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CHARACTERIZATIONS OF CLASSES OF I_0 SETS IN DISCRETE ABELIAN GROUPS

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ABSTRACT. A subset E of a discrete abelian group is an I_0 set if every bounded function on E is the restriction of the Fourier transform of a discrete measure. Special examples of I_0 sets include (real) RI_0 sets and (real) FZI_0 sets (short for Fatou-Zygmund interpolation sets): E is called (real) RI_0 [or (real) FZI_0] if every bounded (real) Hermitian function on E can be interpolated by the transform of a discrete, real [respectively, nonnegative] measure.

The two pairs of classes, (real) RI_0 and (real) FZI_0, are shown to be identical for sets not containing the identity; the class of real RI_0 sets is strictly smaller than the class of I_0 sets, and the class of FZI_0 sets is strictly smaller than the real FZI_0 sets. That completes the problem of determining which of these classes are different and which are the same. Topological characterizations of these classes of sets are given, as are some union results.

1. Introduction and summary of results. Let G be a compact abelian group with discrete, dual group Γ. A subset $E \subseteq \Gamma$ is called a *Sidon set* (respectively *I_0 set*)[1] if every bounded function on E can be interpolated by the Fourier Stieltjes transform of a (respectively discrete) measure on G. There are examples of Sidon sets that are not I_0, but both classes are plentiful. Indeed, every infinite subset of Γ contains an infinite I_0 set. For proofs, see [**2, 3, 7, 10, 12, 13**].

I_0 sets have been extensively studied. During the 1960s and 70s much of the work was related to topological characterizations of the property, cf., [**13, 14, 18, 22**]. This line of research was extended in [**8, 20**] where it was also proven that Sidon sets can be characterized by having proportional I_0 subsets that possess these topological properties in a precise quantitative sense. More recent work, such as [**5–9**] and

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[**16**], has emphasized the study of particular classes of examples of I_0 sets, such as Hadamard sets and ε-Kronecker sets (also called ε-free sets [**4**]).

In this paper we continue the study of subclasses of I_0 sets, requiring that the interpolating measure be real or nonnegative, in addition to discrete. This continues [**7**], where the emphasis was on proving the existence of infinite interpolation sets with those properties. Our emphasis is on characterizing those classes of I_0 sets and determining whether they are the same or different from one another.

1.1. Definitions and results.

Definition 1. A function φ on a subset $E \subset \Gamma$ is *Hermitian* if $\varphi(\chi) = \overline{\varphi(\chi^{-1})}$ for all $\chi \in E$ with $\chi^{-1} \in E$.

Definition 2. A set $E \subseteq \Gamma$ is called:

(1) (real) RI_0 if every (real-valued) bounded Hermitian function φ on E is the restriction of the Fourier-Stieltjes transform of a real, discrete measure to E.

(2) (real) FZI_0 if every (real-valued) bounded Hermitian function φ on E is the restriction of a Fourier-Stieltjes transform of a nonnegative, discrete measure.

The set E is called *asymmetric* if $\gamma \in E \cap E^{-1}$ implies $\gamma = \gamma^{-1}$. Notice that a set E is (real) RI_0 (or FZI_0) if and only if $E \cup E^{-1}$ is the same, so there is no loss in working with asymmetric sets, as we frequently do.

The classes RI_0 and FZI_0 were introduced in [**7**]. Some trivial observations include: RI_0 (or FZI_0) sets are real RI_0 (real FZI_0); and real RI_0 asymmetric sets are I_0. The class of real FZI_0 sets is smaller than the class of RI_0 sets since the singleton consisting of the identity element $\{\mathbf{1}\}$ of Γ is RI_0 but not real FZI_0.

Less trivially, it is shown in [**7**] that E is RI_0 if and only if $E \cup E^{-1}$ is I_0 and, consequently, in contrast to the analogous result for Sidon sets[2], the class of RI_0 sets (even in $\mathbf{Z}$) is strictly smaller than the class of I_0 sets.

Here we complete the argument, determining the classes which are the same and which are different. We show that classes (real) RI_0 and (real) FZI_0 are identical for sets not containing the identity (Proposition 3.1), and that real RI_0 is the same as RI_0 for asymmetric subsets of $\mathbf{Z}$. The three classes I_0, real RI_0/FZI_0, and RI_0/FZI_0 are distinct, as shown by Examples 4.1 and 4.2.[3]

Our results are based, in part, on topological characterizations of (real) RI_0/FZI_0 sets; these are in the spirit of the classical work on I_0 sets. We use these characterizations to study the union problem in Section 5.

