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EXTREME AND SMOOTH POINTS IN LORENTZ AND MARCINKIEWICZ SPACES WITH APPLICATIONS TO CONTRACTIVE PROJECTIONS

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ABSTRACT. We characterize extreme and smooth points in the Lorentz sequence space $d(w,1)$ and in Marcinkiewicz sequence spaces $d_*(w,1)$ and $d^*(w,1)$, which are predual and dual spaces to $d(w,1)$, respectively. We then apply these characterizations for studying the relationship between the existence sets and one-complemented subspaces in $d(w,1)$. We show that a subspace of $d(w,1)$ is an existence set if and only if it is one-complemented.

Marcinkiewicz and Lorentz spaces play an important role in the theory of Banach spaces. They are key objects for instance in the interpolation theory of linear operators. The origins of the Marcinkiewicz spaces go back to the theorem on weak type operators [**24**, Theorem 2.b.15], which was originally due to Marcinkiewicz in the 1930s. The Lorentz spaces introduced by G.G. Lorentz in 1950, have appeared in a natural way as interpolation spaces between suitable Lebesgue spaces by a classical result of Lions and Peetre [**24**, Theorem 2.g.18]. This theory has been developed very extensively thereafter and along with these investigations, the theory of Lorentz and Marcinkiewicz spaces, including the studies of their geometric structure, has evolved independently, e.g., [**7, 8, 23, 25, 27**]. One can observe that these spaces also find applications in other topics of operator theory. It is worth mentioning that Marcinkiewicz spaces $d^*(w,1)$ and their subspaces of order continuous elements $d_*(w,1)$ have emerged recently many times in the context of norm-attaining linear operators. In the papers [**1, 10, 15**] it was shown among others, by using the space $d_*(w,1)$ with specific weight, that the subspace of norm attaining operators is not always dense in the space

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of all bounded operators, contrary to the Bishop-Phelps theorem for linear functionals. For such types of isometric results the knowledge of geometric properties of the ball is of the utmost importance, see e.g., [**10**], where the characterization of complex convexity of the Lorentz spaces was the key factor in the proof of the main result.

In this paper we consider the Lorentz and Marcinkiewicz sequence spaces generated by decreasing weight sequences. In the first two sections we shall characterize the smooth and extreme points of the balls in these spaces. In the last section we shall apply these results to study the relationship between the existence and one-complemented subspaces of Lorentz sequence spaces.

Let's first agree on basic definitions and notations. Throughout the paper any vector space will always be considered over the field of real numbers $\mathbf{R}$. Given a Banach space $(X, \|\cdot\|)$, by S_X and B_X we denote the unit sphere and the unit ball of X, respectively. Recall that $x \in S_X$ is an *extreme point* of the ball B_X whenever $x = (x_1 + x_2)/2$ with $x_i \in S_X$, $i = 1, 2$, implies that $x = x_1 = x_2$. An element $x \in X$ is called a *smooth point* of X if there exists a unique bounded linear functional $\phi \in S_{X^*}$ such that $\phi(x) = \|x\|$. Such a functional ϕ is called a *supporting functional* of x.

A symbol $\operatorname{ext} C$ will stand for the set of all extreme points of a convex subset C of X.

Assume that $\{w(n)\}$ is a decreasing sequence of positive numbers such that $\lim_n w(n) = 0$ and $\sum_{n=1}^{\infty} w(n) = \infty$. Let $W(n) = \sum_{i=1}^{n} w(i)$. By $\operatorname{card} A$ we denote cardinality of $A \subset \mathbf{N}$. For a real sequence $x = \{x(n)\}$, by $x^* = \{x^*(n)\}$ we denote its *decreasing rearrangement*. Recall that $x^*(n) = \inf\{s > 0 : d_x(s) \leq n - 1\}$, $n \in \mathbf{N}$, where d_x is a distribution of x, that is, $d_x(s) = \operatorname{card}\{k \in \mathbf{N} : |x(k)| > s\}$, $s \geq 0$. For any $x = \{x(n)\}$, the support of x is the set $\operatorname{supp} x = \{n \in \mathbf{N} : x(n) \neq 0\}$. We say that two sequences are *equimeasurable* whenever their distributions coincide. The *Lorentz sequence space* $d(w, 1)$ is a collection of all real sequences $x = \{x(n)\}$ such that

$$\|x\|_{w,1} = \sum_{n=1}^{\infty} x^*(n) w(n) < \infty.$$

It is well known that $d(w, 1)$ is a Banach space under the norm $\|\cdot\|_{w,1}$. The *Marcinkiewicz* sequence space $d^*(w, 1)$ consists of all real sequences

$x = \{x(n)\}$ satisfying

$$\|x\|_W = \sup_n \frac{\sum_{i=1}^n x^*(i)}{W(n)} < \infty,$$

and the subspace $d_*(w,1)$ of $d^*(w,1)$ is defined as

$$d_*(w,1) = \left\{ x \in d^*(w,1) : \lim_n \frac{\sum_{i=1}^n x^*(i)}{W(n)} = 0 \right\}.$$

Both spaces $d^*(w,1)$ and $d_*(w,1)$, equipped with the norm $\|\cdot\|_W$, are Banach spaces, and $d_*(w,1)$ is a closed subspace of $d^*(w,1)$. It is well known that $d_*(w,1)$ and $d^*(w,1)$ are predual and dual spaces of $d(w,1)$, respectively. Note also that, by the assumptions on the weight w, each space $d(w,1)$, $d^*(w,1)$ and $d_*(w,1)$ is contained in the space c_0, and thus for any element x in any of these spaces, the distribution function d_x is always finite. For more details on the Lorentz and Marcinkiewicz spaces, see e.g., [**18, 20, 23**]. Given $a \in \mathbf{R}$, $\operatorname{sign} a = 1$ if $a \geq 0$ and $\operatorname{sign} a = -1$ if $a < 0$.

1. Smooth points. In this section we characterize smooth points in Lorentz and Marcinkiewicz sequence spaces. We start with some auxiliary lemmas.

Lemma 1.1. *Let $\phi = \{a(n)\} \in d(w,1)$ be a supporting functional at $x \in S_{d^*(w,1)}$. If there is an $m \in \mathbf{N}$ such that*

$$\sum_{i=1}^m x^*(i) < W(m),$$

then $a^*(m) = a^*(m+1)$.

Proof. Suppose, on the contrary, that $a^*(m) > a^*(m+1)$. Since x is an element of the unit sphere of $d^*(w,1)$, we have $S^*(n) := \sum_{i=1}^n x^*(i) \leq W(n)$ for all $n \in \mathbf{N}$. Thus, in view of $S^*(m) < W(m)$

and by summation by parts, for every $l > m$,

$$\begin{aligned}\sum_{i=1}^{l} a^*(i)x^*(i) &= \sum_{i=1}^{m}(a^*(i)-a^*(i+1))S^*(i)\\ &\quad + \sum_{i=m+1}^{l-1}(a^*(i)-a^*(i+1))S^*(i) + a^*(l)S^*(l)\\ &< \sum_{i=1}^{m}(a^*(i)-a^*(i+1))W(i)\\ &\quad + \sum_{i=m+1}^{l-1}(a^*(i)-a^*(i+1))W(i) + a^*(l)W(l)\\ &= \sum_{i=1}^{l} a^*(i)w(i) = \|\phi\|.\end{aligned}$$

In view of

$$\lim_l \sum_{i=m+1}^{l-1}(a^*(i)-a^*(i+1))S^*(i) \le \lim_l \sum_{i=m+1}^{l-1}(a^*(i)-a^*(i+1))W(i)$$

and

$$\lim_l a^*(l)S^*(l) \le \lim_l a^*(l)W(l),$$

it follows that

$$\sum_{n=1}^{\infty} a^*(n)x^*(n) < \sum_{n=1}^{\infty} a^*(n)w(n) = \|\phi\|. \tag{1.1}$$

Since ϕ is a supporting functional at x, applying the Hardy inequality [**20**], we obtain that

$$\|\phi\| = \phi(x) = \sum_{n=1}^{\infty} a(n)x(n) \le \sum_{n=1}^{\infty} a^*(n)x^*(n).$$

This is a contradiction to inequality (1.1) and the proof is done. □

Corollary 1.2. *Suppose* $\phi = \{a(n)\} \in d(w,1)$ *is a supporting functional at* $x \in S_{d_*(w,1)}$. *Then it is finite, i.e.,* $a(n) = 0$ *except for finite numbers of* $n \in \mathbf{N}$.

