TOKYO J. MATH.
VOL. 24, NO. 2, 2001

Orbits of Isotropy Groups of Compact Symmetric Spaces

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(Communicated by S. Kaneyuki)

Abstract. In this paper, for a compact symmetric space M we consider orbits of the linear isotropy action of M and the canonical action on M of its isotropy group. We reduce some geometric conditions of these orbits to those of their starting points. Consequently we get some relations among the geometric conditions of orbits.

Introduction.

We consider the following two pairs (M, K) of a Riemannian manifold M and a Lie transformation group K acting on M as isometries:

(1) K is the linear isotropy group of an irreducible Riemannian symmetric space N of compact type at a point o and M is the tangent space of N at o,

(2) M is an irreducible Riemannian symmetric space of compact type and K is the isotropy group at a point.

Many mathematicians have investigated several geometric properties of the orbits of such K in M from a viewpoint of differential geometry, e.g., [4], [2], [5] and [7].

In this paper we consider relations among several conditions of orbits of K in M. For x in M we put

$$K_x = \{k \in K \mid kx = x\}.$$

The conditions of the orbit Kx at x we consider are as follows:

(a) The pair $(K_0, K_0 \cap K_x)$ of the identity component K_0 of K and $K_0 \cap K_x$ is a symmetric pair.

(b) The pair $(\mathfrak{k}, \mathfrak{k}_x)$ of the Lie algebras $\mathfrak{k}$ and $\mathfrak{k}_x$ of K and K_x is an orthogonal symmetric Lie algebra.

(c) The normal homogeneous Riemannian metric g_n on K/K_x and the induced Riemannian metric g_i on Kx as a submanifold in M are proportional.

(d) The Levi-Civita connections D of g_n and ∇ of g_i coincide.

Moreoveer we consider the conditions:

(e) The orbit $Kx \subset M$ is a canonical embedding of a symmetric R-space, in the case (1) and

(f) The orbit Kx is a totally geodesic submanifold in M

Received May 15, 2000

in the case (2).

The following equivalent properties have been studied in the case (1): (b) and (e) by Nagano [8], (a) and (b) by Takeuchi [10], Ohnita [9], (a) and (c) by Takeuchi and Kobayashi [12], Olmos and Heintz [2]. We describe a necessary and sufficient condition for each from (a) to (f) by the use of root systems. In the case (1) we get a conclusion the conditions from (a) to (e) are mutually equivalent (Theorem 3.2). Although many parts of Theorem 3.2 have been already obtained, in order to compare the cases (1) and (2) we give a complete proof of Theorem 3.2. In the case (2) those conditions are not equivalent, but we can show some relations among the conditions above (Theorem 4.2).

The authors would like to thank the referee for pointing out some misprints in the manuscript.

1. Preliminaries.

Let G be a compact connected semisimple Lie group and θ an involutive automorphism of G. We denote by G_θ the closed subgroup consisting of all fixed points of θ in G. For a closed subgroup K of G which lies between G_θ and the identity component of G_θ, (G, K) is a Riemannian symmetric pair. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. The involutive automorphism θ of G induces an involutive automorphism of $\mathfrak{g}$, also denoted by θ. We have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

An inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which is invariant under the actions of $\mathrm{Ad}(G)$ and θ induces a biinvariant Riemannian metric on G and G-invariant Riemannian metric on the homogeneous space $M = G/K$, which are also denoted by the same symbol $\langle\ ,\ \rangle$. Moreover we assume that the linear isotropy representation of K on $T_o(M)$ is irreducible. Then M is an irreducible Riemannian symmetric space of compact type with respect to $\langle\ ,\ \rangle$. Conversely any irreducible Riemannian symmetric space of compact type is constructed in this way. Put

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Since θ is involutive, we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

This decomposition is called a canonical decomposition of the orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k})$. The tangent space $T_o(M)$ of M at o is identified with $\mathfrak{m}$ through the differential $\pi_* = (\pi_*)_e$ of the natural projection $\pi : G \to M$. Take and fix a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and a maximal Abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ including $\mathfrak{a}$. Put

$$\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}.$$

Since $\mathfrak{t}$ is θ-invariant, we get an orthogonal direct sum decomposition of $\mathfrak{t}$:

$$\mathfrak{t} = \mathfrak{b} + \mathfrak{a}.$$

For $\alpha \in \mathfrak{t}$ we put

$$\tilde{\mathfrak{g}}_\alpha = \{X \in \mathfrak{g}^{\mathbf{C}} \mid [H, X] = \sqrt{-1}\langle \alpha, H\rangle \quad X (H \in \mathfrak{t})\}$$

and define the root system $\tilde{R}(\mathfrak{g})$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$ by

$$\tilde{R}(\mathfrak{g}) = \{\alpha \in \mathfrak{t} \setminus \{0\} \mid \tilde{\mathfrak{g}}_\alpha \neq \{0\}\} \subset \mathfrak{t}.$$

We simply write $\tilde{R}$ for $\tilde{R}(\mathfrak{g})$. For $\lambda \in \mathfrak{a}$ we put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}^{\mathbf{C}} \mid [H, X] = \sqrt{-1}\langle \lambda, H\rangle \quad X (H \in \mathfrak{a})\}$$

and define the restricted root system $R(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{a}$ by

$$R(\mathfrak{g}, \mathfrak{k}) = \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{g}_\lambda \neq \{0\}\} \subset \mathfrak{a}.$$

We simply write R for $R(\mathfrak{g}, \mathfrak{k})$. Put

$$\tilde{R}_0(\mathfrak{g}) = \tilde{R}(\mathfrak{g}) \cap \mathfrak{b}$$

and denote the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$ by $H \mapsto \bar{H}$. Then we have

$$R(\mathfrak{g}, \mathfrak{k}) = \{\bar{\alpha} \mid \alpha \in \tilde{R}(\mathfrak{g}) \setminus \tilde{R}_0(\mathfrak{g})\}.$$

We define lexicographic orderings > on $\mathfrak{a}$ and $\mathfrak{t}$ with respect to which $\bar{H} > 0$ implies $H > 0$ for $H \in \mathfrak{t}$. We denote by $\tilde{R}_+$, R_+ the set of all positive roots in $\tilde{R}$, R respectively and by $\tilde{F}$, F the fundamental root systems of $\tilde{R}$, R respectively. We have

$$R_+ = \{\bar{\alpha} \mid \alpha \in \tilde{R}_+ \setminus \tilde{R}_0\}, \qquad F = \{\bar{\alpha} \mid \alpha \in \tilde{F} \setminus \tilde{R}_0\}.$$

We put

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} \mid [X, \mathfrak{a}] = \{0\}\}$$

and for $\lambda \in R_+$

$$\mathfrak{k}_\lambda = \mathfrak{k} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}), \qquad \mathfrak{m}_\lambda = \mathfrak{m} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}).$$

Then we obtain the following lemma ([11, Chap. II, §5]).

LEMMA 1.1. (1) $\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda\in R_+} \mathfrak{k}_\lambda$, $\mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{m}_\lambda$ *are orthogonal direct sum decompositions.*

(2) *For each* $\alpha \in \tilde{R}_+ \setminus \tilde{R}_0$ *there exist* $S_\alpha \in \mathfrak{k}$ *and* $T_\alpha \in \mathfrak{m}$ *such that*

$$\{S_\alpha \mid \alpha \in \tilde{R}_+, \bar{\alpha} = \lambda\}, \qquad \{T_\alpha \mid \alpha \in \tilde{R}_+, \bar{\alpha} = \lambda\}$$

are orthonormal bases of $\mathfrak{k}_\lambda$, $\mathfrak{m}_\lambda$ *respectively and that for* $H \in \mathfrak{a}$ *they satisfy*

$$[H, S_\alpha] = \langle \alpha, H\rangle T_\alpha, \quad [H, T_\alpha] = -\langle \alpha, H\rangle S_\alpha, \quad [S_\alpha, T_\alpha] = \bar{\alpha},$$
$$\mathrm{Ad}(\exp H) S_\alpha = \cos\langle \alpha, H\rangle S_\alpha + \sin\langle \alpha, H\rangle T_\alpha,$$
$$\mathrm{Ad}(\exp H) T_\alpha = -\sin\langle \alpha, H\rangle S_\alpha + \cos\langle \alpha, H\rangle T_\alpha.$$

(3) *We define a real Lie subalgebra* $\mathfrak{g}^\sharp$ *of* $\mathfrak{g}^{\mathbf{C}}$ *by*

$$\mathfrak{g}^\sharp = \mathfrak{k} + \sqrt{-1}\mathfrak{m}.$$

For $\lambda \in R$ *we put*

$$\mathfrak{g}^\sharp_\lambda = \mathfrak{g}^\sharp \cap \mathfrak{g}_\lambda.$$

Then for $\lambda \in R_+$ *we have*

$$\mathfrak{g}^\sharp_\lambda + \mathfrak{g}^\sharp_{-\lambda} = \mathfrak{k}_\lambda + \sqrt{-1}\mathfrak{m}_\lambda .$$

(4) *For* $\alpha \in \tilde{R}_+ \setminus \tilde{R}_0$ *we put*

$$X_\alpha = S_\alpha - \sqrt{-1}T_\alpha , \quad X_{-\alpha} = S_\alpha + \sqrt{-1}T_\alpha .$$

Then for $\lambda \in R$

$$\{X_\alpha \mid \alpha \in \tilde{R}, \bar{\alpha} = \lambda\}$$

is a basis of $\mathfrak{g}^\sharp_\lambda$ *and*

$$(\mathfrak{g}^\sharp_\lambda)^{\mathbf{C}} = \mathfrak{g}_\lambda = \sum_{\substack{\alpha \in \tilde{R} \\ \bar{\alpha} = \lambda}} \tilde{\mathfrak{g}}_\alpha$$

holds.