The first sets where these kinds of interpolation properties were studied were the *Hadamard sets* $E = \{n_j\} \subset \mathbf{N}$, where there is a $1 < q$ such that $q \le n_{j+1}/n_j$ for all j. Adaptation of the arguments in [**6, 16**] shows that Hadamard sets are FZI_0 (see also [**7, 15, 22**]). Other examples of FZI_0 sets include ε-Kronecker sets with $\varepsilon < \sqrt{2}$, see [**5, 6, 9**], and independent sets [**7**]. A main result of [**7**] is that every infinite subset of Γ contains an FZI_0 set of the same cardinality.

2. Preliminaries.

2.1. Notation. For a compact abelian group G, G_d denotes the corresponding group with the discrete topology. The Bohr compactification of Γ is denoted by $\overline{\Gamma}$. If $E \subset \Gamma$, $\overline{E}$ denotes the closure of E in $\overline{\Gamma}$. We write $B(\ell^\infty(E))$ for the unit ball of $\ell^\infty(E)$. A superscript r or $+$ on a space of measures will denote the real-valued, respectively positive, measures in that class, and the subscript d denotes discrete measures.

The following result is proved, with slightly more generality, in [**7**, 2.1 and 2.4]. The analogous result for I_0 sets removes the constraints that μ should be real and φ Hermitian.

Proposition 2.1. *Let G be a compact group and $E \subset \Gamma$. The following properties are equivalent.*

(1) *E is RI_0 (respectively FZI_0).*

(2) *There is a constant N such that, for all Hermitian $\varphi \in B(\ell^\infty(E))$, there exists a $\mu \in M_d^r(G)$ (respectively $M_d^+(G)$) with $\|\mu\| \le N$ and $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$.*

(3) *There exists* $0 < \varepsilon < 1$ (*equivalently, for every* $0 < \varepsilon < 1$) *and integer* N *such that for all Hermitian* $\varphi \in B(\ell^\infty(E))$, *there exists a* $\mu \in M_d^r(G)$ (*respectively* $M_d^+(G)$) *with* $\|\mu\|_{M(G)} \leq N$ *and* $|\widehat{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ *for all* $\gamma \in E$.

(4) *There exists a* $0 < \varepsilon < 1$ (*equivalently, for all* $0 < \varepsilon < 1$) *such that for all Hermitian* $\varphi \in B(\ell^\infty(E))$, *there exists a* $\mu \in M_d^r(G)$ (*respectively* $M_d^+(G)$), *with* $|\widehat{\mu}(\gamma) - \varphi(\gamma)| < \varepsilon$ *for all* $\gamma \in E$.

(5) *There exists a* $0 < \varepsilon < 1$, *equivalently, for every* $0 < \varepsilon < 1$, *such that, for each pair of Hermitian functions* $r : E \to \{\pm 1\}$ *and* $s : E \to \{0, \pm i\}$, *there are measures* $\mu_1, \mu_2 \in M_d^r(G)$ (*respectively* $M_d^+(G)$), *such that*

$$|\widehat{\mu_1}(\chi) - r(\chi)| < \varepsilon \text{ and } |\widehat{\mu_2}(\chi) - s(\chi)| < \varepsilon \text{ for all } \chi \in E.$$

We call the least of the constants N satisfying (2) the RI_0 (respectively FZI_0) constant of E.

Similar equivalencies hold for real RI_0 and real FZI_0 with "Hermitian φ" replaced by "real-valued Hermitian φ." Consequently, an asymmetric real RI_0 set is I_0.

We shall use the following result [**7**, Theorem 2.3] at several points.

Theorem 2.2. $E \subset \Gamma$ *is* RI_0 *if and only if* $E \cup E^{-1}$ *is* I_0.

3. Properties and examples of (real) RI_0 and real FZI_0 sets.

3.1. When (real) RI_0 sets are (real) FZI_0. We recall that a closed subset E of the locally compact abelian group Γ is *Helson* if, for every element $f \in C_0(E)$, there exists a measure μ on G such that $\widehat{\mu} = f$ on E. See, for example, [**11**, Chapter 2] for properties of Helson sets. In our context, the important fact is that the closure of an I_0 set in the Bohr compactification $\overline{\Gamma}$ is a Helson subset of $\overline{\Gamma}$.[4] Using that observation about the Bohr closures of I_0 sets, we will prove the following proposition.

