philosophical mould' (p. 173). It would take more than the 'rough winds' borrowed from Shakespeare's sonnet to blow insight for the reader into that image.

The discussion of the axioms of infinity and choice that follows suffers from the same flaws. He describes Ramsey as having cast 'a covetous eye' in the direction of these axioms (p. 175). It would be as helpful to claim that he cast 'a cold eye' on axioms, certainly among the most difficult things to covet. In the same way, Sahlin notes that 'it is odd that the founders of logicism wouldn't budge an inch when it came to this ontological assumption [the axiom of infinity]' (p. 176). Somehow the axiom of 'infinity ± an inch' seems hard to motivate.

While some of Sahlin's comments are legitimate reconstructions of Ramsey's arguments, there are other observations which mislead beyond anything that Ramsey said. For example, Sahlin writes, 'Actually, it is interesting to note that one can see with some precision how Ramsey gradually departs from logicism by giving up its axioms one at a time' (p. 177). Logicism is not a view that can be abandoned in this piecemeal fashion, although uncertainty about axioms (like choice) could lead to a rejection of the whole logicist programme. (One may see a reflection of this movement in French intuitionists like Borel and Baire.) Sahlin's picture strikes one as little more plausible than the idea of giving up an axiom one symbol at a time.

Sahlin's exposition of Ramsey's views is the victim of the absence of critical support, an abundance of overblown metaphors, and simple misuses of language. This is not to say that his discussion fails of its purpose, to send the reader back to Ramsey. It is likely that the reader will hurry back to Ramsey to find out what he could have said to produce Sahlin's prose. The comments that Sahlin makes about the structure and development of Ramsey's ideas lose their effectiveness in such surroundings.

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Do revolutions occur in mathematics? This is the question the contributors to this collection of essays have set out to answer. Michael Crowe's 1975 article "Ten 'laws' concerning patterns of change in the history of mathematics"[1] serves as a starting point for the debate. The last of Crowe's ten laws of historical change is the surprisingly blunt assertion that "revolutions never occur in mathematics." Nevertheless, this has also been the conclusion of many eminent mathematicians. Crowe quotes Fourier's 1822 *Théorie analytique de la chaleur*: "this difficult science [mathematics] is formed slowly, but it preserves every principle it has once acquired; it grows and strengthens itself in the midst of many variations and errors of the human mind," and cites a similar statement by Truesdall over a century later: "while imagination, fancy and invention are the soul of mathematical research, in mathematics there has never yet been a revolution" (C. Truesdall, *Essays in the history of mathematics*, 1968).

Crowe defines a 'revolution"as a radical rejection of earlier entities:

> ...this law depends upon at least the minimal stipulation that a necessary characteristic of a revolution is that some previously existing entity (be it king, constitution, or theory) must be overthrown and irrevocably discarded.

The overthrow of monarchy is the paradigmatic example of a political revolution: England 1640-49, France 1789, or Russia 1917. All three of these upheavals were, Crowe observes, followed by, first, the establishment of a dictatorship, and, in the first two instances, the restoration of monarchy with a reduction in actual power. Notably, the restoration of monarchy did not occur in Russia, and, the deluded nostalgia of the growing ranks of embittered neo-Tsarists notwith-standing, is unlikely to ever take place. Tatyana Tolstaya recently wrote that "one woman I know swore to me that at the beginning of the century her grandfather, a simple worker, a typesetter, drank up most of his weekly pay in the inns, and with the leftover change was still able to buy gold rings with emeralds to appease his wife." Russia's current constitutional crisis may lead to a tragic relapse into totalitarian government, but I doubt we will see a return of the Tsars.

The two key modern classics of the history of science and mathematics to which the contributors repeatedly refer, are Thomas Kuhn's *The structure of scientific revolutions* (1962) and Imre Lakatos' *Proofs and refutations* (1976). As the American press has wryly observed, Kuhn's notion of a 'paradigm-shift' has—in its popularity among the educated classes—now even come to describe the sharp break in President Clinton's economic policies with those of his Republican predecessors. In science, the leading idea of historical change as a succession of periods of orthodoxy, crisis, and revolutionary, discontinuous change has been fruitfully applied to the history of astronomy, physics, chemistry and geology. The Copernican overthrow of the Aristotelian-Ptolemaic astronomical model was followed by the establishment of the Newtonian paradigm with the publication of Newton's *Philosophiae naturalis principia mathematica* in 1687. In chemistry, the phlogiston-loss theory of combustion was replaced by Lavoisier's new oxygen-based paradigm. And in 1915, Einstein's General Relativity Theory supplanted the Newtonian paradigm in mechanics.

It is intriguing to observe that these three scientific 'paradigm-shifts' coincided with the three political revolutions mentioned earlier. Britain's Glorious Revolution occurred a year after the publication of *Principia*, Lavoisier published his treatise *Traite élémentaire de la chimie* in 1789, and the Russian Revolution took place two years after the publication of Einstein's revolutionary work. These apparent coincidences suggest there may be stronger links between political and scientific change than we might think.

The antithesis to Crowe's claim in what has come to be known as the Crowe-Dauben debate is presented in the 1984 article "Conceptual revolutions and the history of mathematics," by Joseph Dauben, and an earlier paper by Herbert Mehrtens, "T.S. Kuhn's theories and mathematics" (1976). Using the Greek discovery of incommensurables and Cantor's invention of transfinite set theory as his primary examples, Dauben argues convincingly that radical transformations of mathematics have occurred, and that Crowe's denial of this fact arises from his overly restrictive definition of the term 'revolution.' Dauben finds the first application of the term to mathematics by Bernard de Fontenelle. In an eulogy of Rolle, included in his *Histoire de l'Academie Royale des Sciences* (1719), Fontenelle wrote of the decisive role of l'Hôpital's work in marking the point at which the infinitesimal calculus had become accepted by the majority of mathematicians.