We define a convex region C in $\mathfrak{a}$ by

$$C = \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle > 0 \quad (\lambda \in F)\} .$$

Then we have

$$\mathfrak{m} = \bigcup_{k \in K_0} \mathrm{Ad}(k)\bar{C} ,$$

where K_0 is the identity component of K. So in order to consider the K-orbits of H in $\mathfrak{m}$ we may suppose H belongs to $\bar{C}$. The closure of C is given by

$$\bar{C} = \{H \in \mathfrak{a} \mid \langle \lambda, H \rangle \geq 0 \quad (\lambda \in F)\} .$$

For a subset $\Delta \subset F$ we define

$$C^\Delta = \{H \in \bar{C} \mid \langle \lambda, H \rangle > 0 \quad (\lambda \in \Delta), \langle \mu, H \rangle = 0 \quad (\mu \in F \setminus \Delta)\} .$$

In particular $C^\Delta = \{0\}$ if $\Delta = \emptyset$.

LEMMA 1.2. $\bar{C} = \bigcup_{\Delta \subset F} C^\Delta$ *is a disjoint union.*

For each $\alpha \in \tilde{F}$ we take $\tilde{H}_\alpha \in \mathfrak{t}$ satisfying

$$\langle \beta, \tilde{H}_a \rangle = \begin{cases} 1 & (\alpha = \beta, \beta \in \tilde{F}) \\ 0 & (\alpha \neq \beta, \beta \in \tilde{F}) \end{cases}$$

and for each $\lambda \in F$ we take $H_\lambda \in \mathfrak{a}$ satisfying

$$\langle \mu, H_\lambda \rangle = \begin{cases} 1 & (\lambda = \mu, \mu \in F) \\ 0 & (\lambda \neq \mu, \mu \in F) . \end{cases}$$

Then we obtain

$$\bar{C} = \left\{ \sum_{\lambda \in F} t_\lambda H_\lambda \,\middle|\, t_\lambda \geq 0 \right\}$$

and for a subset $\Delta \subset F$

$$C^\Delta = \left\{ \sum_{\lambda \in \Delta} t_\lambda H_\lambda \,\middle|\, t_\lambda > 0 \right\} .$$

Let δ be the highest root of R and consider

$$\mathcal{F} = F \cup \{\delta\}.$$

We define a convex region Q in $\mathfrak{a}$ by

$$Q = \{H \in \mathfrak{a} \mid 0 < \langle \lambda, H \rangle < \pi \quad (\lambda \in \mathcal{F})\}.$$

For $\Delta \subset \mathcal{F}$ we put

$$Q^{\Delta} = \left\{ H \in \bar{Q} \,\middle|\, \begin{matrix} 0 < \langle \lambda, H \rangle \ (\lambda \in \Delta \cap F), & \langle \delta, H \rangle < \pi \ (\text{if } \delta \in \Delta), \\ 0 = \langle \mu, H \rangle \ (\mu \in F \setminus \Delta), & \langle \delta, H \rangle = \pi \ (\text{if } \delta \notin \Delta) \end{matrix} \right\}$$

For example $Q^{\mathcal{F}} = Q \subset \mathfrak{a}$. The condition $Q^{\Delta} \neq \emptyset$ holds if and only if $\Delta \neq \emptyset$. A subset $\Delta \subset \mathcal{F}$ satisfying this condition is said to be admissible. For an admissible subset $\Delta \subset \mathcal{F}$, Q^{Δ} is a convex cell in $\bar{Q}$. Thus we will consider only admissible subsets $\Delta \subset \mathcal{F}$.

LEMMA 1.3.

$$\bar{Q} = \bigcup_{\Delta \subset \mathcal{F}} Q^{\Delta}$$

is a disjoint union.

LEMMA 1.4. *If $\Delta \subset \mathcal{F}$ and $H \in Q^{\Delta}$, then the subsets*

$$\{\lambda \in R_+ \mid \langle \lambda, H \rangle \in \pi \mathbf{Z}\}, \quad \{\alpha \in \tilde{R}_+ \mid \langle \alpha, H \rangle \in \pi \mathbf{Z}\}$$

are not dependent on $H \in Q^{\Delta}$, but on Δ. We denote by $\mathcal{R}_+^{\Delta}$, $\tilde{\mathcal{R}}_+^{\Delta}$ these subsets.

2. Some properties of root systems.

We shall now give some general properties of an irreducible root system used in later sections. Let R be an irreducible root system and $F = \{\alpha_1, \cdots, \alpha_r\}$ a fundamental root system of R.

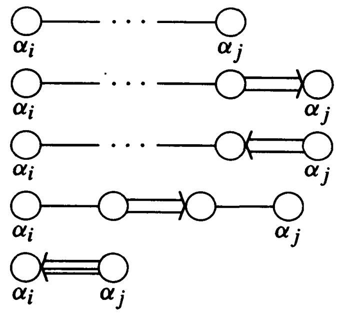
PROPOSITION 2.1. *For $1 \leq i, j \leq r, i \neq j$, there exist positive roots*

$$\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r, \quad \beta = b_1\alpha_1 + \cdots + b_r\alpha_r$$

such that

$$a_i = 1, \quad b_j = 1, \quad \alpha + \beta \in R.$$

PROOF. Two simple roots α_i and α_j are connected in the Dynkin diagram in the following way.



Let α' be the sum of all simple roots which lie between α_i and α_j in the diagram. Put $\alpha = \alpha_i + \alpha'$ and $\beta = \alpha_j$. In general the sum of roots in a connected subdiagram of a Dynkin diagram is a positive root. Therefore α and $\alpha + \beta$ are in R.

We denote the highest root of R by $\alpha_0 = n_1\alpha_1 + \cdots + n_r\alpha_r$.

PROPOSITION 2.2. *If $n_i \geq 2$, there exist positive roots*

$$\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r\,, \quad \beta = b_1\alpha_1 + \cdots + b_r\alpha_r$$

such that

$$a_i \geq 1\,, \quad b_i \geq 1\,, \quad \alpha + \beta \in R\,.$$

PROOF. By the assumption R is not of type A_r. We put

$$\alpha = \alpha_1 + \cdots + \alpha_r\,, \quad \beta = \alpha_0 - \alpha\,.$$

Using the classification of root systems we have $\beta \in R$ and the proposition is proved.

We have the following due to Helgason [3, p. 460].

PROPOSITION 2.3. *For any positive root α there exits a sequence of positive roots $\beta_1, \cdots, \beta_n = \alpha$ such that β_1 and $\beta_i - \beta_{i-1}$ are simple roots.*

From the above we have also the following.

PROPOSITION 2.4. *If $n_i \geq 3$, there exists a positive root*

$$\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r$$

such that

$$a_i = 1\,, \quad \alpha + \alpha_i \in R\,.$$

In particular $\alpha + \alpha_i$ is not the highest root.

3. Orbits of linear isotropy groups.

Let $M = G/K$ be an irreducible Riemannian symmetric space of compact type. For $H \in T_o(M)$, we consider the orbit $K_*H \subset T_o(M)$ of the linear isotropy group K_*. Identifying $T_o(M)$ with $\mathfrak{m}$ in a natural manner, we consider $\mathrm{Ad}(K)H \subset \mathfrak{m}$. The submanifold $\mathrm{Ad}(K)H$ in $\mathfrak{m}$ is connected (see [4, Prop. 2.1]). We may assume that H is in $\bar{C}$.

We define a closed subgroup Z_K^H of K by

$$Z_K^H = \{k \in K \mid \mathrm{Ad}(k)H = H\}.$$

Then the mapping

$$\Phi : K/Z_K^H \to \mathrm{Ad}(K)H;\ kZ_K^H \mapsto \mathrm{Ad}(k)H$$

is a diffeomorphism between K/Z_K^H and $\mathrm{Ad}(K)H$. Hence $\mathrm{Ad}(K)H$ has two Riemannian metrics in the following way. The normal homogeneous Riemannian metric g_n on K/Z_K^H which is induced by the biinvariant Riemannian metric on G and the induced Riemannian metric g_i on $\mathrm{Ad}(K)H \subset \mathfrak{m}$. Take a fundamental root system $F = \{\lambda_1, \cdots, \lambda_l\}$ of R and denote by $\delta = m_1\lambda_1 + \cdots + m_l\lambda_l$ the highest root of R. For the sake of simplicity, we set $H_i = H_{\lambda_i}$.

DEFINITION 3.1. *Put*

$$H = xH_i\,, \quad m_i = 1 \quad (x > 0)\,.$$

Since the involutive automorphism

$$\sigma : K_0 \to K_0;\ k \mapsto (\exp \pi H_i)k(\exp(-\pi H_i))$$

defines a compact symmetric pair $(K_0, Z_{K_0}^H)$, K/Z_K^H is a compact symmetric space, which we call a symmetric R-space. $\Phi : K/Z_K^H \to \mathfrak{m}$ is called the canonical embedding of the symmetric R-space.

We denote by D and ∇ the Levi-Civita connections with respect to the Riemannian metrics g_n and g_i, respectively.