Proposition 3.1. *Let* $E \subset \Gamma \setminus \{\mathbf{1}\}$. *Then* E *is* (*real*) RI_0 *if and only if* E *is* (*real*) FZI_0.

Proof. Suppose E is RI_0. We must show that for every bounded Hermitian function φ on $E \cup E^{-1}$ there exists $\nu \in M_d^+(G)$ with $\widehat{\nu} = \varphi$ on $E \cup E^{-1}$.

Let $A = E \cap E^{-1}$ and $B = E \setminus A$, so $E \cup E^{-1} = A \cup B \cup B^{-1}$ is a disjoint union of I_0 sets. Applying Theorem 2.2, the closures of A, B and B^{-1} are disjoint Helson sets in the Bohr compactification $\overline{\Gamma}$. Let $\overline{E}$ denote the closure. Because $E \cup E^{-1}$ is I_0, we may extend φ to a continuous Hermitian function φ' on $\overline{E} \cup (\overline{E})^{-1}$.

By a theorem of Smith [**11**, 2.5.1] or [**21**], for every continuous Hermitian function φ' on the Helson set $\overline{E} \cup (\overline{E})^{-1}$, there exists a nonnegative $\nu \in M_d(G) = L^1(G_d)$ such that $\widehat{\nu} = \varphi'$ on $\overline{E} \cup (\overline{E})^{-1}$. (Here we apply Smith's theorem to the case where his G, our $\overline{\Gamma}$, is compact.)

Then $\nu \in M_d^+(G)$ has $\widehat{\nu} = \varphi$ on $E \cup E^{-1}$.

The argument is similar if E is real RI_0. The possibility that $E \cup E^{-1}$ is not I_0 is irrelevant here, because if $\mu \in M_d^r(G)$ interpolates a bounded real-valued Hermitian φ on E, then $\widehat{\mu}$ is continuous and Hermitian on all of $\overline{\Gamma}$ and hence the same on the Helson set $\overline{E} \cup \overline{E^{-1}}$, whether or not that union is of disjoint sets.

It is obvious that if E is (real) FZI_0 then E is (real) RI_0. $\square$

3.2. Topological characterizations of real RI_0/FZI_0 sets. It is well known that I_0 sets can be characterized by topological properties. We list below the main classical results and refer the reader to [**13, 18, 20**] for proofs and further discussion.

Theorem 3.2. *For $E \subseteq \Gamma$, the following are equivalent:*

(1) *E is I_0;*

(2) *For every subset $F \subseteq E$, the sets F and $E \setminus F$ have disjoint closures in $\overline{\Gamma}$;*

(3) *For every subset $F \subseteq E$ there exists a $\sigma \in M_d(G)$ such that $\widehat{\sigma}(F)$ and $\widehat{\sigma}(E \setminus F)$ have disjoint closures in $\mathbf{C}$.*

(4) *Every $0, 1$ valued E-function can be extended to a continuous function on $\overline{E}$, equivalently, $\overline{\Gamma}$.*

The equivalence of these properties is due to the facts that $\overline{\Gamma}$ is a normal space and that $C(\overline{\Gamma})$ is the uniform closure of $\{\widehat{\mu} : \mu \in M_d(G)\}$.

In this section we obtain similar characterizations for real RI_0 sets.

It is convenient first to introduce some notation.

Notation 3.3. Let $\widetilde{\Gamma}$ denote the quotient space of $\overline{\Gamma}$ where we identify each χ with its inverse χ^{-1}. We give $\widetilde{\Gamma}$ the quotient topology induced by the natural map $q : \Gamma \to \widetilde{\Gamma}$.

As q is both closed and continuous, it follows that $\widetilde{\Gamma}$ is compact and normal.

Theorem 3.4. *For an asymmetric subset $E \subseteq \Gamma$ the following are equivalent*:

(1) *E is real RI_0*;

(2) *For every $F \subseteq E$, $q(F)$ and $q(E \setminus F)$ have disjoint closures in $\widetilde{\Gamma}$.*

(3) *For every $F \subseteq E$ there exists a $\sigma \in M_d^r(G)$, with $\widehat{\sigma}$ real-valued, such that $\widehat{\sigma}(F)$ and $\widehat{\sigma}(E \setminus F)$ have disjoint closures in* $\mathbf{C}$;

(4) *Every $\{0,1\}$ valued E-function can be extended to a continuous, real-valued function on $q(\overline{E})$ (equivalently to $\overline{\Gamma}$).*

To prove Theorem 3.4, we need the following lemma.