> In those days the book of the Marquis de l'Hôpital had appeared, and almost all the mathematicians began to turn to the side of the new geometry of the infinite, until then hardly known at all. The surpassing universality of its methods, the elegant brevity of its demonstrations, the finesse and directness of the most difficult solutions, its singular and unprecedented novelty, it all embellishes the spirit and has created, in the world of geometry, an unmistakable revolution.

Fontenelle was not "invoking any displacement principle — any rejection of earlier mathematics — before the revolutionary nature of the new geometry of the infinite could be proclaimed." The term was being used to refer to the overwhelming character and magnitude of a *qualitative* change. Since mathematics is a purely deductive science, the changes brought about by a revolution are different from the results of 'paradigm-shift' in the empirical sciences. A revolution in mathematics does not reject earlier results, but the introduction of radically new concepts and methods does displace existing theory, so that "many of the old theorems and discoveries" are "relegated to a significantly lesser position."

Book X of Euclid's *Elements*, for example, presents a new theory of proportions that replaced the older mathematics of discrete numbers and their ratios. Briefly, the new idea involved calling a measure incommensurable if the Euclidean algorithm — originally designed to find the greatest common measure of two magnitudes — fails to terminate. The algorithm is cleverly transformed into a decision procedure for incommensurability! While no previously existing entity or result has been irrecovably discarded (and hence the change fails to be revolutionary in Crowe's terminology), a radically new concept is introduced, and there is a reorganization of mathematics great enough to warrant labelling the change a 'revolution' (under Fontenelle's original sense of the term). As Crowe notes in an "Afterword" to the book, the question of whether revolutions occur in mathematics is in some sense merely definitional. But, apart from clarifying the semantics, thinking about the question leads to an exploration of many key events in the history of mathematics, and a sharpening of the philosophical issues at stake in the historiography of the subject.

In an appendix to his 1984 paper, Dauben adds two more examples of revolutions in mathematics: the new standards of rigour introduced by Cauchy in the nineteenh century, and the creation of nonstandard analysis by Abraham Robinson. Not merely new sets of concepts, each "represents a new way of doing mathematics, by means of which its face and framework were dramatically altered in ways that indeed proved to be revolutionary." Robinson saw his rigorous definition of infinitesimals — which Gödel said had done more than anything else to bring mathematics and logic together — as a justification of the ideas of Leibniz and Cauchy.

The rest of the book, with the exception of Caroline Dunmore's short, lucid essay, "Meta-level revolutions in mathematics," is taken up by eight case studies in the history of mathematics:

Paolo Mancosu, "Descartes's *Géométrie* and revolutions in mathematics"
Emily Grosholz, "Was Leibniz a mathematical revolutionary?"
Giulio Giorello, "The 'fine structure' of mathematical revolutions: metaphysics, legitimacy, and rigour. The case of the calculus from Newton to Berkeley and MacLaurin"
Yuxin Zheng, "Non-Euclidean geometry and revolutions in mathematics"
Luciano Boi, "The 'revolution' in the geometrical vision of space in the nineteenth century, and the hermeneutical epistemology of mathematics"
Jeremy Gray, "The nineteenth-century revolution in mathematical ontology"
Herbert Breger, "A restoration that failed: Paul Finsler's theory of sets"
Donald Gillies, "The Fregean revolution in Logic"

These essays are the real meat of the book in my view, and contain a wealth of historical detail and philosophical puzzles. Mancosu carries out a careful study of sections of

Descartes' *La Géométrie*; the text itself is inexpensively available for reference from Dover, with the original French, and English translation, on facing pages. Giorello's essay is particularly lively and entertaining, and includes correspondance between Newton and his contemporaries, including this letter by Samuel Pepys of 8 April 1661:

> Here we supped very merry, and late to bed; Sir Wm. telling me that old Edgeborrow, his predecessor, did die and walk in my chamber — did make me somewhat afeared, but not so much as for mirth sake I did seem.

Read the book to find out the relevance of this ghostly visit for the history of mathematics! (Hint: think of fluxions.) Dunmore, who holds advanced degrees in mathematics and philosophy (but currently works as a business analyst) characterizes conceptual revolutions in mathematics as a dichotomy between 'object' and 'meta' levels: At the level of mathematical objects, there is no real discontinuity, no revolution in Crowe's sense; the great conceptual transformations only discard the methodologies and metaphysics of the old paradigm, no actual results are lost as in the physical sciences. Grosholz traces the path of Leibniz's mathematical development, the flowering of his extraordinary ability to generalize and fruitfully combine results from geometry, algebra and number theory to produce original and fertile solutions, culminating in the creation of the infinitesimal calculus. Breger's essay shows how Finsler's axioms for set theory were a failed attempt to return to nineteenth century ideas and standards. Gillies compares Frege's *Begriffsschrift* and modern logic textbooks like Mendelson's 1964 *Introduction to mathematical logic* and Bell and Machover's *A course in mathematical logic* (1977) to evaluate the extent of the Fregean revolution in logic.

Finally, there is an extensive bibliography of classic and modern works in the history and philosophy of mathematics. Overall, I think the book would be enjoyable for anyone with an interest in these subjects. I found myself disagreeing with the radically Kuhnian views that question the objectivity of mathematical knowledge, especially with Herbert Mehrtens, who also believes that the 'critical distance' of historians of science means they should never use the word 'great.' But, these disagreements only add spice to the reading experience. The writers are all demonstrably able scholars, and Giorello and Mancosu are also practicing mathematicians (Giorello in functional analysis, Mancosu in logic). The book would make a good addition to any library.