,
(b) $\|\tilde{z}\|^2 + |\tilde{z}^2| \leq \|z\|^2 + |z^2|$,
(c) $L(\tilde{z}) \leq L(z)$,
(d) $N_1(\tilde{z}) \leq N_1(z)$.

PROOF. (a) We have

$$\begin{aligned}\|z\|^2 - |z^2| &= \|\tilde{z}\|^2 + \|\hat{z}\|^2 - |\tilde{z}^2 + \hat{z}^2| \\ &\geq \|\tilde{z}\|^2 + \|\hat{z}\|^2 - |\tilde{z}^2| - |\hat{z}^2| \geq \|\tilde{z}\|^2 - |\tilde{z}^2|.\end{aligned}$$

(b) We have

$$\begin{aligned}\|z\|^2 + |z^2| &= \|\tilde{z}\|^2 + \|\hat{z}\|^2 + |\tilde{z}^2 + \hat{z}^2| \\ &\geq \|\tilde{z}\|^2 + \|\hat{z}\|^2 + |\tilde{z}^2| - |\hat{z}^2| \geq \|\tilde{z}\|^2 + |\tilde{z}^2|.\end{aligned}$$

(c) By (3), (a) and (b) imply $L(z)^2 \geq L(\tilde{z})^2$, and $L(z) \geq L(\tilde{z})$.
(d) By Lemma 1, (b) implies $N_1(z)^2 \geq N_1(\tilde{z})^2$, and $N_1(z) \geq N_1(\tilde{z})$. □

LEMMA 4. (a) *For $a \geq b \geq 0$ and $z \in \mathbf{C}^n$ we have*

$$aL(\tilde{z}) + bM(\tilde{z}) \leq aL(z) + bM(z).$$

(b) *For $z, \zeta \in \mathbf{C}^n$ we have*

$$L(\tilde{z})L(\zeta) + M(\tilde{z})M(\zeta) \leq L(z)L(\zeta) + M(z)M(\zeta).$$

PROOF. (a) By Lemma 3 (c) and (d) we have

$$\begin{aligned}aL(\tilde{z}) + bM(\tilde{z}) &= (a-b)L(\tilde{z}) + 2bN_1(\tilde{z}) \\ &\leq (a-b)L(z) + 2bN_1(z) = aL(z) + bM(z).\end{aligned}$$

(b) Because $L(\zeta) \geq M(\zeta)$ (Lemma 1 (a)), (b) results from (a). □

We denote by $O(n)$ the orthogonal group. It is known that $O(n)$ acts transitively on the real unit sphere $\mathbf{S} = \{x \in \mathbf{R}^n;\ \|x\| = 1\}$. For complex vectors we have the following

LEMMA 5. *For any $z \in \mathbf{C}^n$ there exists $T \in O(n)$ such that $Tz \in \mathbf{C}^2$.*

PROOF. For $z = x+iy,\ x, y \in \mathbf{R}^n$ take $T_1 \in O(n)$ such that $a = T_1x = (a_1, 0, \cdots, 0)$. Then we can find $T_2 \in O(n)$ such that

$$T_2e_1 = e_1 = (\text{the first unit vector}),$$
$$b = T_2(T_1y) = (b_1, b_2, 0, \cdots, 0).$$

Take $T = T_2T_1$. Then we have $Tz = a + ib = (a_1 + ib_1, ib_2, 0, \cdots, 0)$. □

LEMMA 6. *For $T \in O(n)$ and $z \in \mathbf{C}^n$ we have*

$$L(Tz) = L(z), \quad M(Tz) = M(z), \quad N_1(Tz) = N_1(z).$$

PROOF. Take $z = x+iy,\ \zeta = \xi + i\eta, x, y, \xi, \eta \in \mathbf{R}^n$. Then we have $Tz = Tx + iTy$, $T\zeta = T\xi + iT\eta$, and hence

$$\begin{aligned} Tz \cdot T\zeta &= Tx \cdot T\xi - Ty \cdot T\eta + i(Tx \cdot T\eta + Ty \cdot T\xi) \\ &= x \cdot \xi - y \cdot \eta + i(x \cdot \eta + y \cdot \xi) = z \cdot \zeta. \end{aligned}$$