Lemma 3.5. *Let $E, F \subseteq \Gamma$, and suppose that $q(F)$ and $q(E)$ have disjoint closures in $\widetilde{\Gamma}$. Then, for any $\varepsilon > 0$, there exists a $\mu \in M_d^r(G)$, with $\widehat{\mu}$ real-valued, such that $|\widehat{\mu}(F) - 1| < \varepsilon$ and $|\widehat{\mu}(E)| < \varepsilon$.*

Proof of Lemma 3.5. By normality, there exists a continuous function $f : \widetilde{\Gamma} \to \mathbf{R}$ such that $f(q(F)) = 0$ and $f(q(E)) = 1$.

Let $S = \{\widehat{\mu} : \mu \in M_d^r(G)$ and $\widehat{\mu}$ is real-valued$\}$. Since a real measure μ with real-valued transform has the property that $\widehat{\mu}(\chi) = \widehat{\mu}(\chi^{-1})$, any $\widehat{\mu} \in S$ can be viewed as a function on $\widetilde{\Gamma}$. Moreover, such functions are continuous with respect to the quotient topology on $\widetilde{\Gamma}$.

The set S is a subalgebra of real-valued, continuous functions on $\widetilde{\Gamma}$ that contains the real constants, namely $\widehat{k\delta_e}$. If $q(\chi) \neq q(\psi)$, then $\chi \neq \psi, \psi^{-1}$, and there exists a $\widehat{\mu} \in S$ separating the points $q(\chi)$ and $q(\psi)$. (One way to see this is to note that $\{\chi, \chi^{-1}\}$ and $\{\psi, \psi^{-1}\}$ are disjoint closed sets in $\overline{\Gamma}$ and hence there exists a $\mu \in M_d(G)$ whose Fourier transform is 0 on the first set and 1 on the second. Replacing μ by $(\mu + \overline{\mu}) + (\widetilde{\mu + \overline{\mu}})$, we see that we have a real measure with real transform.)

The Stone-Weierstrass theorem implies that S is dense in $C_{\mathbf{R}}(\widetilde{\Gamma})$, and, hence, that f can be approximated by some $\widehat{\mu} \in S$. □

Proof of Theorem 3.4. (1) $\Rightarrow$ (4). We interpolate a given $\{0,1\}$ E-function by the Fourier-Stieltjes transform of $\mu \in M_d^r(G)$ and take $\nu = (1/2)(\mu + \tilde{\mu})$, so $\widehat{\nu}$ is real.

(4) $\Rightarrow$ (3). Since the function that is 1 on F and 0 on $E \setminus F$ has a continuous extension to $q(\overline{E})$, the sets $q(F)$ and $q(E \setminus F)$ must have disjoint closures. By the lemma, there exists a $\mu \in M_d^r(G)$ with $\widehat{\mu}$ real-valued such that $\widehat{\mu}(E)$ and $\widehat{\mu}(E \setminus F)$ have disjoint closures.

(3) $\Rightarrow$ (2). Since $\widehat{\sigma}$ can be viewed as a continuous function on $\widetilde{\Gamma}$, the conclusion follows.

(2) $\Rightarrow$ (1) follows easily from Lemma 3.5 by standard arguments. A similar argument for I_0 sets can be found in [**15**, page 129]. □

Corollary 3.6. *An asymmetric set E is real RI_0 if and only if the quotient map q is one-to-one on $\overline{E}$ (equivalently, $\overline{E}$ is asymmetric) and for all subsets $F \subseteq E$, F and $E \setminus F$ have disjoint closures in $\overline{\Gamma}$.*

Proof. ($\Leftarrow$). The two assumptions imply property (2) of the theorem.