Proof. In view of $x \in d_*(w,1)$, there exists an $N \in \mathbf{N}$ such that

$$N = \max\left\{n : \frac{\sum_{i=1}^{n} x^*(i)}{W(n)} = 1\right\}.$$

Then $\sum_{i=1}^{k} x^*(i) < W(k)$ for all $k > N$, and by Lemma 1.1, $a^*(N+1) = a^*(N+2) = \cdots = 0$, since $\{a(n)\}$ is an element of c_0. □

Proposition 1.3. *If* x *is an element of* $S_{d^*(w,1)}$ *such that*

$$2 \leq \operatorname{card}\left\{m : \frac{\sum_{i=1}^{m} x^*(i)}{W(m)} = 1\right\} < \infty,$$

then there exist two different norm-one supporting functionals in $d(w,1)$ *at* x.

Proof. Let

$$M = \max\left\{m : \frac{\sum_{i=1}^{m} x^*(i)}{W(m)} = 1\right\}$$

by the assumption $M < \infty$. Suppose that $N < M$ such that

$$\frac{\sum_{i=1}^{N} x^*(i)}{W(N)} = 1 = \frac{\sum_{i=1}^{M} x^*(i)}{W(M)}.$$

Notice that

$$w(M) + \sum_{i=1}^{M-1} (w(i) - x^*(i)) = x^*(M) \geq w(M) \geq w(M+1).$$

Notice also that

$$\sum_{i=1}^{M+1} x^*(i) < \sum_{i=1}^{M+1} w(i).$$

So we have $x^*(M+1) < w(M+1)$. Therefore, $x^*(M) > x^*(M+1)$. Hence, there is a permutation σ on $\mathbf{N}$ such that $|x(\sigma(k))| = x^*(k)$ for all $k = 1, \dots, M$. Now let, for $y \in d^*(w,1)$,

$$\phi_1(y) = \frac{1}{W(N)} \sum_{i=1}^{N} \operatorname{sign}(x(\sigma(i))y(\sigma(i)))$$

and

$$\phi_2(y) = \frac{1}{W(M)} \sum_{i=1}^{M} \operatorname{sign}(x(\sigma(i))y(\sigma(i))).$$

It is clear that $\phi_1 \neq \phi_2$, $\phi_1(x) = \phi_2(x) = 1$ and $\|\phi_1\| = \|\phi_2\| = 1$. Thus, ϕ_1 and ϕ_2 are two different norm-one supporting functionals in $d(w,1)$ at x. $\square$

Proposition 1.4. *Let x be an element of $S_{d^*(w,1)}$. If*

$$\operatorname{card}\left\{m : \frac{\sum_{i=1}^m x^*(i)}{W(m)} = 1\right\} = 1,$$

then there is a unique norm-one supporting functional ψ in $d(w,1)$ at x.

Proof. Suppose that

$$\frac{\sum_{i=1}^m x^*(i)}{W(m)} = 1$$

holds for some $m \in \mathbf{N}$. Then $x^*(m) > x^*(m+1)$. Indeed, if $m = 1$, then $x^*(1) = w(1)$. Since $x^*(1) + x^*(2) < W(2)$, so $x^*(2) < w(2) \leq w(1) = x^*(1)$. If $m \geq 2$, then $\sum_{i=1}^{m-1} x^*(i) \leq W(m-1)$ and $\sum_{i=1}^m x^*(i) = W(m)$. Hence, $x^*(m) \geq w(m)$. Thus, $W(m) + x^*(m+1) = \sum_{i=1}^{m+1} x^*(i) < W(m+1)$, that is, $x^*(m+1) < w(m+1) \leq w(m) \leq x^*(m)$.

Let $\phi = \{a(n)\}$ be a norm-one supporting functional at x, where $\{a(n)\}$ is an element of $d(w,1)$, that is,

$$\phi(x) = \sum_{i=1}^{\infty} a(i)x(i) = \|\phi\| = 1.$$

Then, by Lemma 1.1,

$$a^*(1) = a^*(2) = \cdots = a^*(m),$$

and

$$a^*(m+1) = a^*(m+2) = \cdots = 0.$$

Set $a = a^*(1)$. Then there exists a finite set $\{j_1, \ldots, j_m\}$ such that $\lambda_k = \operatorname{sign} a(j_k)$, $k = 1, \ldots, m$, $a(j_k) = a\lambda_k$ and $a(i) = 0$ otherwise. Thus,

$$\phi(y) = \sum_{k=1}^{m} a\lambda_k y(j_k),$$

for every $y \in d^*(w,1)$. Since ϕ is a supporting functional at x,

$$\begin{aligned}1 = \|\phi\| = \sum_{k=1}^{m} aw(k) = \phi(x) = \sum_{k=1}^{m} a\lambda_k x(j_k) \le \sum_{k=1}^{m} ax^*(k)\\ = \sum_{k=1}^{m} aw(k) = 1.\end{aligned}$$

This implies that $a = 1/W(m)$ and that

$$\sum_{k=1}^{m} a\lambda_k x(j_k) = \sum_{k=1}^{m} ax^*(k).$$

Then, in view of $x^*(k) > x^*(m+1)$ for every $k = 1, \ldots, m$, we find a permutation $\{j_{\sigma(1)}, \ldots, j_{\sigma(m)}\}$ of $\{j_1, \ldots, j_m\}$ such that $|x(j_{\sigma(1)})| \ge \cdots \ge |x(j_{\sigma(m)})|$ and

$$\sum_{k=1}^{m} a\lambda_k x(j_k) = \sum_{k=1}^{m} a\lambda_{\sigma(k)} x(j_{\sigma(k)}) = \sum_{k=1}^{m} ax^*(k).$$

Thus,

$$\lambda_{\sigma(k)} x(j_{\sigma(k)}) = x^*(k), \quad k = 1, \ldots m.$$

Hence $|x(j_k)| = \lambda_k x(j_k)$ and so $\lambda_k = \operatorname{sign}(x(j_k))$ for every $k = 1, \ldots, m$. Thus, for $y \in d^*(w,1)$,

$$\phi(y) = \frac{1}{W(m)} \sum_{k=1}^{m} \operatorname{sign}(x(j_k)) y(j_k).$$

On the other hand, there is a permutation π on $\mathbf{N}$ such that $|x(\pi(k))| = x^*(k)$ for $k = 1, \dots, m$, because $x^*(m) > x^*(m+1)$. Then the linear functional ψ, defined by

$$\psi(y) = \frac{1}{W(m)} \sum_{k=1}^{m} \operatorname{sign}(x(\pi(k)))y(\pi(k)),$$

is a norm-one supporting functional at x. Since $x^*(m) > x^*(m+1)$ and $|x(\pi(k))| = x^*(k) = |x(j_k)|$ for $k = 1, \dots, m$, so

$$\{\pi(k) : k = 1, \dots, m\} = \{j_k : k = 1, \dots, m\}.$$

However, it implies that $\phi = \psi$ and completes the proof. □

Theorem 1.5. *Let x be an element of $S_{d_*(w,1)}$. Then x is a smooth point of $B_{d_*(w,1)}$ if and only if*

$$\operatorname{card}\left\{m : \frac{\sum_{i=1}^{m} x^*(i)}{W(m)} = 1\right\} = 1.$$

Proof. The necessity follows from Proposition 1.3 and sufficiency from Proposition 1.4. □

Since $d_*(w,1)$ is M-embedded, so $(d^*(w,1))^* = d(w,1) \oplus_1 F$, where F is the set of singular functionals. If $\xi \in F$, then it vanishes on $d_*(w,1)$, [**16**, Examples III.1.4]; see also [**18**].