THEOREM 3.2. *Let $M = G/K$ be an irreducible Riemannian symmetric space of compact type. Then the following conditions are equivalent.*

(1) $\mathrm{Ad}(K)H \subset \mathfrak{m}$ *is a canonical embedding of a symmetric R-space.*
(2) *The pair $(K_0, Z_{K_0}^H)$ is a compact symmetric pair.*
(3) *The pair $(\mathfrak{k}, \mathfrak{z}_K^H)$ is an orthogonal symmetric Lie algebra, where we put $\mathfrak{z}_K^H = \mathcal{L}(Z_K^H)$.*
(4) *The Riemannian metrics g_n and g_i are proportional.*
(5) $\nabla = D$.

PROOF. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and that (4) $\Rightarrow$ (5). In order to show this theorem, we prove lemmas needed later.

Let $\Delta \subset F$, $H \in C^\Delta$. From Lemma 1.1, we have

$$T_H(\mathrm{Ad}(K)H) = [\mathfrak{k}, H] = \sum_{\lambda \in R_+ \setminus R_+^\Delta} \mathfrak{m}_\lambda\,,$$

$$T_{o'}(K/Z_K^H) = (\mathfrak{z}_K^H)^\perp = \sum_{\lambda \in R_+ \setminus R_+^\Delta} \mathfrak{k}_\lambda\,,$$

where o' is the origin of K/Z_K^H and we denote by $(\cdot)^\perp$ the orthogonal complement of $(\cdot)$ in $\mathfrak{k}$. The linear isomorphism $(\Phi_*)_{o'}$ is given by

$$(\Phi_*)_{o'} : (\mathfrak{z}_K^H)^\perp \to [\mathfrak{k}, H];\ X \mapsto [X, H]. \tag{3.1}$$

LEMMA 3.3. *The condition* (3) *in Theorem* 3.2 *holds if and only if for* $\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta$ *the following two conditions hold.*

$$\alpha + \beta \notin \tilde{R},$$
$$\langle \alpha - \beta, H\rangle \neq 0 \textit{ implies } \alpha - \beta \notin \tilde{R}.$$

PROOF. Let $H \in C^\Delta$.

$$\begin{aligned} &(\mathfrak{k}, \mathfrak{z}_K^H) \text{ is an orthogonal symmetric Lie algebra} \\ \Leftrightarrow\ & [(\mathfrak{z}_K^H)^\perp, (\mathfrak{z}_K^H)^\perp] \subset \mathfrak{z}_K^H \\ \Leftrightarrow\ & [[(\mathfrak{z}_K^H)^\perp, (\mathfrak{z}_K^H)^\perp], H] = 0 \\ \Leftrightarrow\ & \langle \alpha, H\rangle[T_\alpha, S_\beta] - \langle \beta, H\rangle[T_\beta, S_\alpha] = 0 \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta). \end{aligned}$$

From Lemma 1.1, we have

$$[T_\alpha, S_\beta] = \frac{\sqrt{-1}}{4}([X_\alpha, X_\beta] - [X_{-\alpha}, X_{-\beta}] - [X_{-\alpha}, X_\beta] + [X_\alpha, X_{-\beta}]),$$

which implies that

$$\begin{aligned} &(\mathfrak{k}, \mathfrak{z}_K^H) \text{ is an orthogonal symmetric Lie algebra} \\ \Leftrightarrow\ & \begin{cases} \langle \alpha + \beta, H\rangle[X_\alpha, X_\beta] = 0, \\ \langle \alpha - \beta, H\rangle[X_\alpha, X_{-\beta}] = 0 \end{cases} \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta) \\ \Leftrightarrow\ & \begin{cases} [\mathfrak{g}_\lambda^\sharp, \mathfrak{g}_\mu^\sharp] = \{0\}, \\ \langle \alpha - \beta, H\rangle \neq 0 \Rightarrow [\mathfrak{g}_\lambda^\sharp, \mathfrak{g}_{-\mu}^\sharp] = \{0\} \end{cases} \quad (\bar{\alpha} = \lambda, \bar{\beta} = \mu) \\ \Leftrightarrow\ & \begin{cases} [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \{0\}, \\ \langle \alpha - \beta, H\rangle \neq 0 \Rightarrow [\mathfrak{g}_\lambda, \mathfrak{g}_{-\mu}] = \{0\} \end{cases} \quad (\bar{\alpha} = \lambda, \bar{\beta} = \mu) \\ \Leftrightarrow\ & \begin{cases} \alpha + \beta \notin \tilde{R}_+, \\ \langle \alpha - \beta, H\rangle \neq 0 \Rightarrow \alpha - \beta \notin \tilde{R} \end{cases} \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta). \end{aligned}$$

LEMMA 3.4. *The condition* (4) *in Theorem* 3.2 *holds if and only if* $\langle \alpha, H\rangle$ *is a constant for* $\alpha \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta$. *Moreover this is equivalent to the condition* (1) *in Theorem* 3.2.

PROOF. From (3.1), the condition (4) holds if and only if there exists a positive constant c such that

$$c\langle X, Y\rangle = \langle [X, H], [Y, H]\rangle \quad \text{for any } X, Y \in (\mathfrak{z}_K^H)^\perp.$$

From Lemma 1.1,

$$\{S_\alpha \mid \alpha \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta\}, \quad \{T_\alpha \mid \alpha \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta\}$$

are orthonormal bases of $T_{o'}(K/Z_K^H)$ and $T_H(\mathrm{Ad}(K)H)$, respectively. Hence we have

$$(4) \Leftrightarrow \langle \lambda, H\rangle \quad \text{is a constant for } \lambda \in R_+ \setminus R_+^{\Delta}.$$

It is clear that if $H = xH_i$ ($m_i = 1$), then the two Riemannian metrics are proportional. Conversely we assume that the condition (4). For any restricted simple root $\lambda_i \in R_+ \setminus R_+^{\Delta}$, we have

$$\langle \lambda_i, H\rangle = \text{constant } x (> 0).$$

For the restricted highest root $\delta \in R_+ \setminus R_+^{\Delta}$, we have

$$\langle \delta, H\rangle = \sum_{i \in \{i \mid \lambda_i \in \Delta\}} m_i x = x,$$

which implies that $H = xH_i$ ($m_i = 1$).

LEMMA 3.5. *The condition* (5) *in Theorem* 3.2 *holds if and only if for* $\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^{\Delta}$, $\langle \alpha - \beta, H\rangle \neq 0$ *implies* $\alpha \pm \beta \notin \tilde{R}$.

In order to prove this lemma, we prepare the following. For $X \in \mathfrak{k}$, we define a Killing vector field X^* on $\mathrm{Ad}(K)H$ by

$$(X^*)_x = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tX)x = [X, x].$$

We also denote by the same symbol X^* the corresponding vector field on K/Z_K^H.

LEMMA 3.6. *For* $X, Y \in \mathfrak{k}$,

$$(\nabla_{X^*}Y^*)_H = [Y, [X, H]]_{[\mathfrak{k},H]},$$
$$(D_{X^*}Y^*)_{o'} = -[X_{(\mathfrak{z}_K^H)^\perp}, Y_{\mathfrak{z}_K^H}] - \frac{1}{2}[X_{(\mathfrak{z}_K^H)^\perp}, Y_{(\mathfrak{z}_K^H)^\perp}]_{(\mathfrak{z}_K^H)^\perp},$$

where $(\cdot)_{[\mathfrak{k},H]}$ *is the* $[\mathfrak{k}, H]$*-component of* $(\cdot)$, *and where* $(\cdot)_{(\mathfrak{z}_K^H)^\perp}$ *and* $(\cdot)_{\mathfrak{z}_K^H}$ *are the* $(\mathfrak{z}_K^H)^\perp$*- and* $\mathfrak{z}_K^H$*-component of* $(\cdot)$ *in* $\mathfrak{k}$, *respectively.*

PROOF. For the first equation we refer to [5]. For the second we refer to [6, p. 176].

PROOF OF LEMMA 3.5.

$$\begin{aligned}
\nabla = D &\Leftrightarrow (\nabla_{X^*}X^*)_H = (\Phi_*)_{o'}((D_{X^*}X^*)_{o'}) \quad (X \in \mathfrak{k})\\
&\Leftrightarrow [X, [X, H]]_{[\mathfrak{k},H]} = -[[X_{(\mathfrak{z}_K^H)^\perp}, X_{\mathfrak{z}_K^H}], H] \quad (X \in \mathfrak{k})\\
&\Leftrightarrow [X_{(\mathfrak{z}_K^H)^\perp}, [X_{(\mathfrak{z}_K^H)^\perp}, H]]_{[\mathfrak{k},H]} = 0 \quad (X \in \mathfrak{k})\\
&\Leftrightarrow [X, [Y, H]]_{[\mathfrak{k},H]} + [Y, [X, H]]_{[\mathfrak{k},H]} = 0 \quad (X, Y \in (\mathfrak{z}_K^H)^\perp)\\
&\Leftrightarrow \langle [X, [Y, H]] + [Y, [X, H]], [\mathfrak{k}, H]\rangle = 0 \quad (X, Y \in (\mathfrak{z}_K^H)^\perp)\\
&\Leftrightarrow [[X, [Y, H]], H] + [[Y, [X, H]], H] = 0 \quad (X, Y \in (\mathfrak{z}_K^H)^\perp)\\
&\Leftrightarrow [[S_\alpha, [S_\beta, H]], H] + [[S_\beta, [S_\alpha, H]], H] = 0 \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^{\Delta})\\
&\Leftrightarrow \langle \alpha - \beta, H\rangle \neq 0 \Rightarrow [S_\alpha, S_\beta] = 0 \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^{\Delta}).
\end{aligned}$$

By a similar argument in the proof of Lemma 3.3, we have

$$\begin{aligned}\nabla = D &\Leftrightarrow \langle \alpha - \beta, H\rangle \neq 0 \Rightarrow [X_\alpha, X_\beta] = [X_\alpha, X_{-\beta}] = 0 \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta) \\ &\Leftrightarrow \langle \alpha - \beta, H\rangle \neq 0 \Rightarrow \alpha \pm \beta \notin \tilde{R} \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{R}_+^\Delta).\end{aligned}$$

We return to the proof of Theorem 3.2.