($\Rightarrow$). Property (2) of the theorem certainly implies $\overline{F} \cap \overline{E \setminus F} = \varnothing$. If q is not one-to-one on $\overline{E}$, then there is some $\chi, \chi^{-1} \in \overline{E}$ with $\chi \neq \chi^{-1}$. Thus, χ, χ^{-1} must belong to the closure of disjoint subsets of E, say F and $E \setminus F$ respectively. But then $q(\chi)$ belongs to the closure of both $q(F)$ and $q(E \setminus F)$. Thus (2) of the theorem implies both conditions. □

Corollary 3.7. *An asymmetric set E is real RI_0 if and only if E is I_0 and $\overline{E}$ is asymmetric.*

3.3. Topological characterizations of RI_0/FZI_0 sets. In this subsection we topologically characterize RI_0 sets.

Theorem 3.8. *An asymmetric E is RI_0 if and only if:*

(1) *For every $F \subseteq E$, there exists a $\sigma \in M_d^r(G)$, with $\widehat{\sigma}$ real-valued, such that $\widehat{\sigma}(F)$ and $\widehat{\sigma}(E \setminus F)$ have disjoint closures in $\mathbf{C}$ and*

(2) *The Bohr closure of $\{\chi \in E : \chi^2 \neq 1\}$ in $\overline{\Gamma}$ does not contain any elements of order 2.*

Remark 3.9. Condition (2) of the theorem holds, for example, if there exists a $\sigma \in M_d^r(G)$ such that $\widehat{\sigma}(E) = i$. Example 4.1 shows that it is not always the case that (1) implies (2).

Proof. Assume E is RI_0. Since E is asymmetric, it follows that, for each bounded Hermitian E-function φ, there exists a $\sigma \in M_d^r(G)$ such that $\widehat{\sigma}|_E = \varphi$. In particular, we can take $\varphi(F) = \{0\}$, $\varphi(E \setminus F) = \{1\}$. Replacing σ by $\sigma + \widetilde{\sigma}$ we see that condition (1) holds.

Let $E_1 = \{\chi \in E : \chi^2 \neq 1\}$, and suppose that there is a $\gamma \in \overline{E_1}$ of order 2. Suppose $\sigma \in M_d^r(G)$ interpolates the Hermitian function $\varphi = i$ on E_1 and 0 otherwise on E. For any such σ, we have $\widehat{\sigma}(\gamma) = i$.

As $\overline{\widehat{\sigma}(\chi)} = \widehat{\sigma}(\chi^{-1})$, we have $\widehat{\sigma}(\chi) = -i$ for $\chi \in E_1^{-1}$ and hence also on the closure. But $\gamma \in \overline{E_1^{-1}}$, which gives a contradiction.

Now assume that conditions (1) and (2) hold. Then (1) implies E is I_0, so it is enough to prove $E \cup E^{-1}$ is I_0 by Theorem 2.2. As E^{-1} is also I_0, it is enough to prove that E and $E^{-1} \setminus E$ have disjoint closures.

So assume there are nets $\{\chi_\alpha\}, \{\psi_\alpha\}$ in E and $E^{-1} \setminus E$, respectively, that have the same limit γ. Since E is I_0, $\gamma \notin \Gamma$. In particular, $\gamma \neq 1$. If $\psi_\alpha = \psi_\alpha^{-1}$, then $\psi_\alpha \in E^{-1} \cap E$, and this is not the case; thus, ψ_α is not of order 2. As $\gamma^{-1} = \lim \psi_\alpha^{-1}$, it follows from condition (2) that γ is not of order 2. Consequently, $\{\chi_\alpha\}, \{\psi_\alpha^{-1}\}$ are nets in E with different limits and so, without loss of generality, they are distinct (eventually they belong to disjoint neighborhoods of γ and γ^{-1}, respectively).

Let $F = \{\chi_\alpha\}$ and obtain $\sigma \in M_d^r(G)$ as in hypothesis (1), separating F and $E \setminus F$. As $\gamma \in \overline{F}$, $\widehat{\sigma}(\gamma) \in \overline{\widehat{\sigma}(F)}$ and, similarly, since $\gamma^{-1} \in \overline{E \setminus F}$, $\widehat{\sigma}(\gamma^{-1}) \in \overline{\widehat{\sigma}(E \setminus F)}$. But, as σ is a real measure with real-valued transform, $\widehat{\sigma}(\gamma^{-1}) = \widehat{\sigma}(\gamma)$, and this contradicts the assumption that $\widehat{\sigma}(F)$ and $\widehat{\sigma}(E \setminus F)$ have disjoint closures. $\square$

Corollary 3.10. *Suppose G is divisible. An asymmetric set $E \subset \Gamma$ is RI_0 if and only if it is real RI_0.*

Proof. If E is real RI_0, then property (1) of Theorem 3.8 holds.