Suppose that ϕ is a supporting functional at $x \in d^*(w,1)$. Then it has a unique representation $\phi = \psi + \xi$, where $\psi = \{a(n)\} \in d(w,1)$, and ξ is a singular linear functional. By the M-ideal property we have

$$\|\phi\| = \|\psi\| + \|\xi\| \geq \psi(x) + \xi(x) = \phi(x) = \|\phi\|.$$

Therefore, both ψ and ξ are supporting functionals at x.

Proposition 1.6. *Let x be an element of $S_{d^*(w,1)}$. Suppose that*

$$\frac{\sum_{k=1}^{n} x^*(k)}{W(n)} < 1$$

holds for all $n \in \mathbf{N}$. Then a supporting functional ϕ at x is singular.

Proof. Let $\phi = \psi + \xi$ be a unique decomposition, where $\psi = \{a(n)\}$ and ξ is a singular linear functional. Then $\psi = \{a(n)\}$ is a supporting functional at x, and by Lemma 1.1, $a^*(1) = a^*(2) = \cdots = 0$. Thus, $\psi = 0$. $\square$

Proposition 1.7. *Let x be an element of $S_{d^*(w,1)}$. If*

$$\limsup_n \frac{\sum_{k=1}^n x^*(k)}{W(n)} = 1,$$

then there exist two different norm-one supporting functionals of x.

We need the following lemma.

Lemma 1.8. *Let x be an element of $S_{d^*(w,1)}$. If*

$$\limsup_n \frac{\sum_{k=1}^n x^*(k)}{W(n)} = 1,$$

then there is a decomposition $x = x_1 + x_2$ such that $|x_1| \wedge |x_2| = 0$ and $\|x_1\|_W = \|x_2\|_W = 1$.

Proof of Lemma 1.8. For any subset G of $\mathbf{N}$, define the characteristic function $\chi_G : \mathbf{N} \to \mathbf{R}$ as $\chi_G(i) = 0$ if $i \notin G$ and $\chi_G(i) = 1$ if $i \in G$.

We claim that for every nonempty finite set $F \subset \mathbf{N}$ and for every $\varepsilon > 0$, there is a finite set $G \subset \mathbf{N}$ with $G \cap F = \varnothing$ and such that $\|x\chi_G\|_W \geq 1-\varepsilon$. Indeed, by assumption, there is an unbounded strictly increasing sequence $\{m_n\}$ of $\mathbf{N}$ such that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{m_n} x^*(k)}{W(m_n)} = 1.$$

Since F is finite and $x^* \in c_0$, we can clearly assume that

$$m := \min\{|x(s)| : s \in F \cap \operatorname{supp} x\} > 0,$$

and so there is an N such that $x^*(k) < m$ for every $k > N$. Since $m_N \geq N$, and for $n \geq N$, we have

$$\frac{\sum_{k=1}^{m_n} x^*(k)}{W(m_n)} = \frac{\sum_{k=1}^{m_N} x^*(k)}{W(m_n)} + \frac{\sum_{k=m_N+1}^{m_n} x^*(k)}{W(m_n)},$$

and the first sequence in the above sum converges to zero, the second one converges to 1. This implies that $\|x\chi_{\mathbf{N}\setminus F}\|_W = 1$, so there is a finite set $G \subset \mathbf{N} \setminus \mathbf{F}$ so that $\|x\chi_G\|_W > 1 - \varepsilon$ as we wanted to show. By applying the above claim, we can construct a sequence of disjoint sets $\{F_n\}$ of natural numbers such that $\lim_n \|x\chi_{F_n}\|_W = 1$. Finally, by setting $G_1 = \cup_n F_{2n-1}$ and $G_2 = \mathbf{N} \setminus G_1$, we have for all $n \in \mathbf{N}$,

$$\|x\chi_{F_{2n}}\|_W \le \|x\chi_{G_2}\|_W \le 1, \quad \|x\chi_{F_{2n-1}}\|_W \le \|x\chi_{G_1}\|_W \le 1,$$

and so the elements $x_1 = x\chi_{G_1}$, $x_2 = x\chi_{G_2}$ satisfy $\|x_1\|_W = \|x_2\|_W = 1$, $x = x_1 + x_2$ and $|x_1| \wedge |x_2| = 0$. □

Proof of Proposition 1.7. By Lemma 1.8, any $x \in S_{d^*(w,1)}$ can be decomposed as $x = x_1 + x_2$, $|x_1| \wedge |x_2| = 0$ and $\|x_1\|_W = \|x_2\|_W = 1$. Let $Y_i = \{z \in d^*(w,1) : \operatorname{supp} z \subset \operatorname{supp} x_i\}$, $i = 1, 2$. Notice that

$$\operatorname{dist}(x, Y_2) = \operatorname{dist}(x_1, Y_2) = 1 = \|x_1\|_W.$$

By a well-known corollary of the Hahn-Banach theorem, there is a functional $\phi_1 \in (d^*(w,1))^*$ such that $\|\phi_1\| = 1 = \phi_1(x_1)$ and $\phi_1(Y_2) = \{0\}$. Thus, $\phi_1(x) = \|x\|_W$. Analogously there exists $\phi_2 \in (d^*(w,1))^*$ such that $\|\phi_2\| = 1 = \phi(x_2)$ and $\phi_2(Y_1) = \{0\}$. Hence, ϕ_1 and ϕ_2 are distinct norm-one supporting functionals of x. □

Theorem 1.9. *An element x in $S_{d^*(w,1)}$ is a smooth point of $B_{d^*(w,1)}$ if and only if there is an $m \in \mathbf{N}$ such that*

$$\frac{\sum_{k=1}^m x^*(k)}{W(m)} = 1 > \sup_{n \ne m} \left\{ \frac{\sum_{k=1}^n x^*(k)}{W(n)} \right\}.$$

Proof. The necessity follows from Propositions 1.3 and 1.7. In order to show the sufficiency, suppose the inequality in the hypothesis is satisfied. Then $x^*(m) > x^*(m+1)$ and there exists a permutation σ on $\mathbf{N}$ such that $|x(\sigma(k))| = x^*(k)$ for $k = 1, \dots, m$. Let $\phi = \psi + \xi$ be a norm-one supporting functional at x, where $\psi \in d(w,1)$ and ξ is singular. If $\xi \ne 0$, then setting

$$t = \max\{\sigma(k) : k = 1, \dots, m\},$$

it is clear that $\|x\chi_{\mathbf{N}\setminus\{1,\dots,t\}}\|_W < 1$. Therefore,

$$\|\xi\| = \xi(x) = \xi(x\chi_{\mathbf{N}\setminus\{1,\dots,t\}}) < \|\xi\|,$$

and so it is a contradiction. Hence, $\xi = 0$. By Proposition 1.4, the norm-one supporting functional $\psi \in d(w,1)$ at x is unique, and the proof is done. $\square$

Below we provide a characterization of smooth points in the Lorentz space $d(w,1)$.

Theorem 1.10. *An element x of the unit sphere of $d(w,1)$ is a smooth point if and only if* $\operatorname{supp} x$ *is infinite and the following condition is satisfied*:

(1.2) *Whenever there is a* $k \geq 1$ *such that* $w(k) > w(k+1)$, *we get* $x^*(k) > x^*(k+1)$.

Proof. We shall show first the necessity. It is easy to see that if $\operatorname{supp} x$ is finite, then there are infinitely many supporting functionals at x. Thus, assume that $\operatorname{supp} x$ is infinite. We shall show that if $x^*(k_0) = x^*(k_0+1)$ and $w(k_0) > w(k_0+1)$ for some $k_0 \in \mathbf{N}$, then we can obtain two different supporting functionals at x. It is well known that there is a one-to-one and onto mapping $\sigma : \mathbf{N} \to \operatorname{supp} x$ such that $x^* = |x \circ \sigma|$. Choose two sequences y_1 and y_2 defined by

$$y_1(k) = \begin{cases} \operatorname{sign}(x(k)) \cdot w(\sigma^{-1}(k)) & \text{for } k \in \operatorname{supp} x; \\ 0 & \text{otherwise,} \end{cases}$$

$$y_2(k) = \begin{cases} y_1(k), & \\ \qquad k \neq \sigma(k_0) \text{ and } k \neq \sigma(k_0+1); & \\ \operatorname{sign}(x(\sigma(k_0))) \cdot (w(k_0) + w(k_0+1))/2, & \\ \qquad k = \sigma(k_0); & \\ \operatorname{sign}(x(\sigma(k_0+1))) \cdot (w(k_0) + w(k_0+1))/2, & \\ \qquad k = \sigma(k_0+1). & \end{cases}$$

Notice that $\|y_1\|_W = \|y_2\|_W = 1$. It is also easy to check that y_1 and y_2 are two different supporting functionals at x.