Because $\overline{Tz} = T\bar{z}$, we have $\|Tz\|^2 = Tz \cdot \overline{Tz} = Tz \cdot T\bar{z} = z \cdot \bar{z} = \|z\|^2$. Therefore, Lemma results from (3). □

THEOREM 7. *For $z, \zeta \in \mathbf{C}^n$ we have*

$$|z \cdot \zeta| \le L(z)N_1(\zeta).$$

PROOF. Suppose first $z, \zeta \in \mathbf{C}^2$. Then we have

$$\|z\|^4 - |z^2|^2 = (z_1\bar{z}_1 + z_2\bar{z}_2)^2 - (z_1^2 + z_2^2)(\bar{z}_1^2 + \bar{z}_2^2) = -(z_1\bar{z}_2 - z_2\bar{z}_1)^2.$$

Therefore,

$$L(z) = \max\sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 \pm i(z_1\bar{z}_2 - z_2\bar{z}_1)} = \max\{|z_1 \pm iz_2|\}.$$

Similarly, we have

$$M(z) = \min\{|z_1 \pm iz_2|\}, \quad N_1(z) = \frac{1}{2}(|z_1 + iz_2| + |z_1 - iz_2|).$$

On the other hand, we have

$$z \cdot \zeta = z_1\zeta_1 + z_2\zeta_2 = \frac{1}{2}\{(z_1 + iz_2)(\zeta_1 - i\zeta_2) + (z_1 - iz_2)(\zeta_1 + i\zeta_2)\}$$

and hence

$$\begin{aligned} |z \cdot \zeta| &\le \frac{1}{2}\{|z_1 + iz_2||\zeta_1 - i\zeta_2| + |z_1 - iz_2||\zeta_1 + i\zeta_2|\} \\ &\le \max\{|z_1 \pm iz_2|\} \times \frac{1}{2}(|\zeta_1 - i\zeta_2| + |\zeta_1 + i\zeta_2|) = L(z)N_1(\zeta). \end{aligned}$$

Suppose now $z, \zeta \in \mathbf{C}^n$. By Lemma 5 there exists $T \in O(n)$ such that $\alpha = T\zeta \in \mathbf{C}^2$. Put $w = Tz$. By the rotation invariance, we have $|z \cdot \zeta| = |w \cdot \alpha|$. Then by the first step, we have

$$|w \cdot \alpha| = |\tilde{w} \cdot \alpha| \leq L(\tilde{w}) N_1(\alpha).$$

By Lemma 3 (c) and Lemma 6 we have

$$|z \cdot \zeta| = |w \cdot \alpha| \leq L(\tilde{w}) N_1(\alpha) \leq L(w) N_1(\alpha) = L(z) N_1(\zeta). \qquad \square$$

THEOREM 8. *For $\zeta \in \mathbf{C}^n$ we have*

$$\begin{aligned} N_1(\zeta) &= \sup\{|z \cdot \zeta|;\ z \in \mathbf{C}^n, L(z) = 1\} \\ &= \sup\{|x \cdot \zeta|;\ x \in \mathbf{R}^n, \|x\| = 1\}. \end{aligned}$$

PROOF. By Theorem 7 we have

$$\begin{aligned} N_1(\zeta) &\geq \sup\{|z \cdot \zeta|;\ z \in \mathbf{C}^n, L(z) = 1\} \\ &\geq \sup\{|x \cdot \zeta|;\ x \in \mathbf{R}^n, \|x\| = 1\}. \end{aligned}$$