If $\gamma^2 = 1$, then $\gamma(x^2) = 1$ for all $x \in G$. Since G is divisible, for every $g \in G$, there exists some $x \in G$ such that $g = x^2$. Hence, $\gamma \equiv 1$, property (2) of Theorem 3.8 is vacuous, and so E is RI_0. $\square$

Corollary 3.11. *For an asymmetric subset $E \subseteq \mathbf{Z}$ the following are equivalent*:

(1) *E is RI_0;*

(2) *E is real RI_0;*

(3) *For every $F \subseteq E$, there exists a $\sigma \in M_d^r(\mathbf{T})$ and $\varepsilon < \delta$ such that $|\widehat{\sigma}(F)| \leq \varepsilon$ and $|\widehat{\sigma}(E \setminus F)| \geq \delta$.*

If $\mathbf{0} \notin E$, then the preceding are equivalent to E being FZI_0.

4. Two examples.

Example 4.1. A set that is I_0 but not real RI_0:

Consider $E_1 = E \cup F$ where

$$E = \{10^j + 10j + 1 : j \geq 1\} \text{ and } F = \{-10^j - 1 : j \geq 1\}.$$

E and F are both FZI_0 sets, being Hadamard [**7**]. If we put $b = 2\pi/10$, then $\widehat{\delta_b}(E) = \{e^{2\pi i/10}\}$ while $\widehat{\delta_b}(F) = \{e^{-2\pi i/10}\}$. Thus, $\overline{E} \cap \overline{F}$ is empty, and, hence, E_1 is I_0.

However, $\overline{E} \cap \overline{F^{-1}}$ is not empty, so $E_1 \cup E_1^{-1}$ is not I_0 and therefore E_1 is not RI_0 by Theorem 2.2. It also follows from this that $q(E)$ and $q(F)$ do not have disjoint closures, and, although q is one-to-one on the asymmetric set E, the mapping q is not one-to-one on the closure of E_1. Thus, the failure of Theorem 3.4 (2) implies that E_1 is not real RI_0.

Example 4.2. A set that is real FZI_0 (hence real RI_0) but not RI_0:

Consider $\Gamma = \mathbf{Z} \times \mathbf{D}_2$, and take $E = \{(j, \gamma_j) : j \in \mathbf{N}\}$, where $\{\gamma_j\}$ is an independent set in $\mathbf{D}_2$. Proposition 2.1 (5) implies that E is real FZI_0 since the independence of $\{\gamma_j\}$ ensures we can interpolate ± 1 valued sequences by positive measures having real transforms. This also shows that (1) of Theorem 3.8 is satisfied and that E is I_0.

Now $E^{-1} = \{(-j, \gamma_j) : j \in \mathbf{N}\}$. If we choose a net $\{j_\alpha\}$ of positive integers that tends to $0 \in \overline{\mathbf{Z}}$, then $\{-j_\alpha\}$ also tends to 0 and so E and E^{-1} do not have disjoint closures. Thus, $E \cup E^{-1}$ is not I_0 and so, by Theorem 2.2, E is not RI_0.

One can also directly see the failure of hypothesis (2) of Theorem 3.8 (and hence hypothesis (1) of Theorem 3.8 does not imply hypothesis (2)). Indeed, any cluster point of the net $\{(j_\alpha, \gamma_{j_\alpha})\}$ is of order 2 and is contained in the Bohr closure of $\{\chi \in E : \chi \text{ not of order } 2\}$.

5. Union results. In contrast to the situation for Sidon sets, it is not in general true that the union of two I_0 sets is again I_0; see Example 4.1. Indeed, the union of two I_0 sets is I_0 if and only if the sets have disjoint closures in $\overline{\Gamma}$. Similar results hold for unions of real RI_0 and RI_0/FZI_0 sets.

Proposition 5.1. *Suppose that E and F are* (*real*) *RI_0 sets, and assume that $q(E)$ and $q(F)$ have disjoint closures in $\widetilde{\Gamma}$. Then $E \cup F$ is* (*real*) *RI_0.*

Proof. Let N be the larger of the (real) RI_0 constant of E and F. Apply Lemma 3.5 to choose $\mu \in M_d^r(G)$, with real transform, such that $|\widehat{\mu}(F) - 1| < \varepsilon$ and $|\widehat{\mu}(E)| < \varepsilon$ for $\varepsilon < 1/2N$. Given (real) Hermitian $\varphi \in B((l^\infty(E \cup F))$, obtain real, discrete measures μ_1, μ_2 with $\widehat{\mu_1} = \varphi$ on E and

$$\widehat{\mu_2} = \frac{\varphi - \widehat{\mu_1}}{\widehat{\mu}} \text{ on } F.$$