Now let $x \in S_{d(w,1)}$ satisfy condition (1.2), and let $y \in S_{d^*(w,1)}$ be a supporting functional of x. Then

$$1 = \sum_{k=1}^{\infty} x(k)y(k) = \sum_{k=1}^{\infty} \operatorname{sign}(x(\sigma(k))) \cdot x^*(k)y(\sigma(k)),$$

where $x^* = |x \circ \sigma|$. Taking $S(n) = \sum_{k=1}^{n} \operatorname{sign}(x(\sigma(k))) \cdot y(\sigma(k))$ and $S'(n) = \sum_{k=1}^{n} y^*(k)$, we have $S(n) \le S'(n) \le W(n)$ for every $n \in \mathbf{N}$, in view of the Hardy inequality and $\|y\|_W = 1$. We shall show by induction that, for every $n \in \mathbf{N}$,

$$y(\sigma(n)) = \operatorname{sign}(x(\sigma(n))) \cdot y^*(n) = \operatorname{sign}(x(\sigma(n))) \cdot w(n).$$

Since $\lim_{n\to\infty} x^*(n) = 0$, there is an m such that $m = \max\{k \ge 1 : x^*(1) = x^*(k)\}$. If $S(m) < W(m)$, then by the summation by parts, we get

$$\begin{aligned}
1 &= \sum_{i=1}^{\infty} x^*(i)y(\sigma(i))\operatorname{sign}(x(\sigma(i))) \\
&= \sum_{i=1}^{m} (x^*(i) - x^*(i+1))S(i) \\
&\quad + \lim_{l\to\infty} \left\{ \sum_{i=m+1}^{l-1} (x^*(i) - x^*(i+1))S(i) + x^*(l)S(l) \right\} \\
&< \sum_{i=1}^{m} (x^*(i) - x^*(i+1))W(i) \qquad (1.3)\\
&\quad + \lim_{l\to\infty} \left\{ \sum_{i=m+1}^{l-1} (x^*(i) - x^*(i+1))W(i) + x^*(l)W(l) \right\} \\
&= \sum_{i=1}^{\infty} x^*(i)w(i) = 1,
\end{aligned}$$

which is a contradiction. So $S(m) = W(m)$. Notice that $\sup_k |y(k)| \le y^*(1) \le w(1)$. Since $x^*(i) = x^*(j)$ for every $1 \le i,\ j \le m$, we also have

$w(1) = \cdots = w(m)$ by assumption (1.2). This and $S(m) \le S'(m) \le W(m)$ imply that

$$S(m) = S'(m) = W(m) = m \cdot w(1),$$

and $\operatorname{sign}(x(\sigma(k))) \cdot y(\sigma(k)) = y^*(k) = w(k)$ for $k = 1, \dots, m$. Hence,

$$y(\sigma(1)) = \operatorname{sign}(x(\sigma(1))) \cdot y^*(1) = \operatorname{sign}(x(\sigma(1))w(1).$$

For the inductive step, assume that for every $k \le n$, we have

$$y(\sigma(k)) = \operatorname{sign}(x(\sigma(k))) \cdot y^*(k) = \operatorname{sign}(x(\sigma(k))) \cdot w(k).$$

Let now $m = \max\{k \ge n+1 : x^*(n+1) = x^*(k)\}$. If $S(m) < W(m)$, then the inequality (1.3) yields a contradiction, and so $S(m) = W(m)$. By the induction hypothesis and by (1.2), we get $S(m) - S(n) = W(m) - W(n) = (m-n)w(n+1)$ and $w(n+1) \ge \sup_{j \ge n+1} |y(\sigma(j))| = \sup_{j \ge n+1} y^*(j)$. Thus, for $n+1 \le j \le m$

$$y(\sigma(j)) = \operatorname{sign}(x(\sigma(j))) \cdot y^*(j) = \operatorname{sign}(x(\sigma(j))) \cdot w(j).$$

This completes the induction and the uniqueness of the supporting functional at x has been proved. $\square$

2. Extreme points. A Banach space $(X, \|\cdot\|)$, a collection of real sequences, is said to be an *r.i. sequence space* if for any $x = \{x(n)\} \in X$ we have $\|x\| = \|x^*\|$, and for any $y = \{y(n)\}$ such that $|y(n)| \le |x(n)|$ for every $n \in \mathbf{N}$, we have that $y \in X$ and $\|y\| \le \|x\|$. It is clear that all spaces $d(w,1)$, $d^*(w,1)$ and $d_*(w,1)$ are r.i. sequence spaces. An r.i. space X is *strictly monotone* if for $x, y \in X$ such that $|x(n)| \le |y(n)|$ for all $n \in \mathbf{N}$ and $|x(m)| < |y(m)|$ for some $m \in \mathbf{N}$, we have that $\|x\| < \|y\|$.

Proposition 2.1. *Let $(X, \|\cdot\|)$ be an r.i. sequence space, and let $x \in S_X$ be such that its distribution d_x is a finite valued function.*

(i) *If* $\operatorname{supp} x$ *is finite or equal to* $\mathbf{N}$, *then x is an extreme point of B_X if and only if x^* is an extreme point.*

(ii) *If x is an extreme point of B_X, then x^* is also an extreme point of B_X. If, in addition, X is strictly monotone, then the converse statement is also satisfied.*

Proof. Since $d_x(\theta) < \infty$ for all $\theta > 0$, $\lim_n x^*(n) = 0$ and there exist $M \subset \mathbf{N}$ and a one-to-one and onto mapping $\sigma : \mathbf{N} \to M$ such that, for all $n \in \mathbf{N}$,

$$x^*(n) = |x(\sigma(n))| = \lambda_n x(\sigma(n)), \tag{2.1}$$

where $\lambda_n = \operatorname{sign} x(\sigma(n))$. In fact, we can take $M = \mathbf{N}$ if $\operatorname{supp} x$ is finite and $M = \operatorname{supp} x$ otherwise.

(i) Under the assumptions, σ is a permutation of $\mathbf{N}$ and then the operator

$$Ty(n) = \lambda_n y(\sigma(n)), \quad y \in X,$$

is an isometry on X such that $Tx = x^*$. We get the conclusion immediately since T preserves extreme points.

(ii) Suppose $x^* \in S_X$ is not an extreme point of B_X. Then there exist $y, z \in S_X$ such that $y \neq z$ and $x^* = (y+z)/2$. Hence,

$$x(\sigma(n)) = \frac{\lambda_n y(n) + \lambda_n z(n)}{2},$$

and so we get for every $n \in \operatorname{supp} x$,

$$x(n) = \frac{\beta_n y(\sigma^{-1}(n)) + \beta_n z(\sigma^{-1}(n))}{2},$$

where $\sigma^{-1} : M \to \mathbf{N}$ is one-to-one and onto mapping and $\beta_n = \operatorname{sign} x(n)$. Thus, setting

$$\overline{y}(n) = \begin{cases} \beta_n y(\sigma^{-1}(n)) & n \in M, \\ 0 & \text{otherwise;} \end{cases}$$

$$\overline{z}(n) = \begin{cases} \beta_n z(\sigma^{-1}(n)) & n \in M, \\ 0 & \text{otherwise;} \end{cases}$$

we have that $\overline{y}$ and $\overline{z}$ are equimeasurable with y and z, respectively. Hence, $\|x\| = \|\overline{y}\| = \|\overline{z}\|$. Moreover, $x = (\overline{y} + \overline{z})/2$ and $\overline{y} \neq \overline{z}$, since there exists an $m \in M$ such that $y(\sigma^{-1}(m)) \neq z(\sigma^{-1}(m))$ by the assumption that $y \neq z$. Thus, x is not an extreme point of B_X as well.