We first prove the conditions (3) and (5) are equivalent. It is clear from Lemmas 3.3 and 3.5 that (3) implies (5). In order to prove the converse, it is sufficient to prove that for $\alpha, \beta \in \tilde{R} \setminus \tilde{R}_+^\Delta$ if $\langle \alpha - \beta, H\rangle = 0$ then $\alpha + \beta \notin \tilde{R}$. If $\alpha + \beta \in \tilde{R}$, then $\alpha + \beta \in \tilde{R} \setminus \tilde{R}_+^\Delta$ and $\langle (\alpha + \beta) - \alpha, H\rangle \neq 0$. Hence by the assumption we have $\beta = (\alpha + \beta) - \alpha \notin \tilde{R}$, which is a contradiction.

We second prove that the condition (5) implies (1). In the case when M is of type I (see [3, p. 379] for the definition), $\tilde{R}$ is irreducible. Take a fundamental root system $\tilde{F} = \{\alpha_1, \cdots, \alpha_r\}$ of $\tilde{R}$ and denote by $\alpha_0 = n_1\alpha_1 + \cdots + n_r\alpha_r$ the highest root of $\tilde{R}$. Assume that $\nabla = D$, which is equivalent to the condition that $(\mathfrak{k}, \mathfrak{z}_K^H)$ is an orthogonal symmetric Lie algebra. Since

$$\bar{C} \subset \{H \in \mathfrak{t} \mid \langle \alpha, H\rangle \geq 0 \ (\alpha \in \tilde{F})\},$$

there exist nonnegative constants $y_1, \cdots, y_r$ such that

$$H = y_1\tilde{H}_1 + \cdots + y_r\tilde{H}_r \quad (y_i \geq 0).$$

If there exist two positive constants $y_i > 0, y_j > 0$ $(i \neq j)$, then from Proposition 2.1 there exist $\alpha, \beta \in \tilde{R}_+$, such that

$$\langle \alpha, \tilde{H}_{\alpha_i}\rangle, \quad \langle \beta, \tilde{H}_{\alpha_j}\rangle > 0, \quad \alpha + \beta \in \tilde{R}_+,$$

which contradicts Lemma 3.3. Hence $H = y_i\tilde{H}_i$.

If $n_i \geq 2$, then from Proposition 2.2 there exist $\alpha, \beta \in \tilde{R}_+$ such that

$$\langle \alpha, H\rangle, \quad \langle \beta, H\rangle > 0, \quad \alpha + \beta \in \tilde{R}_+,$$

which contradicts Lemma 3.3. Hence $n_i = 1$. In this case it is clear that the two Riemannian metrics g_i and g_n are proportional. Hence (1) holds. In the case when M is of type II (see [3, p. 379] for the definition), M is a compact connected simple Lie group L furnished with a biinvariant Riemannian metric $\langle\ ,\ \rangle$ (see [3, p. 439]). Take a maximal Abelian subalgebra $\mathfrak{t}(\mathfrak{l})$ of $\mathfrak{l}$. We denote by $\tilde{R}(\mathfrak{l})$, $\tilde{F}(\mathfrak{l}) = \{\alpha_i\}_{1 \leq i \leq r}$ and $\alpha_0 = \sum m_i\alpha_i$ the root system of $\mathfrak{l}$ with respect to $\mathfrak{t}(\mathfrak{l})$, a fundamental root system of $\tilde{R}(\mathfrak{l})$ and the highest root of $\tilde{R}(\mathfrak{l})$, respectively. We denote by $\{H_i\}$ the dual system of $\tilde{F}(\mathfrak{l})$, i.e., $\langle H_i, \alpha_j\rangle = \delta_{ij}$. Then $\mathrm{Ad}(L)H \subset \mathfrak{l}(H \in \mathfrak{l})$ is a canonical embedding of symmetric R-space if and only if $H = xH_i$ $(x > 0, m_i = 1)$. On the other hand, $\nabla = D$ if and only if

$$\alpha, \beta \in \tilde{R}_+(\mathfrak{l}), \langle \alpha, H\rangle > 0, \langle \beta, H\rangle > 0, \langle \alpha - \beta, H\rangle \neq 0 \Rightarrow \alpha \pm \beta \notin \tilde{R}(\mathfrak{l}).$$

Hence by a similar argument above we can prove that the condition (5) implies (1).

4. Orbits of isotropy groups.

For $p \in M$, we consider the orbit $Kp \subset M$ of the isotropy group K. B. Y. Chen and T. Nagano [1] proved that for a fixed point p of the geodesic symmery s_o with respect to o, Kp is connected. Using a similar method, we have

PROPOSITION 4.1. *For any p in M, $Kp = K_0 p$. In particular, Kp is connected.*

PROOF. Since M is complete ([3, p. 205]), there exists a geodesic $c(t)$ such that $c(0) = o$ and $c(1) = p$. For any k in K, $kc(t)$ is also a geodesic. Take maximal tori A and A' in M containing the images c and kc, respectively. There exists $k_0 \in K_0$ such that $A' = k_0 A$. Then the images c and $k_0^{-1}kc$ are contained in A, therefore the initial vectors $c'(0)$ and $\mathrm{Ad}(k_0^{-1}k)c'(0)$ are in $\mathfrak{a}$. By using [3, p. 285, Prop. 2.2], there exists k_1 in K_0 such that $\mathrm{Ad}(k_1)c'(0) = \mathrm{Ad}(k_0^{-1}k)c'(0)$, which implies that $kp = k_0 k_1 p \in K_0 p$.

Since $M = K \operatorname{Exp} \bar{Q}$ ([3, p. 323, Theorem 8.6]), we may assume that $p = \operatorname{Exp} H$, $H \in \bar{Q}$.

We define a closed subgroup N_K^H of K by

$$N_K^H = \{k \in K \mid k \operatorname{Exp} H = \operatorname{Exp} H\}$$
$$= \{k \in K \mid \exp(-H) k \exp H \in K\}.$$

Then the mapping

$$\Psi : K/N_K^H \to K \operatorname{Exp} H;\ kN_K^H \mapsto k \operatorname{Exp} H$$

is a diffeomorphism between K/N_K^H and $K \operatorname{Exp} H$. Hence $K \operatorname{Exp} H$ has two Riemannian metrics in a natural manner: the normal homogeneous Riemannian metric g_n on K/N_K^H which is induced by the biinvariant Riemannian metric on G and the induced Riemannian metric g_i on $K \operatorname{Exp} H \subset M$. We denote by $\mathbf{D}$ and ∇ the Levi-Civita connections with respect to the Riemannian metrics g_n and g_i, respectively. We can write

$$H = x_1 H_1 + \cdots + x_l H_l \in \bar{Q}, \quad x_i \geq 0, \quad \sum_{i=1}^{l} m_i x_i \leq \pi .$$

THEOREM 4.2. *Let $M = G/K$ be an irreeducible Riemannian symmetric space of compact type.*

(1) *The orbit $K \operatorname{Exp} H \subset M$ is a totally geodesic submanifold if and only if one of the following conditions holds.*

$$\begin{cases} \text{(i)} & H = (\pi/2)H_i & (m_i = 2), \\ \text{(ii)} & H = (\pi/2)H_i & (m_i = 1), \\ \text{(iii)} & H = (\pi/2)(H_i + H_j) & (m_i = m_j = 1). \end{cases}$$

(2) *The pair* $(\mathfrak{k}, \mathfrak{n}_K^H)$ *is an orthogonal symmetric Lie algebra, where we put* $\mathfrak{n}_K^H = \mathcal{L}(N_K^H)$ *if and only if one of the following conditions holds.*

$$\begin{cases} \text{(i)} & H = (\pi/2)H_i & (m_i = 2)\,, \\ \text{(ii)}' & H = \pi x H_i & (m_i = 1, 0 < x < 1)\,, \\ \text{(iii)}' & H = \pi(xH_i + (1-x)H_j) & (m_i = m_j = 1, 0 < x < 1)\,. \end{cases}$$

(3) *The Riemannian metrics* g_n *and* g_i *are proportional if and only if one of the following conditions holds.*

$$\begin{cases} \text{(i), (ii)', (iii)' in (2) and} \\ \text{(iv)} & H = (\pi/3)H_i & (m_i = 2)\,, \\ \text{(v)} & H = (\pi/3)H_i & (m_i = 3)\,, \\ \text{(vi)} & H = (\pi/3)(H_i + H_j) & (m_i = 1, m_j = 2)\,, \\ \text{(vii)} & H = (\pi/3)(H_i + H_j) & (m_i = m_j = 1)\,, \\ \text{(viii)} & H = (\pi/3)(H_i + H_j + H_k) & (m_i = m_j = m_k = 1)\,. \end{cases}$$

(4) $\nabla = \mathbf{D}$ *if and only if the condition* (3) *holds.*

In particular, the condition (1) *implies* (2) *and* (2) *implies* (3).