Suppose first $\zeta = (\zeta_1, \zeta_2, 0, \cdots, 0) \in \mathbf{C}^2$. Then we have

$$\begin{aligned} &\sup\{|x \cdot \zeta|;\ x \in \mathbf{R}^n, \|x\| = 1\} \\ &\geq \sup\left\{\frac{1}{2}|(x_1 + ix_2)(\zeta_1 - i\zeta_2) + (x_1 - ix_2)(\zeta_1 + i\zeta_2)|;\ x \in \mathbf{R}^2, \|x\| = 1\right\} \\ &= \sup\left\{\frac{1}{2}|e^{i\theta}(\zeta_1 - i\zeta_2) + e^{-i\theta}(\zeta_1 + i\zeta_2)|;\ \theta \in \mathbf{R}\right\} \\ &= \frac{1}{2}(|\zeta_1 - i\zeta_2| + |\zeta_1 + i\zeta_2|) = N_1(\zeta). \end{aligned}$$

Suppose now $\zeta \in \mathbf{C}^n$. By Lemma 5 there exists $T \in O(n)$ such that $T\zeta \in \mathbf{C}^2$. Then by Lemma 6 and the first step we have

$$\begin{aligned} &\sup\{|x \cdot \zeta|;\ x \in \mathbf{R}^n, \|x\| = 1\} \\ &\quad \geq \sup\{|Tx \cdot T\zeta|;\ x \in \mathbf{R}^n, \|x\| = 1\} \\ &\quad \geq \sup\{|y \cdot T\zeta|;\ y \in \mathbf{R}^n, \|y\| = 1\} = N_1(T\zeta) = N_1(\zeta) \qquad \square \end{aligned}$$

THEOREM 9. *For $z \in \mathbf{C}^n$ we have*

$$L(z) = \sup\{|z \cdot \zeta|;\ \zeta \in \mathbf{C}^n, N_1(\zeta) = 1\}.$$

PROOF. By Theorem 7 we have

$$L(z) \geq \sup\{|z \cdot \zeta|;\ \zeta \in \mathbf{C}^n, N_1(\zeta) = 1\}.$$

Suppose first $z \in \mathbf{C}^2$. Then we have

$$\begin{aligned}
&\sup\{|z \cdot \zeta|;\ \zeta \in \mathbf{C}^n, N_1(\zeta) = 1\} \\
&\quad \geq \sup\{|z \cdot \zeta|;\ \zeta \in \mathbf{C}^2, N_1(\zeta) = 1\} \\
&\quad = \sup\left\{\frac{1}{2}|(z_1 + iz_2)(\zeta_1 - i\zeta_2) + (z_1 - iz_2)(\zeta_1 + i\zeta_2)|;\right. \\
&\qquad \left.\frac{1}{2}(|\zeta_1 + i\zeta_2| + |\zeta_1 - i\zeta_2|) = 1\right\} \\
&\quad = \max\{|z_1 \pm iz_2|\} = L(z)\,.
\end{aligned}$$

Suppose now $z \in \mathbf{C}^n$. By Lemma 5 there exists $T \in O(n)$ such that $Tz \in \mathbf{C}^2$. Then by Lemma 6 and the first step, we have

$$\begin{aligned}
&\sup\{|z \cdot \zeta|;\ \zeta \in \mathbf{C}^n, N_1(\zeta) = 1\} \\
&\quad = \sup\{|Tz \cdot T\zeta|;\ \zeta \in \mathbf{C}^n, N_1(\zeta) = 1\} \\
&\quad = \sup\{|Tz \cdot \alpha|;\ \alpha \in \mathbf{C}^n, N_1(\alpha) = 1\} = L(Tz) = L(z)\,. \qquad \square
\end{aligned}$$

COROLLARY 10. *$L(z)$ and $N_1(z)$ are norms on $\mathbf{C}^n$ and dual to each other.*

PROOF. In view of Lemmas 1 and 2, to show $L(z)$ (resp. $N_1(z)$) is a norm on $\mathbf{C}^n$ we have only to show the subadditivity, which results from Theorem 9 (resp. Theorem 8). □

N.B. The Lie norm $L(z)$ is equal to the cross norm of the Euclidean norm $\|x\|$ on $\mathbf{R}^n$:

$$L(z) = \inf\left\{\sum_{j=1}^{M} |\lambda_j|\|x_j\|;\ z = \sum_{j=1}^{M} \lambda_j x_j\,, \lambda_j \in \mathbf{C}, x_j \in \mathbf{R}^n\right\}.$$

This important fact can be proved by the method of this section (see [1] or [5]).