Suppose now that X is strictly monotone, and let x not be an extreme point of B_X. Then there exist $y, z \in S_X$ such that $y \neq z$, and for all $n \in \mathbf{N}$,

$$x(n) = \frac{y(n) + z(n)}{2}.$$

It follows that $\operatorname{supp} y \cup \operatorname{supp} z \subset \operatorname{supp} x$. Indeed, if there is an $m \in \mathbf{N}$ such that $x(m) = 0$ and $y(m) \neq 0$, then $z(m) \neq 0$ and setting $\tilde{y} = y\chi_{\operatorname{supp} x}$ and $\tilde{z} = z\chi_{\operatorname{supp} x}$, we have $\|\tilde{y}\| < \|y\|$ and $\|\tilde{z}\| < \|z\|$, by strict monotonicity of X. However, $x = (\tilde{y} + \tilde{z})/2$ and so $\|x\| \leq (\|\tilde{y}\| + \|\tilde{z}\|)/2 < (\|y\| + \|z\|)/2 = \|x\|$, a contradiction.

By (2.1), for all $n \in \mathbf{N}$,

$$x^*(n) = \frac{\lambda_n y(\sigma(n)) + \lambda_n z(\sigma(n))}{2}.$$

Since supports of y and z are included in $\operatorname{supp} x$, $|y \circ \sigma|$ and $|z \circ \sigma|$ are equimeasurable with y and z, respectively. It is also clear that they are different. Thus, taking $y_0(n) = \lambda_n y(\sigma(n))$ and $z_0(n) = \lambda_n z(\sigma(n))$, we have that $\|y_0\| = \|z_0\| = 1$, $y_0 \neq z_0$ and $x^* = (y_0 + z_0)/2$. Thus, x^* is not an extreme point, which completes the proof. □

Theorem 2.2. *An element $x \in S_{d^*(w,1)}$ is an extreme point of $B_{d^*(w,1)}$ if and only if $x^* = w$.*

Proof. Recall that whenever $x \in d^*(w,1)$ then $d_x(\theta) < \infty$ for every $\theta > 0$. Assume that $x^* \neq w$, where $\|x\|_W = 1$. In view of Proposition 2.1 (ii) it is enough to show that if x^* is an extreme point, then $x^* = w$. We shall prove it by use of induction. Suppose, on the contrary, that $x^*(1) < w(1)$. We have three possible cases.

Case (1). Suppose first that $x^*(1) > x^*(2) > x^*(3)$ holds. Then choose an $\varepsilon > 0$ such that $x^*(1) + \varepsilon < w(1)$,

$$x^*(1) + \varepsilon > x^*(2) - \varepsilon > x^*(3)$$

and

$$x^*(1) - \varepsilon > x^*(2) + \varepsilon > x^*(3).$$

Then, by setting

$$y = (x^*(1) + \varepsilon, x^*(2) - \varepsilon, x^*(3), \dots)$$

and

$$z = (x^*(1) - \varepsilon, x^*(2) + \varepsilon, x^*(3), \dots),$$

we have that $\|y\|_W, \|z\|_W \le 1$ and $x^* = (y+z)/2$. Hence, it is a contradiction to the assumption that x^* is an extreme point.

Case (2). Suppose that $x^*(1) = x^*(2) = \cdots = x^*(m) > x^*(m+1)$ for some $m \ge 2$. Then, for every $1 \le k < m$,

$$kx^*(k) < w(1) + \cdots + w(k).$$

Indeed, if $kx^*(1) = w(1) + \cdots + w(k)$ for some $1 \le k < m$, then $w(k) < x^*(1)$. Hence,

$$mx^*(1) = w(1) + \cdots + w(k) + (m-k)x^*(1) > w(1) + \cdots + w(m),$$

which is a contradiction to the fact that $\|x\|_W = 1$.

Now choose $\varepsilon > 0$ such that $(k-1)x^*(1) + \varepsilon < w(1) + \cdots + w(k-1)$ for every $1 \le k < m$ and $x^*(1) - \varepsilon > x^*(m+1)$. Setting

$$\begin{aligned} y &= (x^*(1) + \varepsilon, x^*(1), \dots, x^*(1) - \varepsilon, x^*(m+1), \dots), \\ z &= (x^*(1) - \varepsilon, x^*(1), \dots, x^*(1) + \varepsilon, x^*(m+1), \dots), \end{aligned}$$

we have that $\|y\|_W = \|z\|_W \le 1$ and $x^* = (y+z)/2$. This is also a contradiction.

Case (3). Suppose that $x^*(1) > x^*(2) = x^*(3) = \cdots = x^*(m) > x^*(m+1)$ for some $m \ge 3$. Then, for $1 \le k < m$,

$$x^*(1) + (k-1)x^*(2) < w(1) + \cdots + w(k).$$

Indeed, if $x^*(1) + (k-1)x^*(2) = w(1) + \cdots + w(k)$ for some $2 \le k < m$, then $(k-1)x^*(2) \ge w(1) - x^*(1) + (k-1)w(k) > (k-1)w(k)$. Thus, $x^*(2) > w(k)$ and so

$$\begin{aligned} x^*(1) + (m-1)x^*(2) &= w(1) + \cdots + w(k) + (m-k)x^*(2) \\ &> w(1) + \cdots + w(k) + w(k+1) + \cdots + w(m). \end{aligned}$$

It is however impossible since x^* has norm one.

Now choose $\varepsilon > 0$ such that $x^*(1) > x^*(2) + \varepsilon$, $x^*(2) - \varepsilon > x^*(m+1)$ and

$$x^*(1) + (k-2)x^*(2) + \varepsilon < w(1) + \cdots + w(k-1) \quad \text{for every} \quad 1 \le k < m.$$

Let

$$y = (x^*(1), x^*(2) + \varepsilon, \dots, x^*(2) - \varepsilon, x^*(m+1), \dots)$$

and

$$z = (x^*(1), x^*(2) - \varepsilon, \dots, x^*(2) + \varepsilon, x^*(m+1), \dots).$$

Then $\|y\|_W = \|z\|_W \leq 1$ and $x^* = (y+z)/2$, which is impossible since x^* is an extreme point.

Therefore, we have shown that $x^*(1) = w(1)$. For the use of induction, suppose now that $x^*(k) = w(k)$ for $1 \leq k \leq n$. If $x^*(n+1) < w(n+1)$, then exactly the same argument as for cases (1), (2) and (3), shows that it is a contradiction. Hence, $x^*(n+1) = w(n+1)$. Therefore, if x^* is an extreme point, then $x^* = w$.

Now we show that x is an extreme point of the unit ball of $d^*(w,1)$ if $x^* = w$. Let $A = \operatorname{supp} x$ and $\beta : A \to A$ be a one-to-one and onto mapping such that the sequence $\{|x(\beta(n))|\}$ is decreasing on A, that is, $|x(\beta(j))| \leq |x(\beta(i))|$ whenever $i < j$. Consider $\gamma : \mathbf{N} \to \mathbf{N}$ defined as $\gamma(n) = \beta(n)$ for $n \in A$ and $\gamma(n) = n$ for $n \notin A$. Then

$$(Ty)(n) = \operatorname{sign} x(\gamma(n)) y(\gamma(n))$$

is a linear isometry on $d^*(w,1)$ such that

$$(Tx)(n) = |x(\gamma(n))|.$$

Hence, x is an extreme point whenever $|x \circ \gamma|$ is an extreme point. In view of that we can assume that $x \in S_{d^*(w,1)}$ is nonnegative, decreasing whenever restricted to its support A, and $x^* = w$. Now let $y, z \in S_{d^*(w,1)}$ be such that, for all $n \in \mathbf{N}$,

$$x(n) = \frac{y(n) + z(n)}{2}.$$

In view of the assumptions on x, letting $A = \{n_1, n_2, \dots\}$, where $n_1 < n_2 < \cdots$, we have