PROOF. From Lemma 1.3, there exists an admissible subset $\Delta \subset \mathcal{F}$ such that $H \in Q^\Delta$. From Lemma 1.1, we have

$$T_{\operatorname{Exp} H}(K \operatorname{Exp} H) = (\exp H)_*(\operatorname{Ad}(\exp(-H))(\mathfrak{k}))_{\mathfrak{m}} = (\exp H)_* \sum_{\lambda \in R_+ \setminus \mathcal{R}_+^\Delta} \mathfrak{m}_\lambda$$
$$\cong \sum_{\lambda \in R_+ \setminus \mathcal{R}_+^\Delta} \mathfrak{m}_\lambda\,.$$

We denote by o' the origin of K/N_K^H. Then we have:

$$T_{o'}(K/N_K^H) = (\mathfrak{n}_K^H)^\perp = \sum_{\lambda \in R_+ \setminus \mathcal{R}_+^\Delta} \mathfrak{k}_\lambda\,. \tag{4.2}$$

Hence

$$\{T_\alpha (\cong (\exp H)_* T_\alpha) \mid \alpha \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^\Delta\}\,, \quad \{S_\alpha \mid \alpha \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^\Delta\}$$

are orthonormal bases of $T_{\operatorname{Exp} H}(K \operatorname{Exp} H)$ and $T_{o'}(K/N_K^H)$, respectively. The linear isomorphism $(\Psi_*)_{o'}$ is given by

$$(\Psi_*)_{o'} : (\mathfrak{n}_K^H)^\perp \to (\operatorname{Ad}(\exp(-H))\mathfrak{k})_{\mathfrak{m}};\ X \mapsto (\operatorname{Ad}(\exp(-H))X)_{\mathfrak{m}}\,.$$

In order to show this theorem, we prove some lemmas.

LEMMA 4.3. *The condition* (1) *in Theorem* 4.2 *holds if and only if* $\sin\langle \alpha, H\rangle \in \{0, 1\}$ *for any* $\alpha \in \tilde{R}_+$. *Moreover this is equivalent to the condition* $\sin\langle \lambda, H\rangle \in \{0, 1\}$ *for any* $\lambda \in R_+$.

PROOF. We denote by $\boldsymbol{h}$ the second fundamental form of $K \operatorname{Exp} H \subset M$. It is known by [13, p. 122] that

$$\boldsymbol{h}_{\operatorname{Exp} H}(T_\alpha, T_\beta) = \cot\langle \beta, H\rangle ([T_\alpha, S_\beta])^T \quad \text{for} \quad \alpha, \beta \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^\Delta\,,$$

where $(\cdot)^T$ is the $(\mathfrak{a} + \sum_{\lambda \in \mathcal{R}_+^\Delta} \mathfrak{m}_\lambda)$-component of $(\cdot)$. In particular,

$$\boldsymbol{h}_{\mathrm{Exp}\,H}(T_\alpha, T_\alpha) = -\cot\langle\alpha, H\rangle\bar{\alpha} \quad \text{for} \quad \alpha \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^\Delta .$$

Hence we get the conclusion.

We assume that M is of type I for the time being.

LEMMA 4.4. *The condition* (1) *in Theorem* 4.2 *holds if and only if one of the following conditions holds.*

$$\begin{cases} [\text{i}] & H = (\pi/2)\tilde{H}_i \in \mathfrak{a} & (n_i = 2)\,, \\ [\text{ii}] & H = (\pi/2)\tilde{H}_i \in \mathfrak{a} & (n_i = 1)\,, \\ [\text{iii}] & H = (\pi/2)(\tilde{H}_i + \tilde{H}_j) \in \mathfrak{a} & (n_i = n_j = 1)\,. \end{cases}$$

Moreover these are equivalent to the following conditions.

$$\begin{cases} (\text{i}) & H = (\pi/2)H_i & (m_i = 2)\,, \\ (\text{ii}) & H = (\pi/2)H_i & (m_i = 1)\,, \\ (\text{iii}) & H = (\pi/2)(H_i + H_j) & (m_i = m_j = 1)\,. \end{cases}$$

PROOF. From Lemma 4.3, we have

$$\begin{aligned} & K\,\mathrm{Exp}\,H \subset M \text{ is a totally geodesic submanifold} \\ \Leftrightarrow\ & \sin\langle\lambda, H\rangle \in \{0, 1\} \qquad (\lambda \in R_+) \\ \Leftrightarrow\ & \langle\lambda, H\rangle \in \{0, \pi/2, \pi\} \quad (\lambda \in R_+) \\ \Leftrightarrow\ & H \text{ is one of (i), (ii) and (iii)}\,. \end{aligned}$$

Since

$$H \in \bar{Q} \subset \{H \in \mathfrak{t} \mid 0 \leq \langle\alpha, H\rangle \leq \pi \ (\alpha \in \tilde{F} \cup \{\alpha_0\})\}\,,$$

a similar argument implies that

$$\begin{aligned} & K\,\mathrm{Exp}\,H \subset M \text{ is a totally geodesic submanifold} \\ \Leftrightarrow\ & H \text{ is one of [i], [ii] and [iii]}\,. \end{aligned}$$

LEMMA 4.5. *The condition* (2) *in Theorem* 4.2 *holds if and only if for* $\alpha, \beta \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^\Delta$ *the following two conditions hold.*

$$\langle\alpha + \beta, H\rangle \neq \pi \text{ implies } \alpha + \beta \notin \tilde{R}_+\,,$$
$$\langle\alpha - \beta, H\rangle \neq 0 \text{ implies } \alpha - \beta \notin \tilde{R}\,.$$

PROOF. $(\mathfrak{k}, \mathfrak{n}_K^H)$ is an orthogonal symmetric Lie algebra

$$\begin{aligned} \Leftrightarrow\ & [(\mathfrak{n}_K^H)^\perp, (\mathfrak{n}_K^H)^\perp] \subset \mathfrak{n}_K^H \\ \Leftrightarrow\ & [\mathrm{Ad}(\exp(-H))S_\alpha, \mathrm{Ad}(\exp(-H))S_\beta] \subset \mathfrak{k} \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^\Delta) \\ \Leftrightarrow\ & \cos\langle\alpha, H\rangle \sin\langle\beta, H\rangle [S_\alpha, T_\beta] + \sin\langle\alpha, H\rangle \cos\langle\beta, H\rangle [T_\alpha, S_\beta] = 0\,. \end{aligned}$$

From Lemma 1.1, we have

$$\begin{aligned}
&\cos\langle\alpha, H\rangle\sin\langle\beta, H\rangle[S_\alpha, T_\beta] + \sin\langle\alpha, H\rangle\cos\langle\beta, H\rangle[T_\alpha, S_\beta] \\
&= \frac{\sqrt{-1}}{4}\cos\langle\beta, H\rangle\sin\langle\alpha, H\rangle([X_\alpha, X_\beta] - [X_{-\alpha}, X_{-\beta}] - [X_{-\alpha}, X_\beta] + [X_\alpha, X_{-\beta}]) \\
&\quad + \frac{\sqrt{-1}}{4}\cos\langle\alpha, H\rangle\sin\langle\beta, H\rangle([X_\beta, X_\alpha] - [X_{-\beta}, X_{-\alpha}] - [X_{-\beta}, X_\alpha] + [X_\beta, X_{-\alpha}]) \\
&= \frac{\sqrt{-1}}{4}\sin\langle\alpha+\beta, H\rangle([X_\alpha, X_\beta] - [X_{-\alpha}, X_{-\beta}]) \\
&\quad + \frac{\sqrt{-1}}{4}\sin\langle\alpha-\beta, H\rangle([X_\alpha, X_{-\beta}] - [X_{-\alpha}, X_\beta])\,.
\end{aligned}$$

Hence we get

$$(\mathfrak{k}, \mathfrak{n}_K^H) \text{ is an orthogonal symmetric Lie algebra}$$
$$\Leftrightarrow \begin{cases}\sin\langle\alpha+\beta, H\rangle[X_\alpha, X_\beta] = 0\,, \\ \sin\langle\alpha-\beta, H\rangle[X_\alpha, X_{-\beta}] = 0\end{cases} \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^{\Delta})\,.$$

By a similar argument in the proof of Lemma 3.3, we have

$$(\mathfrak{k}, \mathfrak{n}_K^H) \text{ is an orthogonal symmetric Lie algebra}$$
$$\Leftrightarrow \begin{cases}\langle\alpha+\beta, H\rangle \neq \pi \Rightarrow \alpha+\beta \notin \tilde{R}_+\,, \\ \langle\alpha-\beta, H\rangle \neq 0 \Rightarrow \alpha-\beta \notin \tilde{R}\end{cases} \quad (\alpha, \beta \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^{\Delta})\,.$$

LEMMA 4.6. *The condition* (2) *in Theorem* 4.2 *holds if and only if one of the following conditions holds.*

$$\begin{cases}
[\mathrm{i}] & H = (\pi/2)\tilde{H}_i \in \mathfrak{a} & (n_i = 2)\,, \\
[\mathrm{ii}]' & H = \pi x \tilde{H}_i \in \mathfrak{a} & (n_i = 1, 0 < x < 1)\,, \\
[\mathrm{iii}]' & H = \pi(x\tilde{H}_i + (1-x)\tilde{H}_j) \in \mathfrak{a} & (n_i = n_j = 1, 0 < x < 1)\,.
\end{cases}$$

Moreover these are equivalent to the following conditions.

$$\begin{cases}
(\mathrm{i}) & H = (\pi/2)H_i & (m_i = 2)\,, \\
(\mathrm{ii})' & H = \pi x H_i & (m_i = 1, 0 < x < 1)\,, \\
(\mathrm{iii})' & H = \pi(xH_i + (1-x)H_j) & (m_i = m_j = 1, 0 < x < 1)\,.
\end{cases}$$

PROOF. First we prove that $(\mathfrak{k}, \mathfrak{n}_K^H)$ is an orthogonal symmetric Lie algebra if and only if H is one of [i], [ii]′ and [iii]′. It is clear that if H is one of [i], [ii]′ and [iii]′, then $(\mathfrak{k}, \mathfrak{n}_K^H)$ is an orthogonal symmetric Lie algebra. Hence we prove the converse.