2. Norms between $L(z)$ and $N_1(z)$.

LEMMA 11. *For $z, \zeta \in \mathbf{C}^n$ we have*

$$2|z \cdot \zeta| \leq L(z)L(\zeta) + M(z)M(\zeta)\,. \tag{4}$$

PROOF. Suppose first $z, \zeta \in \mathbf{C}^2$. By Proof of Theorem 7, we have only to show

$$\begin{aligned}
&|z_1 + iz_2||\zeta_1 - i\zeta_2| + |z_1 - iz_2||\zeta_1 + i\zeta_2| \\
&\quad \leq \max\{|z_1 \pm iz_2|\}\max\{|\zeta_1 \pm i\zeta_2|\} + \min\{|z_1 \pm iz_2\}\min\{|\zeta_1 \pm i\zeta_2|\}
\end{aligned}$$

which can be checked easily.

Suppose now $z, \zeta \in \mathbf{C}^n$. By Lemma 5 there exists $T \in O(n)$ such that $\alpha = T\zeta \in \mathbf{C}^2$. Put $w = Tz$. Then we have $|w \cdot \alpha| = |\tilde{w} \cdot \alpha|$ and

$$\begin{aligned}
2|z \cdot \zeta| &= 2|Tz \cdot T\zeta| = 2|w \cdot \alpha| = 2|\tilde{w} \cdot \alpha| \\
&\leq L(\tilde{w})L(\alpha) + M(\tilde{w})M(\alpha) \\
&\leq L(w)L(\alpha) + M(w)M(\alpha) = L(z)L(\zeta) + M(z)M(\zeta)\,,
\end{aligned}$$

where we used the first step and Lemmas 4 and 6. □

For $p \geq 1$ and $z \in \mathbf{C}^n$ we define

$$N_p(z) = \left(\frac{1}{2}(L(z)^p + M(z)^p)\right)^{1/p},$$

where $L(z)$ and $M(z)$ are defined in (3). By Lemma 6 $N_p(z)$ is invariant by rotations; that is, $N_p(Tz) = N_p(z)$ for any $T \in O(n)$ and $z \in \mathbf{C}^n$.

If $1 \leq p \leq r$, then we have $N_p(z) \leq N_r(z)$ (see [3]). In fact, by the Hölder inequality we have

$$\begin{aligned} N_p(z)^p &= \frac{1}{2}(L(z)^p + M(z)^p) = \left(\frac{L(z)^r}{2}\right)^\alpha \left(\frac{1}{2}\right)^{1-\alpha} + \left(\frac{M(z)^r}{2}\right)^\alpha \left(\frac{1}{2}\right)^{1-\alpha} \\ &\leq \left(\frac{L(z)^r}{2} + \frac{M(z)^r}{2}\right)^\alpha \left(\frac{1}{2} + \frac{1}{2}\right)^{1-\alpha} = N_r(z)^p, \end{aligned}$$

where $\alpha = p/r, 0 < \alpha \leq 1$.

For $z \in \mathbf{C}^2$ we have

$$N_p(z) = \left(\frac{1}{2}(|z_1 + iz_2|^p + |z_1 - iz_2|^p)\right)^{1/p}, \tag{5}$$

which is equivalent to the Lebesgue L^p norm. By the Hölder inequality we have

$$\begin{aligned} &|z_1 + iz_2||\zeta_1 - i\zeta_2| + |z_1 - iz_2||\zeta_1 + i\zeta_2| \\ &\quad \leq (|z_1 + iz_2|^p + |z_1 - iz_2|^p)^{1/p}(|\zeta_1 + i\zeta_2|^q + |\zeta_1 - i\zeta_2|^q)^{1/q} \end{aligned} \tag{6}$$

that is, $|z \cdot \zeta| \leq N_p(z)N_q(\zeta)$.