$$x(n_k) = w(k), \quad k \in \mathbf{N}.$$

We shall show first that

$$y(n_k) = z(n_k) = x(n_k), \quad k \in \mathbf{N}.$$

Let further

$$\tilde{y} = y\chi_A \quad \text{and} \quad \tilde{z} = z\chi_A.$$

We shall apply mathematical induction. Let $n = n_1$. Since $y, z \in B_{d^*(w,1)}$, so $|y(n_1)| \leq w(1)$ and $|z(n_1)| \leq w(1)$. Then $y(n_1) = z(n_1) = w(1)$ in view of $w(1) = x(n_1)$. Notice also that we have $\tilde{y}^*(1) = y(n_1) = \tilde{z}^*(1) = z(n_1)$. Assume now that

$$\tilde{y}^*(i) = y(n_i) = w(i) = z(n_i) = \tilde{z}^*(i)$$

for all $i = 1, \ldots, m$ and some $m > 1$. Then

$$\frac{\sum_{i=1}^{m+1} \tilde{y}^*(i)}{W(m+1)} = \frac{\sum_{i=1}^{m} w(i) + \tilde{y}^*(m+1)}{W(m+1)} \leq 1,$$

and so $\tilde{y}^*(m+1) \leq w(m+1)$. Hence, for all $i \geq m+1$,

$$\tilde{y}^*(i) \leq w(m+1),$$

and since there exists a $j \geq m+1$ such that $\tilde{y}^*(j) = y(n_{m+1})$, we have

$$|y(n_{m+1})| \leq w(m+1).$$

Analogously, we can show that

$$|z(n_{m+1})| \leq w(m+1),$$

and so

$$\tilde{y}^*(m+1) = y(n_{m+1}) = z(n_{m+1}) = \tilde{z}^*(m+1) = w(m+1) = x(n_{m+1}).$$

This completes the induction. It remains to show that $y(i) = z(i) = 0$ for all $i \notin A$. Notice that $\tilde{y}^* \leq y^*$, and we have for every $n \geq 1$,

$$1 \geq \frac{\sum_{i=1}^{n} y^*(i)}{W(n)} \geq \frac{\sum_{i=1}^{n} \tilde{y}^*(i)}{W(n)} = 1.$$

Hence, $\tilde{y}^* = y^*$ and in view of $\lim_n y^*(n) = 0$, we have $y(i) = 0$. Similarly $z(i) = 0$ for every $i \notin A$. $\square$

Remark 2.3. The extreme points for the unit ball of finite-dimensional Marcinkiewicz sequence spaces are characterized in [**9**].

Lemma 2.4. *The Lorentz space $d(w,1)$ is strictly monotone.*

Proof. If $0 \le x < y$ and $x, y \in d(w,1)$, then $x^* \le y^*$ and there exists an $m \in \mathbf{N}$ such that $x^*(m) < y^*(m)$. This results from the simple fact that d_y is finite. It follows that $\|x\|_{w,1} < \|y\|_{w,1}$. $\square$

The next result follows immediately from Proposition 2.1 and Lemma 2.4.

Corollary 2.5. *An element $x \in S_{d(w,1)}$ is an extreme point of the ball $B_{d(w,1)}$ if and only if x^* is an extreme point of $B_{d(w,1)}$.*

Theorem 2.6. *An element $x \in S_{d(w,1)}$ is an extreme point of the ball $B_{d(w,1)}$ if and only if there exists an $n_0 \in \mathbf{N}$ such that $x^*(i) = 1/W(n_0)$ for $i = 1, \dots, n_0$, $x^*(i) = 0$ for $i > n_0$ and $w(1) > w(n_0)$, provided that $n_0 > 1$.*

Proof. In view of Corollary 2.5 we assume that $x = x^*$. Suppose first that $x \in S_{d(w,1)}$ is an extreme point of $B_{d(w,1)}$, and let

$$n_0 = \sup\{n \in \mathbf{N} : x(n) = x(1)\}.$$

Since $d(w,1) \subset c_0$, it is clear that $n_0 \in \mathbf{N}$. We shall show that $x(n_0+1) = 0$. Let, on the contrary, $x(n_0+1) > 0$, and set

$$n_1 = \max\{n \in \mathbf{N} : x(n) = x(n_0+1)\}.$$

Setting $d = \min\{x(1) - x(n_0+1), x(n_0+1) - x(n_1+1)\}$, we have $d > 0$. Fix $b > 0$ such that

$$b\left(1 + \frac{W(n_0)}{W(n_1) - W(n_0)}\right) < d.$$

Define

$$y = \left(x(1) - b, \dots, x(n_0) - b, x(n_0+1) + \frac{bW(n_0)}{W(n_1) - W(n_0)}, \dots, \right.$$
$$\left. x(n_1) + \frac{bW(n_0)}{W(n_1) - W(n_0)}, x(n_1+1), x(n_1+2) \dots \right)$$

and

$$z = \left(x(1) + b, \ldots, x(n_0) + b, x(n_0 + 1) - \frac{bW(n_0)}{W(n_1) - W(n_0)}, \ldots, \right.$$
$$\left. x(n_1) - \frac{bW(n_0)}{W(n_1) - W(n_0)}, x(n_1 + 1), x(n_1 + 2) \ldots \right).$$

Note that $y \neq z$ and $x = (y + z)/2$. By the choice of b and d, $y = y^*$ and $z = z^*$. Hence,

$$\begin{aligned} \|y\|_{w,1} &= \sum_{j=1}^{\infty} y(j)w(j) = \sum_{j=1}^{\infty} x(j)w(j) \\ &\quad - bW(n_0) + \frac{bW(n_0)}{W(n_1) - W(n_0)} \sum_{j=n_o+1}^{n_1} w(j) \\ &= \sum_{j=1}^{\infty} x(j)w(j) = 1. \end{aligned}$$

Analogously, we can show that $\|z\|_{w,1} = 1$. It contradicts the assumption that x is an extreme point and consequently $x(n_0 + 1) = 0$, as required. If $n_0 > 1$ and $w(1) = w(n_0)$, define for $0 < b < x(1)$,

$$y_b = (x(1) + b, x(2), \ldots, x(n_0 - 1), x(n_0) - b, x(n_0 + 1), \ldots)$$

and

$$z_b = (x(1) - b, x(2), \ldots, x(n_0 - 1), x(n_0) + b, x(n_0 + 1), \ldots).$$

It is easy to see that $\|y_b\|_{w,1} = \|z_b\|_{w,1} = 1$, $z_b \neq y_b$ and $x = (y_b + z_b)/2$. So, x is not an extreme point, which is a contradiction. Thus, we showed as required that, if $n_0 > 1$, then $w(1) > w(n_0)$.

Now assume $x \in d(w, 1)$ and $n \in \mathbf{N}$ are such that $x(i) = 1/W(n)$ for $i = 1, \ldots, n$, $x(i) = 0$ for $i > n$ and $w(1) > w(n)$ if $n > 1$. We shall show that x is an extreme point of $B_{d(w,1)}$. If $n = 1$, this is an easy consequence of Lemma 2.4. Suppose that $n > 1$. Let $x = (y+z)/2$ with $\|y\|_{w,1} = \|z\|_{w,1} = 1$ and $y \neq z$. By Lemma 2.4, $y(i) = z(i) = 0$ for $i > n$. Indeed, if $y(i) \neq 0$ for some $i > n$, then $z(i) = -y(i)$. Defining $y^1 = (y(1), \ldots y(n), 0, \ldots)$ and $z^1 = (z(1), \ldots z(n), 0, \ldots)$, we have that

$x = (y^1 + z^1)/2$. But, by strict monotonicity, $\|z^1\|_{w,1} < \|z\|_{w,1} = 1$ and $\|y^1\|_{w,1} < \|y\|_{w,1} = 1$ and so $\|x\|_{w,1} < 1$; a contradiction. Define

$$I_1 = \{i = 1, \dots, n : y(i) > 1/W(n)\},$$
$$I_2 = \{i = 1, \dots, n : y(i) = 1/W(n)\}$$

and

$$I_3 = \{i = 1, \dots, n : y(i) < 1/W(n)\}.$$

By strict monotonicity, we have $y, z \geq 0$. Otherwise, we can choose $\tilde{y}$ and $\tilde{z}$ such that $|\tilde{y}| < |y|$, $|\tilde{z}| < |z|$ and $x = (\tilde{y} + \tilde{z})/2$. Hence, $\|\tilde{y}\|_{w,1} < \|y\|_{w,1} < 1$ and $\|\tilde{z}\|_{w,1} < \|z\|_{w,1} < 1$, which is a contradiction to the fact that $x = (y + z)/2$.