We can write

$$H = y_1\tilde{H}_1 + \cdots + y_r\tilde{H}_r\,, \quad y_i \geq 0\,, \quad \sum_{i=1}^{r} n_i y_i \leq \pi\,.$$

Assume that α_0 is not in $\tilde{\mathcal{R}}_+^{\Delta}$ (i.e., $\langle\alpha_0, H\rangle < \pi$). Put

$$\gamma = \alpha_1 + \cdots + \alpha_r \in \tilde{R}_+\,,$$

then $0 < \langle \gamma, H \rangle < \pi$ if and only if $\gamma \notin \tilde{\mathcal{R}}_+^{\Delta}$. If $\tilde{R} \neq A_r$, then $\alpha_0 - \gamma \in \tilde{R}_+$. So by Lemma 4.5, we have

$$\langle \alpha_0 - \gamma, H \rangle = 0 ,$$

which implies that

$$n_i \neq 1 \Rightarrow y_i = 0 .$$

The above relation holds even if $\tilde{R}$ is of type A_r. If $y_i, y_j > 0$ $(i \neq j)$, by Proposition 2.1 there exist $\alpha, \beta \in \tilde{R}$ such that

$$\alpha = \sum_{k=1}^{r} a_k \alpha_k , \quad \beta = \sum_{k=1}^{r} b_k \alpha_k \in \tilde{R}_+ , \quad a_i \geq 1 , \quad b_j \geq 1 , \quad \alpha + \beta \in \tilde{R}_+ .$$

We have

$$0 < \langle \alpha + \beta, H \rangle \leq \langle \alpha_0, H \rangle < \pi ,$$

which contradicts Lemma 4.5. Hence we get [ii]$'$ $H = x\tilde{H}_i$ $(n_i = 1, 0 < x < 1)$.

Next we assume $\alpha_0 \in \tilde{\mathcal{R}}_+^{\Delta}$ (i.e., $\langle \alpha_0, H \rangle = \pi$).

In the case when $H = x\tilde{H}_i$. If $n_i = 1$ then $K \operatorname{Exp} H$ is a single point $\{o\}$. If $n_i = 2$, then [i] holds. If $n_i \geq 3$, then by Proposition 2.4 there exists $\alpha \in \tilde{R}_+$ such that

$$\alpha = \sum_{k=1}^{r} a_k \alpha_k , \quad a_i = 1 , \quad \alpha + \alpha_i \in \tilde{R}_+ .$$

Then

$$\alpha, \alpha_i \notin \tilde{\mathcal{R}}_+^{\Delta} , \quad \langle \alpha + \alpha_i, H \rangle < \langle \alpha_0, H \rangle = \pi ,$$

which is a contradiction.

In the case when $H = y_i \tilde{H}_i + y_j \tilde{H}_j$, by Proposition 2.1 there exist $\alpha, \beta \in \tilde{R}_+$ such that

$$\alpha = \sum_{k=1}^{r} a_k \alpha_k , \quad \beta = \sum_{k=1}^{r} b_k \alpha_k , \quad a_i = 1 , \quad b_j = 1 , \quad \alpha + \beta \in \tilde{R}_+ .$$

Then by Lemma 4.5, we have

$$\langle \alpha + \beta, H \rangle = \pi = \langle \alpha_0, H \rangle .$$

Hence we get $n_i = n_j = 1$ ([iii]$'$).

In the case when $y_i, y_j, y_k > 0$. By changing the indices i, j and k, we can suppose α_k does not lie between α_i and α_j in the Dynkin diagram. Let α' be the sum of all simple roots which lie between α_i and α_j in the Dynkin diagram. Put $\alpha = \alpha_i + \alpha'$, $\beta = \alpha_j$, then we have $\alpha, \beta \notin \tilde{\mathcal{R}}_+^{\Delta}$ and

$$\alpha + \beta \in \tilde{R}_+ , \quad \langle \alpha + \beta, H \rangle < \langle \alpha_0, H \rangle = \pi ,$$

which is a contradiction.

Hence we have $(\mathfrak{k}, \mathfrak{n}_K^H)$ is an orthogonal symmetric Lie algebra if and only if H is one of [i], [ii]$'$ and [iii]$'$.

We will show that H is one of (i), (ii)$'$ and (iii)$'$ if and only if H is one of [i], [ii]$'$ and [iii]$'$ using the following theorem.

THEOREM 4.7 ([11, Chap. II, §5]). *Renumbering α_i if necessary, let*

$$\tilde{F} = \left\{ \begin{array}{llll} \alpha_1, & \cdots & , \alpha_k, & \underbrace{\alpha_{k+1}, \cdots, \alpha_l}_{\mathfrak{a}}, \quad \underbrace{\alpha_{l+1}, \cdots, \alpha_m}_{\mathfrak{b}}, \\ \updownarrow & & \updownarrow & \\ \alpha_{p(1)}, & \cdots & , \alpha_{p(k)} & \end{array} \right\}.$$

Here $\alpha_i \leftrightarrow \alpha_{p(i)}$ means that α_i and $\alpha_{p(i)}$ are transformed each other by Satake involution. Then the dual system $\{H_i\}_{1 \le i \le l}$ of

$$F = \left\{ \begin{array}{c} \lambda_1 = \bar{\alpha}_1, \cdots, \lambda_k = \bar{\alpha}_k, \\ \lambda_{k+1} = \bar{\alpha}_{k+1} = \alpha_{k+1}, \cdots, \lambda_l = \bar{\alpha}_l = \alpha_l \end{array} \right\}$$

is given by

$$H_i = \tilde{H}_i + \tilde{H}_{p(i)} \quad (1 \le i \le k),$$
$$H_j = \tilde{H}_j \quad (k+1 \le j \le l).$$

(i) If $H = (\pi/2)H_i$ ($m_i = 2$), then $K \operatorname{Exp} H$ is a totally geodesic submanifold in M. Hence by Lemma 4.4, H is one of [i], [ii] and [iii]. Since $\langle \alpha_0, H \rangle = \langle \delta, H \rangle = \pi$, H is [i] or [iii].

(ii)′ If $H = \pi x H_i$ ($m_i = 1, 0 < x < 1$), then $K \operatorname{Exp}(\pi/2)H_i$ is a totally geodesic submanifold in M. Since $\langle \alpha_0, (\pi/2)H_i \rangle < \pi$, $(\pi/2)H_i = (\pi/2)\tilde{H}_s$ for some s with $n_s = 1$. Hence H is [ii]′.

(iii)′ If $H = \pi(xH_i + (1-x)H_j)$ ($m_i = m_j = 1, 0 < x < 1$), the above argument implies that $H_i = \tilde{H}_s$, $H_j = \tilde{H}_t$ for some s, t with $n_s = n_t = 1$. Hence H is [iii]′.

[i] If $H = (\pi/2)\tilde{H}_i \in \mathfrak{a}$ ($n_i = 2$), then $K \operatorname{Exp} H$ is a totally geodesic submanifold in M. Since $\langle \alpha_0, H \rangle = \pi$, H is (i) or (iii)′.

[ii]′ If $H = \pi x \tilde{H}_i \in \mathfrak{a}$ ($n_i = 1, 0 < x < 1$), then a similar argument in (ii)′ implies that $\tilde{H}_i = H_s$ for some s with $m_s = 1$. H is (ii)′.

[iii]′ In the case when $H = \pi(x\tilde{H}_i + (1-x)\tilde{H}_j) \in \mathfrak{a}$ ($n_i = n_j = 1, 0 < x < 1$). If $\tilde{H}_i \in \mathfrak{a}$, which is equivalent to $\tilde{H}_j \in \mathfrak{a}$, then a similar argument in (ii)′ implies $\tilde{H}_i = H_s$, $\tilde{H}_j = H_t$ for some s, t with $m_s = m_t = 1$. Hence H is (iii)′. So we assume that $\tilde{H}_i \notin \mathfrak{a}$, which is equivalent to $\tilde{H}_j \notin \mathfrak{a}$. From the lemma below, we have [iii] $H = (\pi/2)\tilde{H}_i + (\pi/2)\tilde{H}_{p(i)}$. Hence H is (i).

Hence we complete the proof of Lemma 4.6.

LEMMA 4.8. *For $\tilde{H}_i, \tilde{H}_j \notin \mathfrak{a}$ $(i < j)$ and $x, y \neq 0$, the condition $H = x\tilde{H}_i + y\tilde{H}_j \in \mathfrak{a}$ implies $j = p(i)$, $x = y$.*

PROOF. Since $\tilde{H}_i, \tilde{H}_j \notin \mathfrak{a}$, we have $\alpha_i, \alpha_j \notin \mathfrak{a}$ by Theorem 4.7. Since $\langle \alpha_i, H \rangle = x \neq 0$ and H is in $\mathfrak{a}$, we have $\alpha_i \notin \mathfrak{b}$. Similarly we have $\alpha_j \notin \mathfrak{b}$. Hence we have

$$i, j \in \{1, \cdots, k, p(1), \cdots, p(k)\}.$$

We write

$$\tilde{H}_s = Y_s + Z_s \quad (Y_s \in \mathfrak{a}, Z_s \in \mathfrak{b}, s \in \{1, \cdots, k, p(1), \cdots, p(k)\}).$$

Since $\{\tilde{H}_s\}$ span $\mathfrak{t}$, $\{Z_1, \cdots, Z_k, Z_{l+1}, \cdots, Z_m\}$ span $\mathfrak{b}$. On the other hand, $\dim \mathfrak{b} = \dim \mathfrak{t} - \dim \mathfrak{a} = k + m - l$. Hence $\{Z_1, \cdots, Z_k, Z_{l+1}, \cdots, Z_m\}$ is a basis of $\mathfrak{b}$. Therefore we get the assertion.