For general n we have the following

THEOREM 12. *For $z, \zeta \in \mathbf{C}^n$ we have*

$$|z \cdot \zeta| \leq N_p(z)N_q(\zeta),$$

where $p, q > 1$ satisfy $1/p + 1/q = 1$.

PROOF. By the Hölder inequality we have

$$\frac{1}{2}(L(z)L(\zeta) + M(z)M(\zeta)) \leq \frac{1}{2}(L(z)^p + M(z)^p)^{1/p}(L(\zeta)^q + M(\zeta)^q)^{1/q} = N_p(z)N_q(\zeta).$$

Hence, Lemma 11 implies Theorem. □

THEOREM 13. *For $z \in \mathbf{C}^n$ we have*

$$N_p(z) = \sup\{|z \cdot \zeta|;\ \zeta \in \mathbf{C}^n, N_q(\zeta) = 1\}, \tag{7}$$

where $1/p + 1/q = 1$.

PROOF. Suppose first $z, \zeta \in \mathbf{C}^2$. Then the equality in the Hölder inequality (6) holds if and only if $a|z_1 + iz_2|^p = b|\zeta_1 - i\zeta_2|^q$ and $a|z_1 - iz_2|^p = b|\zeta_1 + i\zeta_2|^q$ for some $a, b \geq 0$, not both 0. Therefore, (7) is valid if $n = 2$.

Suppose now $z \in \mathbf{C}^n$. Take $T \in O(n)$ such that $w = Tz \in \mathbf{C}^2$. By Theorem 12 we have

$$\begin{aligned} N_p(z) &\geq \sup\{|z \cdot \zeta|;\ \zeta \in \mathbf{C}^n, N_q(\zeta) = 1\} \\ &= \sup\{|Tz \cdot T\zeta|;\ \zeta \in \mathbf{C}^n, N_q(\zeta) = 1\} \\ &= \sup\{|w \cdot \alpha|;\ \alpha \in \mathbf{C}^n, N_q(\alpha) = 1\} \\ &\geq \sup\{|w \cdot \alpha|;\ \alpha \in \mathbf{C}^2, N_q(\alpha) = 1\} \\ &= N_p(w) = N_p(Tz) = N_p(z). \qquad \square \end{aligned}$$

COROLLARY 14. *$N_p(z)$ is a norm on $\mathbf{C}^n$.*

PROOF. We have to show the following three conditions:
(a) $N_p(z) \geq 0$; $N_p(z) = 0$ if and only if $z = 0$.
(b) $N_p(\lambda z) = |\lambda| N_p(z)$ for any $\lambda \in \mathbf{C}$ and $z \in \mathbf{C}^n$.
(c) $N_p(z + w) \leq N_p(z) + N_p(w)$ for any $z, w \in \mathbf{C}^n$.
(a) and (b) are clear. (c) results from Theorem 13. $\square$

Generalizing Lemma 3 we have the following

COROLLARY 15. *For $z \in \mathbf{C}^n$ we denote by $\tilde{z}$ the projection of z to $\mathbf{C}^2$. Then we have*

$$N_p(\tilde{z}) \leq N_p(z). \tag{8}$$

PROOF. Let $z \in \mathbf{C}^n$. By the homogeneity of (8) we may assume $N_p(z) = 1$. By Theorem 13 there exists $\zeta \in \mathbf{C}^2$ such that $N_q(\zeta) = 1$ and $|\tilde{z} \cdot \zeta| = N_p(\tilde{z})$. By Theorem 12 we have

$$\begin{aligned} 1 = N_q(\zeta) &\geq \sup\{|w \cdot \zeta|;\ w \in \mathbf{C}^n, N_p(w) = 1\} \\ &= \sup\{|\tilde{w} \cdot \zeta|;\ w \in \mathbf{C}^n, N_p(w) = 1\} \\ &\geq |\tilde{z} \cdot \zeta| = N_p(\tilde{z}). \end{aligned}$$

Hence, we have $N_p(\tilde{z}) \leq 1$, which proves (8). $\square$

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