Let, for $i = 1, 2, 3$, $k_i = \operatorname{card} I_i$. Since $d(w, 1)$ is strictly monotone, $y \neq x$ and $\|y\|_{w,1} = 1$, so $k_1 > 0$ and $k_3 > 0$. Without loss of generality, permuting the coordinates of y and z, if necessary, we can assume that $y^* = y$. Since $\|y\|_{w,1} = 1$,

$$y(1)w(1) + \cdots + y(n)w(n) = 1.$$

By $\|z\|_{w,1} = 1$ and $x = (y + z)/2$ we have $z(i) = 2/W(n) - y(i)$ for $i = 1, \dots, n$ and $z(i) = 0$ for $i = n + 1, \dots$. Hence, $z^* = (2/W(n) - y(n), \dots, 2/W(n) - y(1), 0, \dots)$. Thus,

$$1 = \|z\|_{w,1} = \left(\frac{2}{W(n)} - y(n)\right)w(1) + \cdots + \left(\frac{2}{W(n)} - y(1)\right)w(n),$$

and so

$$y(1)w(n) + \cdots + y(n)w(1) = 1.$$

Moreover, by the assumption $w(1) > w(n)$, in view of $y(1) > y(n)$ and by the Hardy inequality, we have

$$\begin{aligned}
1 &= y(1)w(1) + \cdots + y(n)w(n) \\
&> y(1)w(n) + y(n)w(1) + (y(2)w(2) + \ldots + y(n-1)w(n-1)) \\
&\geq y(1)w(n) + y(n)w(1) + (y(2)w(n-1) + \cdots + y(n-1)w(2)) \\
&= y(1)w(n) + \cdots + y(n)w(1) = 1,
\end{aligned}$$

which is a contradiction. The proof is complete. $\square$

3. Applications. In this section we shall study the relationship between the existence sets and one-complemented subspaces of the Lorentz space $d(w,1)$, applying the characterization of smooth points in $d(w,1)$ (Theorem 1.10) and extreme points in its dual $d^*(w,1)$ (Theorem 2.2).

Let X be a Banach space, and let $C \subset X$ be a nonempty set. A continuous surjective mapping $P : X \to C$ is called *a projection onto* C, whenever $P|_C = \mathrm{Id}$, that is, $P^2 = P$.

Given a subspace V of a Banach space X, by $P(X,V)$ we denote the set of all linear, bounded projections from X onto V. Recall that a closed subspace V of a Banach space X is called *one-complemented* if there exists a norm one projection $P \in P(X,V)$. Setting, for each $x \in X$,

$$M_C(x) = \{z \in X : \|z - c\| \le \|x - c\| \text{ for any } c \in C\},$$

it is clear that $x \in M_C(x)$ for every $x \in X$ and $M_C(c) = \{c\}$ for every $c \in C$. Letting $\operatorname{Min} C$ be a subset of X consisting of an element x such that $M_C(x) = \{x\}$, we say that $C \subset X$ is *optimal* if $\operatorname{Min} C = C$. Observe that for any $C \subset X$, $C \subset \operatorname{Min} C$.

This notion has been introduced by Beauzamy and Maurey in [**4**], where basic properties concerning optimal sets can be found.

A set $C \subset X$ is called *an existence set of best coapproximation* (*existence set* for brevity), if for any $x \in X$, $R_C(x) \neq \varnothing$, where

$$R_C(x) = \{d \in C : \|d - c\| \le \|x - c\| \text{ for any } c \in C\}. \tag{3.1}$$

It is clear that any existence set is an optimal set. The converse, in general, is not true. However, by [**4**, Proposition 2], if X is one-complemented in X^{**} and strictly convex, then any optimal subset of X is an existence set in X, which, in particular, holds true for strictly convex spaces X, such that $X = Z^*$ for some Banach space Z.

Existence and optimal sets have been studied by many authors from different points of view, mainly in the context of approximation theory and functional analysis (see, e.g., [**2–6, 11, 13, 14, 19, 22, 28**]). There is also a large literature concerning one-complemented subspaces (see, e.g., a survey paper [**26**] and a recent paper [**17**]).

It is obvious that any one-complemented subspace is an existence set. However, the converse, in general, is not true. By a deep result of Lindenstrauss [**22**] there exist a Banach space X and a subspace V of X, with $\operatorname{codim} V = 2$, such that:

(a) V is one-complemented in any Y, where $Y \supset V$ is a hyperplane in X, i.e., $Y = f^{-1}(\{0\})$ for some $f \in X^* \setminus \{0\}$.

(b) V is not one-complemented in X.

This fact together with the simple observation stated as Lemma 3.1 below, gives an example of a subspace being an existence set which is not one-complemented.

Lemma 3.1. *Let X be a Banach space, and let $V \subset X$, $V \neq \{0\}$ be a linear subspace. Then V is an existence set in X if and only if for any $x \in X \setminus V$, there exists a $P_x \in P(Z_x, V)$ with $\|P_x\| = 1$. Here $Z_x = V \oplus [x]$, where $[x]$ denotes the linear span generated by x.*

Proof. Assume that for any $x \in X \setminus V$ there exists a $P_x \in P(Z_x, V)$, $\|P_x\| = 1$. Fix $z \in Z_x$ and $v \in V$. Note that

$$\|P_x z - v\| = \|P_x(z - v)\| \leq \|z - v\|.$$

Hence, $P_x z \in R_V(z)$ and so V is an existence set in X. Now assume that V is an existence set in X, and fix $x \in X \setminus V$. Take any $d \in R_V(x)$. Since any $z \in Z_x$ can be uniquely expressed as $z = \alpha x + v$ for some $v \in V$ and $\alpha \in \mathbf{R}$, we can define $P_x : Z_x \to V$ by

$$P_x z = \alpha d + v.$$

It is easy to see that $P_x \in P(Z_x, V)$. To show that $\|P_x\| = 1$, fix $y = \alpha x + v \in Z_x$, with $\alpha \neq 0$. Since $d \in R_V(x)$,

$$\|P_x y\| = \|\alpha d + v\| = |\alpha| \|d + v/\alpha\| \leq |\alpha| \|x + v/\alpha\| = \|\alpha x + v\| = \|y\|,$$

which completes the proof. $\square$

In [**4**] the following result has been proved.

Theorem 3.2 ([**4**, Proposition 5]). *Let $V \neq \{0\}$ be a linear subspace of a smooth, reflexive and strictly convex Banach space X. If V is* *an optimal set, then V is one-complemented in X. If X is a smooth Banach space, then any subspace of X which is an existence set is one-complemented. Moreover, in both cases a norm-one projection from X onto V is uniquely determined.*

We shall show here that the above result can be true in spaces that are not smooth. We will prove that any subspace of $d(w,1)$ which is an existence set must be one-complemented, which cannot be deduced from Theorem 3.2 because by Theorem 1.10, $d(w,1)$ is not a smooth space. Just recently [**21**], a similar result has been proved for spaces c_0, c, ℓ_1 and a large class of Musielak-Orlicz sequence spaces equipped with the Luxemburg norm. These facts provide a partial answer to the question stated in [**4**, page 125] concerning generalization of Theorem 3.2 to the nonsmooth case.

One of the main tools in our investigations, stated below, has been recently proved in [**21**].

Theorem 3.3. *Let X be a Banach space, and let $V \subset X$ be a linear subspace. Assume that V is an existence set and $V \neq \{0\}$. Put*

$$G_V = \{v \in V \setminus \{0\} : \text{there exists a unique } f \in S_{X^*} : f(v) = \|v\|\}.$$

Assume that the norm closure of G_V in X is equal to V. Then there exists a unique projection $P \in P(X,V)$ such that $\|P\| = 1$. Consequently, V is one-complemented in X.

For further reference we state the next well-known result.