LEMMA 4.9. *The condition* (3) *in Theorem* 4.2 *holds if and only if* $\sin\langle\alpha, H\rangle$ *is a constant for* $\alpha \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^{\Delta}$.

PROOF. The Riemannian metrics g_n and g_i are proportional if and only if there exists a positive constant c such that

$$\langle X, Y\rangle = c\langle \Psi_* X, \Psi_* Y\rangle \quad \text{for} \quad X, Y \in (\mathfrak{n}_K^H)^\perp .$$

Since $\Psi_* S_\alpha = -\sin\langle\alpha, H\rangle T_\alpha$, this condition is equivalent to

$$\sin\langle\alpha, H\rangle = \text{constant} \quad (\alpha \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^{\Delta}) .$$

If H is one of (i)–(viii) in Theorem 4.2, then it is clear that g_n and g_i are proportional by Lemma 4.9. To show the converse, take an admissible subset Δ such that H is in Q^Δ.
(I) In the case when $\langle\delta, H\rangle < \pi$.

$$H = x_{i_1} H_{i_1} + \cdots + x_{i_k} H_{i_k} , \quad x_{i_s} > 0 \; (1 \le s \le k) , \quad \sum_{s=1}^{k} m_{i_s} x_{i_s} < \pi .$$

In the case when $\langle\lambda, H\rangle = \text{constant} \in (0, \pi)$ $(\lambda \in R_+ \setminus \mathcal{R}_+^{\Delta})$. We have

$$x_{i_s} = \langle\lambda_{i_s}, H\rangle = \langle\delta, H\rangle = \sum_{s=1}^{k} m_{i_s} x_{i_s} , \quad (1 \le s \le k) ,$$

which implies that (ii)′ $H = \pi x H_i$ $(m_i = 1, 0 < x < 1)$.

In the case when

$$R_+ \setminus \mathcal{R}_+^{\Delta} = A \cup B , \quad A, B \neq \emptyset , \quad \pi - \langle A, H\rangle = \langle B, H\rangle = c \in \left(0, \frac{\pi}{2}\right) .$$

Since $\delta \in A$, we have

$$c = \pi - \langle\delta, H\rangle = \pi - \sum_{s=1}^{k} m_{i_s} x_{i_s} .$$

If there exists a restricted simple root λ_{i_s} such that $\lambda_{i_s} \in A$, then

$$c = \pi - \langle\lambda_{i_s}, H\rangle = \pi - x_{i_s} .$$

Hence (ii)′ $H = \pi x H_i$ $(m_i = 1, 0 < x < 1)$. Hence we assume that $\{\lambda_{i_1}, \cdots, \lambda_{i_k}\} \subset B$.

$$c = \langle\lambda_{i_s}, H\rangle = x_{i_s} \quad (1 \le s \le k) .$$

From Proposition 2.3, there exist a simple root λ_{i_t} with $0 < \langle\lambda_{i_t}, H\rangle < \pi$ and $\mu' \in \mathfrak{a}$ with $\langle\mu', H\rangle = 0$ such that

$$\mu = \delta - \lambda_{i_t} - \mu' \in R_+ .$$

Since $0 < \langle \mu, H \rangle < \langle \delta, H \rangle$, we have $\mu \in B$. Hence

$$c = \langle \mu, H \rangle = c\left(\sum_{s=1}^{k} m_{i_s} - 1 \right),$$

which implies that

$$\sum_{s=1}^{k} m_{i_s} = 2, \quad c = \frac{\pi}{3}.$$

Hence we have

$$\begin{array}{lll} \text{(iv)} & H = (\pi/3)H_i & (m_i = 2) \quad \text{or} \\ \text{(vii)} & H = (\pi/3)(H_i + H_j) & (m_i = m_j = 1). \end{array}$$

(II) In the case when $\langle \delta, H \rangle = \pi$. We can write

$$H = x_{i_1} H_{i_1} + \cdots + x_{i_k} H_{i_k}, \quad x_{i_s} > 0 \ (1 \le s \le k), \quad \sum_{s=1}^{k} m_{i_s} x_{i_s} = \pi.$$

In the case when $\langle \lambda, H \rangle = c$ ($\lambda \in R_+ \setminus \mathcal{R}_+^{\Delta}$). We have

$$c = \langle \lambda_{i_s}, H \rangle = x_{i_s}, \quad (1 \le s \le k).$$

From Proposition 2.3, there exist a simple root λ_{i_t} with $0 < \langle \lambda_{i_t}, H \rangle < \pi$ and $\mu' \in \mathfrak{a}$ with $\langle \mu', H \rangle = 0$ such that

$$\mu = \delta - \lambda_{i_t} - \mu' \in R_+.$$

If $\langle \mu, H \rangle = 0$, then $\langle \lambda_i, H \rangle = 0$ and $K \operatorname{Exp} H$ is a single point $\{o\}$. Hence we may assume $\mu \in R_+ \setminus \mathcal{R}_+^{\Delta}$. Then

$$c = \langle \mu, H \rangle = c\left(\sum_{s=1}^{k} m_{i_s} - 1 \right) = \langle \delta, H \rangle - c = \pi - c,$$

which implies that

$$c = \frac{\pi}{2}, \quad \sum_{s=1}^{k} m_{i_s} = 2.$$

Hence we have

$$\begin{array}{lll} \text{(i)} & H = (\pi/2)H_i & (m_i = 2) \quad \text{or} \\ \text{(iii)} & H = (\pi/2)(H_i + H_j) & (m_i = m_j = 1). \end{array}$$

In the case when

$$R_+ \setminus \mathcal{R}_+^{\Delta} = A \cup B, \quad A, B \ne \emptyset, \quad \pi - \langle A, H \rangle = \langle B, H \rangle = c \in \left(0, \frac{\pi}{2}\right).$$

Since

$$x_{i_s} = \langle \lambda_{i_s}, H \rangle = \begin{cases} \pi - c & (\lambda_{i_s} \in A), \\ c & (\lambda_{i_s} \in B), \end{cases}$$

we may put

$$H = c(H_{j_1} + \cdots + H_{j_p}) + (\pi - c)(H_{j_{p+1}} + \cdots + H_{j_k}).$$

Since $B \neq \emptyset$, we have $p > 0$. Since $\langle \delta, H \rangle = \pi$, we have

$$\begin{aligned} \pi - c &= \langle \delta, H \rangle \\ &= \sum_{s=1}^{k} m_{j_s} x_{j_s} \\ &= c \sum_{s=1}^{p} m_{j_s} + (\pi - c) \sum_{s=p+1}^{k} m_{j_s}, \end{aligned}$$

which implies that H is (iii)$'$ or

$$H = c(H_{j_1} + \cdots + H_{j_p}), \quad m_{j_1} + \cdots + m_{j_p} > 2.$$

If $m_{j_1} + \cdots + m_{j_p} = 3$, then

$$\pi = \langle \delta, H \rangle = 3c, \quad c = \frac{\pi}{3}$$

and

(v) $H = (\pi/3)H_i$ $(m_i = 3)$,
(vi) $H = (\pi/3)(H_i + H_j)$ $(m_i = 1, m_j = 2)$ or
(viii) $H = (\pi/3)(H_i + H_j + H_k)$ $(m_i = m_j = m_k = 1)$.

If $m_{j_1} + \cdots + m_{j_p} = m > 3$, then by Proposition 2.3 there exist $\lambda, \mu, \nu \in R_+$ such that

$$\langle \lambda, H \rangle = c, \quad \langle \mu, H \rangle = 2c, \quad \langle \nu, H \rangle = 3c.$$

Since $\langle \delta, H \rangle = mc = \pi$, we have $\lambda, \mu, \nu \in R_+ \setminus \mathcal{R}_+^{\Delta}$, which contradicts the assumption.

For $X \in \mathfrak{k}$, we define a Killing vector field X^+ on K/N_K^H by

$$(X^+)_{kN_K^H} = \left.\frac{d}{dt}\right|_{t=0} \exp(tX)kN_K^H.$$

We also denote by the same symbol X^+ the corresponding vector field on $K \operatorname{Exp} H$.

