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CONVERSE JENSEN INEQUALITY

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ABSTRACT. We use Skorokhod's embedding theorem to give a new proof of a converse to Jensen's inequality.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let X be an element of $L^1(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra, and define

$$Y := \mathbf{E}[X|\mathcal{G}],$$

an element of $L^1(\Omega, \mathcal{G}, \mathbf{P})$. If $\varphi : \mathbf{R} \to \mathbf{R}$ is convex, then the conditional form of Jensen's inequality asserts that

$$\mathbf{E}[\varphi(X)|\mathcal{G}] \geq \varphi(\mathbf{E}[X|\mathcal{G}]) = \varphi(Y), \quad \text{a.s.},$$

part of the assertion being that the expectations are almost surely well-defined. In particular,

$$\mathbf{E}[\varphi(X)] \geq \mathbf{E}[\varphi(Y)], \tag{1}$$

with an analogous stipulation. The following converse assertion seems to be well known, see [**2, 3**]. The proof we present may have some claim to novelty. We write $X \overset{\mathrm{d}}{=} Y$ to indicate that random variables X and Y have the same distribution.

Theorem. *Let X and Y be integrable random variables such that* (1) *holds for all convex φ. Then there is a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, a random variable $X' \in L^1(\Omega', \mathcal{F}', \mathbf{P}')$, and a σ-algebra $\mathcal{G}' \subset \mathcal{F}'$ such that $X \overset{\mathrm{d}}{=} X'$ and $Y \overset{\mathrm{d}}{=} \mathbf{E}[X' \mid \mathcal{G}']$.*

Proof. Taking $\varphi(x) = x$ and then $\varphi(x) = -x$, we see that $\mathbf{E}[X] = \mathbf{E}[Y]$. Let $B = (B_t)$ be a one-dimensional Brownian motion defined

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on some filtered probability space $(\Omega', \mathcal{F}', \mathcal{F}'_t, \mathbf{P}')$, such that $B_0 \stackrel{\mathrm{d}}{=} Y$. By Skorokhod's theorem, as presented in [**1**], condition (1) implies that there is an $(\mathcal{F}'_t)$-stopping time T such that $B_T \stackrel{\mathrm{d}}{=} X$. But then $\mathbf{E}[B_T \mid \mathcal{F}'_0] = B_0$. Thus, we take $X' = B_T$, and $\mathcal{G}' = \mathcal{F}'_0$. $\square$

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