Then, for $\omega = \mu_1 + \mu_2 * \mu$, we have $|\widehat{\omega}(\gamma) - \varphi(\gamma)| < 1$ for $\gamma \in E \cup F$. □

Corollary 5.2. *Suppose that E and F are* (*real*) *RI_0 sets, and assume that there is some $\sigma \in M_d^r(G)$, with real transform, such that $\widehat{\sigma}(E)$ and $\widehat{\sigma}(F)$ have disjoint closures. Then $E \cup F$ is* (*real*) *RI_0.*

Example 5.3. A union example: separation by the Fourier transforms of positive, discrete measures is not enough for the union of FZI_0 sets to be FZI_0.

Take E, F as in Example 4.1. Then E and F are FZI_0 sets, but their union is not RI_0. If we put $b = 2\pi/10$, then $\widehat{\delta_b}(E) = e^{2\pi i/10}$, while $\widehat{\delta_b}(F) = e^{-2\pi i/10}$. Consequently, in the union results for RI_0/FZI_0 sets, it is not sufficient for the two sets E and F to have disjoint closures in $\overline{\Gamma}$ or for a positive, discrete measure σ to exist with the property that $\widehat{\sigma}(E)$ and $\widehat{\sigma}(F)$ have disjoint closures.

Here is a union result for FZI_0 sets which is again topological, but of a different flavor.

Proposition 5.4. *Let $E, F \subset \Gamma$. Assume $F = \cup_{j=1}^N F_j$, and that, for all j, $\overline{F_j^{\pm}} \cap \overline{E}$ is empty, $\overline{F_j F_j^{-1}} \cap \overline{E}$ is empty and $1 \notin \overline{E}$. Then there is a $\mu \in M_d^+(G)$ such that $\widehat{\mu} = 0$ on E and $\widehat{\mu} \geq 1/2$ on F. If, in addition, E and F are FZI_0, then so is $E \cup F$.*

Proof. The assumptions on E and F ensure that one can choose a neighborhood V of the identity in $\overline{\Gamma}$ such that

$$(V \cdot V^{-1}) \cap \overline{E} = \varnothing, \quad \overline{(F_i^{\pm 1} \cdot V \cdot V^{-1})} \cap \overline{E} = \varnothing,$$

and

$$\overline{(F_j \cdot F_j^{-1} \cdot V \cdot V^{-1})} \cap \overline{E} = \varnothing.$$

Now set $f = \sum_{i=1}^N f_j$, where

$$f_j = \frac{1}{2\,|V|}\left(1_{(F_j \cdot V) \cup V} * 1_{(F_j^{-1} \cdot V^{-1}) \cup V^{-1}}\right).$$

Then $f_j \geq 1/2$ on F_j and $f_j = 0$ on E. Since f is positive definite on $\overline{\Gamma}$, there is a positive discrete measure μ such that $f = \widehat{\mu}$.

The existence of such a measure certainly ensures that if E and F are FZI_0, then so is their union. ◻

Corollary 5.5. *If E is FZI_0 and F is a finite set with $\mathbf{1} \notin F$, then $E \cup F$ is FZI_0.*

Proof. As $E \cup E^{-1}$ is FZI_0, there is no loss of generality in assuming $F^{\pm} \cap E$ is empty. Take $F_j = \{\lambda_j\}$, $1 \le j \le N$, where $F = \{\lambda_1, \dots, \lambda_N\}$, and use the fact [**19**] that the Bohr closure of an I_0 set has no cluster points in Γ to see that the hypotheses of the proposition are satisfied. ◻

ENDNOTES

1. Short for *interpolation set*.

2. An asymmetric Sidon set in the dual of a connected group has the property that every bounded Hermitian function on the set can be interpolated by a nonnegative measure ([**1**]); the term "Fatou-Zygmund" (or FZ) property was used for such sets in [**17**].

3. It is convenient in some ways to carry on the "distinction" since (real) RI_0 is easier to establish and (real) FZI_0 can be easier to apply. We keep this in mind by writing "(real) RI_0/FZI_0."

4. This is easily seen from the fact that the closure of an I_0 set E in the Bohr compactification is identical to the Stone-Čech compactification of E.

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