LEMMA 4.10. *For $X \in \mathfrak{k}$,*

$$\begin{aligned} (\nabla_{X^+} X^+)_{\operatorname{Exp} H} &= \text{the } (\operatorname{Ad}(\exp(-H))\mathfrak{k})_{\mathfrak{m}}\text{-component of} \\ &\quad -[(\operatorname{Ad}(\exp(-H))X)_{\mathfrak{m}}, (\operatorname{Ad}(\exp(-H))X)_{\mathfrak{k}}], \\ (\mathbf{D}_{X^*} X^*)_{o'} &= -[X_{(\mathfrak{n}_K^H)^\perp}, X_{\mathfrak{n}_K^H}]. \end{aligned}$$

LEMMA 4.11. *The condition* (4) *in Theorem* 4.2 *holds if and only if for $\alpha, \beta \in \tilde{R}_+ \setminus \tilde{\mathcal{R}}_+^{\Delta}$ the condition $\sin\langle \alpha + \beta, H \rangle \sin\langle \alpha - \beta, H \rangle \neq 0$ implies $\alpha \pm \beta \notin \tilde{R}$.*

PROOF. From Lemma 4.10, we have

$$\begin{aligned} \nabla = \mathbf{D} \Leftrightarrow \text{for } X \in (\mathfrak{n}_K^H)^\perp, \\ \operatorname{Ad}(\exp H)[(\operatorname{Ad}(\exp(-H))X)_{\mathfrak{m}}, (\operatorname{Ad}(\exp(-H))X)_{\mathfrak{k}}] \subset \mathfrak{m}. \end{aligned}$$

This condition is equivalent to the following condition:

For $X, Y \in (\mathfrak{n}_K^H)^\perp$,

$$\begin{aligned}\mathrm{Ad}(\exp H)\{[(\mathrm{Ad}(\exp(-H))X)_{\mathfrak{m}},\ (\mathrm{Ad}(\exp(-H))Y)_{\mathfrak{k}}]\\ +[(\mathrm{Ad}(\exp(-H))Y)_{\mathfrak{m}},\ (\mathrm{Ad}(\exp(-H))X)_{\mathfrak{k}}]\}\subset \mathfrak{m}\,.\end{aligned}$$

Using (4.2), we have

$$\begin{aligned}\mathbf{\nabla}=\mathbf{D} &\Leftrightarrow (\sin^2\langle\beta, H\rangle\cos^2\langle\alpha, H\rangle - \sin^2\langle\alpha, H\rangle\cos^2\beta, H)[S_\alpha, S_\beta]=0\\ &\Leftrightarrow \sin\langle\alpha+\beta, H\rangle\sin\langle\alpha-\beta, H\rangle[S_\alpha, S_\beta]=0 \quad (\alpha,\beta\in\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta)\,.\end{aligned}$$

By a similar argument in the proof of Lemma 3.3, we have the assertion.

It is clear that if g_n and g_i are proportional then $\mathbf{\nabla}=\mathbf{D}$. In order to show the converse, we prove the following lemma.

LEMMA 4.12. For any $\alpha, \beta \in \tilde{R}\setminus\tilde{\mathcal{R}}_+^\Delta$ $(\alpha\neq\beta)$, there exists a sequence $\{\gamma_i\}_{1\le i\le k}$ of roots such that

$$\begin{aligned}&\alpha=\gamma_0, \gamma_1, \cdots, \gamma_k=\beta \quad \gamma_i\in\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta\,,\\ &\gamma_{i-1}+\gamma_i\in\tilde{R} \quad \text{or} \quad \gamma_{i-1}-\gamma_i\in\tilde{R} \quad (1\le i\le k)\,.\end{aligned}$$

PROOF. We define an equivalence relation $\sim$ on $\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta$ as follows: For $\alpha,\beta\in\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta$, $\alpha\sim\beta$ if and only if one of the following two conditions holds.

$$\begin{cases}\alpha=\beta\,,\\ \text{There exists a sequence } \gamma_0=\alpha, \gamma_1, \cdots, \gamma_k=\beta \text{ in } \tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta\\ \text{such that } \gamma_{i-1}+\gamma_i\in\tilde{R} \quad \text{or} \quad \gamma_{i-1}-\gamma_i\in\tilde{R} \quad (1\le i\le k)\,.\end{cases}$$

For $a\in\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta$, we denote by $C(\alpha)$ the equivalence class of α. There exists a simple root $\alpha_i\in\tilde{F}$ such that $\alpha_i\in\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta$. If $C(\alpha_i)\neq C(\alpha)$, then $\mathrm{span}C(\alpha_i)\perp\mathrm{span}C(\alpha)$. On the other hand, since the Dynkin diagram of $\tilde{R}$ is connected, $\mathrm{span}C(\alpha_i)=\mathfrak{t}$. Hence we get $C(\alpha_i)=\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta$. Therefore we get the assertion.

We show that if $\mathbf{\nabla}=\mathbf{D}$ then g_i and g_n are proportional. For $\alpha,\beta\in\tilde{R}_+\setminus\tilde{\mathcal{R}}_+^\Delta$ $(\alpha\neq\beta)$, take a sequence $\{\gamma_i\}$ as in Lemma 4.12. Since $\mathbf{\nabla}=\mathbf{D}$, we have

$$\sin\langle\gamma_{i-1}+\gamma_i, H\rangle\sin\langle\gamma_{i-1}-\gamma_i, H\rangle=0\,,$$

which implies that

$$\sin\langle\alpha, H\rangle=\sin\langle\gamma_0, H\rangle=\cdots=\sin\langle\gamma_k, H\rangle=\sin\langle\beta, H\rangle\,.$$

Thus we complete the proof of Theorem 4.2 when M is of type I.

When $M=L$ is of type II, we use the same notation as in §3. For $H\in\mathfrak{t}(\mathfrak{l})$ with $0\le\langle H/2, \tilde{R}_+(\mathfrak{l})\rangle\le\pi$, we consider the submanifold $\exp(\mathrm{Ad}(L)H)\subset L$. By a similar argument above we have the following

$$\begin{aligned}&\exp(\mathrm{Ad}(L)H) \subset L \text{ is a totally geodesic submanifold}\\ &\Leftrightarrow \sin\langle H/2, \alpha\rangle \in \{0, 1\} \quad \text{for} \quad \alpha \in \tilde{R}_+(\mathfrak{l})\\ &\Leftrightarrow \begin{cases} \text{(i)} & H/2 = (\pi/2)H_i \qquad (m_i = 2),\\ \text{(ii)} & H/2 = (\pi/2)H_i \qquad (m_i = 1),\\ \text{(iii)} & H/2 = (\pi/2)(H_i + H_j) \quad (m_i = m_j = 1).\end{cases}\end{aligned}$$

We define a closed subgroup $Z_L^{\exp H}$ of L by

$$Z_L^{\exp H} = \{a \in L \mid a \exp H a^{-1} = \exp H\}.$$

We denote by $\mathfrak{z}_L^{\exp H}$ the Lie algebra of $Z_L^{\exp H}$.

The pair $(\mathfrak{l}, \mathfrak{z}_L^{\exp H})$ is an orthogonal symmetric Lie algebra

$$\begin{aligned}&\Leftrightarrow \text{for } \alpha, \beta \in \tilde{R}_+(\mathfrak{l}) \text{ with } 0 < \langle\alpha, H/2\rangle, \langle\beta, H/2\rangle < \pi,\\ &\quad\begin{cases}\langle\alpha + \beta, H/2\rangle \neq \pi \Rightarrow \alpha + \beta \notin \tilde{R}_+(\mathfrak{l}),\\ \langle\alpha - \beta, H/2\rangle \neq 0 \Rightarrow \alpha - \beta \notin \tilde{R}(\mathfrak{l})\end{cases}\\ &\Leftrightarrow \begin{cases}\text{(i)} & H/2 = (\pi/2)H_i \qquad (m_i = 2),\\ \text{(ii)}' & H/2 = \pi x H_i \qquad (m_i = 1, 0 < x < 1),\\ \text{(iii)}' & H/2 = \pi(xH_i + (1-x)H_j) \quad (m_i = m_j = 1, 0 < x < 1).\end{cases}\end{aligned}$$

The Riemannian metrics g_n and g_i are proportional

$$\begin{aligned}&\Leftrightarrow \sin\langle\alpha, H/2\rangle = \text{constant for } a \in \tilde{R}_+(\mathfrak{l}) \text{ with } 0 < \langle\alpha, H/2\rangle < \pi.\\ &\Leftrightarrow \begin{cases}\text{(i) (ii)}', \text{(iii)}' \text{ or,}\\ \text{(iv)} & H/2 = (\pi/3)H_i \qquad (m_i = 2),\\ \text{(v)} & H/2 = (\pi/3)H_i \qquad (m_i = 3),\\ \text{(vi)} & H/2 = (\pi/3)(H_i + H_j) \quad (m_i = 1, m_j = 2),\\ \text{(vii)} & H/2 = (\pi/3)(H_i + H_j) \quad (m_i = m_j = 1),\\ \text{(viii)} & H/2 = (\pi/3)(H_i + H_j + H_k) \quad (m_i = m_j = m_k = 1).\end{cases}\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla} = \mathbf{D} &\Leftrightarrow \text{for } \alpha, \beta \in \tilde{R}_+(\mathfrak{l}) \text{ with } 0 < \langle\alpha, H/2\rangle, \langle\beta, H/2\rangle < \pi,\\ &\quad \sin\langle\alpha + \beta, H/2\rangle \sin\langle\alpha - \beta, H/2\rangle \neq 0 \Rightarrow \alpha \pm \beta \notin \tilde{R}(\mathfrak{l})\\ &\Leftrightarrow g_n \text{ and } g_i \text{ are proportional}.\end{aligned}$$

Thus the Theorem 4.2 is proved.

COROLLARY 4.13. *When M is an adjoint space (see [3, p. 327] for the definition), then Kp ($p \in M$) is a totally geodesic submanifold in M if and only if $Kp = M^+(p)$, i.e., p is a fixed point of the geodesic symmetry s_o at o.*

PROOF. Put

$$\Gamma(G, K) = \{H \in \mathfrak{a} \mid \exp H \in K\}.$$

Then $p = \mathrm{Exp}\, H$ $(H \in \bar{Q})$ is fixed by s_o if and only if $2H \in \Gamma(G, K)$. When M is an adjoint space, then

$$\Gamma(G, K) = \{H \in \alpha \mid \langle H, R \rangle \subset \pi \mathbf{Z}\}.$$

Hence we get the assertion.

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