Lemma 3.4. *Let X, Y be two Banach spaces, $V \subset X$ a linear subspace, and let $T : X \to Y$ be a linear isometry. Then V is an existence set in X if and only if $T(V)$ is an existence set in $T(X)$. Also V is one-complemented in X if and only if $T(V)$ is one complemented in $T(X)$.*

For $n \in \mathbf{N}$ and a decreasing sequence of positive numbers $\{w(1), \dots, w(n)\}$ define a finite-dimensional Lorentz space

$$d^n(w,1) = (\mathbf{R}^n, \|\cdot\|_{w,1}),$$

where

$$\|x\|_{w,1} = \sum_{j=1}^{n} x^*(j)w(j).$$

Before we state the main result we shall prove several auxiliary lemmas.

Lemma 3.5. *Let* $\{C_j\}_{j\in\mathbf{N}}$ *be a family of finite, nonempty subsets of* $\mathbf{N}$ *such that* $C_i \cap C_j = \varnothing$ *for* $i \neq j$. *Define for* $j \in \mathbf{N}$,

$$X_{C_j} = \{x \in d(w,1) : x(i) = x(k) \text{ for any } i, k \in C_j\}.$$

Let

$$X_C = \bigcap_{j=1}^{\infty} X_{C_j}.$$

Then X_C *is one-complemented in* $d(w,1)$. *The same result applies to* $d^n(w,1)$. *In this case we consider a finite family of nonempty, pairwise disjoint subsets of* $\{1,\dots,n\}$.

Proof. Let for $j \in \mathbf{N}$, $C_j = \{i_1,\dots i_{k_j}\}$, where $k_j = \operatorname{card} C_j$.

Set for $x \in d(w,1)$, $j \in \mathbf{N}$, $P_j x = (z(1),\dots,z(n),\dots)$, where $z(i) = (\sum_{l\in C_j} x(l))/k_j$ if $i \in C_j$, and $z(i) = x(i)$ in the opposite case. It is clear that $P_j \in P(d(w,1), X_{C_j})$. We also have that $\|P_j\| = 1$. Indeed, since for any permutation $\sigma : \mathbf{N} \to \mathbf{N}$, the mapping $T_\sigma : d(w,1) \to d(w,1)$ given by $T_\sigma x = x \circ \sigma$ is a linear, surjective isometry of $d(w,1)$, then by Lemma 3.4, we can assume that $C_j = \{1,\dots,k_j\}$. Let $x \in S_{d(w,1)}$, and set for $l = 2,\dots,k_j$,

$$x^l = (x(l), x(l+1),\dots,x(k_j), x(1),\dots,x(l-1), x(k_j+1),\dots).$$

Then $x + \sum_{l=2}^{k_j} x^l = k_j(P_j x)$, and

$$\|P_j x\|_{w,1} = \|(x + \sum_{l=2}^{k_j} x^l)/k_j\|_{w,1} \leq (\|x\|_{w,1} + \sum_{l=2}^{k_j} \|x^l\|_{w,1})/k_j = 1,$$

since $\|x^l\|_{w,1} = \|x\|_{w,1} = 1$ for $l = 2,\dots,k_j$. Thus, $\|P_j\| = 1$. Now define for $j \in \mathbf{N}$,

$$X_j = \bigcap_{m=1}^{j} X_{C_m},$$

and

$$Q_j = P_1 \circ P_2 \circ \cdots \circ P_j.$$

Since $C_i \cap C_k = \varnothing$, for $i \neq k$, so $Q_j \in P(d(w,1), X_j)$. By the above reasoning, $\|Q_j\| = 1$.

Now, fix $x \in d(w,1)$. Define $Qx = ((Qx)(1), \dots (Qx)(n), \dots)$, where $(Qx)(i) = x(i)$ if $i \notin \cup_{j \in \mathbf{N}} C_j$ and $(Qx)(i) = (\sum_{l \in C_j} x(l))/k_j$ if $i \in C_j$. Since $C_i \cap C_j = \varnothing$ for $i \neq j$,

$$(Qx)(i) = \lim_j (Q_j x)(i),$$

for any $i \in \mathbf{N}$. Now we show that $Qx \in d(w,1)$ for any $x \in d(w,1)$. Indeed, for any $x \in B_{d(w,1)}$ and any $j \in \mathbf{N}$, we have $\|Q_j x\| \leq 1$ since $\|Q_j\| = 1$. In view of $d(w,1) = (d_*(w,1))^*$ and the fact that $d_*(w,1)$ is separable, the weak* topology on $B_{d(w,1)}$ is metrizable. Thus, by the Banach-Alaoglu theorem, there exists a subsequence $\{j_k\}$ and $Rx \in B_{d(w,1)}$, with $Q_{j_k} x \to Rx$ weakly* in $d(w,1)$. In particular, for any $i \in \mathbf{N}$, we have

$$(Rx)(i) = \lim_k (Q_{j_k} x)(i).$$

This shows that $Qx = Rx$, and consequently $Qx \in d(w,1)$. Note also that $Qx \in X_C$ and for any $x \in X_C$, $Qx = x$. Since $Qx = Rx$, $Qx \in B_{d(w,1)}$, for any $x \in B_{d(w,1)}$. Thus, Q is a linear projection from $d(w,1)$ onto X_C with $\|Q\| = 1$, which completes the proof. The case of $d^n(w,1)$ can be proved in a similar way. □

The next lemma is well known but for the sake of completeness we include its proof here.

Lemma 3.6. *Let X be a Banach space, and let $x \in X$. Define*

$$D(x) = \{f \in B_{X^*} : f(x) = \|x\|\}.$$

Then

$$\varnothing \neq \operatorname{ext} D(x) \subset \operatorname{ext} B_{X^*}.$$

Proof. If $x = 0$, then $D(x) = B_{X^*}$ which shows our claim. So assume $x \neq 0$. By the Hahn-Banach theorem, $D(x) \neq \varnothing$. Note that $D(x)$ is a convex, weakly* closed subset of B_X. By the Banach-Alaoglu and the Krein-Milman theorems, $\operatorname{ext} D(x) \neq \varnothing$. We show that $\operatorname{ext} D(x) \subset \operatorname{ext} B_{X^*}$. Let $f \in \operatorname{ext} D(x)$. Assume $f = (f_1 + f_2)/2$ and $f_1, f_2 \in S_{X^*}$. Hence,

$$\|x\| = f(x) = (f_1(x) + f_2(x))/2.$$

Since $\|f_1\| = \|f_2\| = 1$, $f_1(x) = f_2(x) = \|x\|$, which gives $f_1, f_2 \in D(x)$. Since $f \in \operatorname{ext} D(x)$, $f_1 = f_2$, as required. $\square$

Lemma 3.7. *Let $v \in d(w,1) \setminus \{0\}$ be such that $v = v^*$ and $\operatorname{card}(\operatorname{supp} v) = \infty$. Let*

$$D_1 = \{k \in \mathbf{N} : v(1) = v(k)\} \quad \textit{and} \quad n_1 = \max D_1.$$

For $i \geq 2$, let

$$D_i = \{k \in \mathbf{N} : v(n_{i-1} + 1) = v(k)\} \quad \textit{and} \quad n_{i-1} = \max D_{i-1}.$$

Set

$$E(v) = \{f \in \operatorname{ext} B_{d^*(w,1)} : f(v) = \|v\|_{w,1}\}.$$

Then $f \in E(v)$ if and only if $f = w \circ \sigma$, where $\sigma : \mathbf{N} \to \mathbf{N}$ is a permutation such that for any $k \in D_i$ and $l \in D_{i+1}$ we have $w(\sigma(k)) \geq w(\sigma(l))$ and

$$\sum_{k \in D_i} f(k) = \sum_{k \in D_i} w(\sigma(k)) = \sum_{k \in D_i} w(k)$$

for any $i \in \mathbf{N}$.

Proof. Assume $f \in E(v)$. Since $v = v^*$ and $\operatorname{card}(\operatorname{supp}(v)) = \infty$, by Theorem 2.2, $f = w \circ \sigma$ for some permutation $\sigma : \mathbf{N} \to \mathbf{N}$. Now we will check that $w(\sigma(k)) \geq w(\sigma(l))$ for every $k \in D_i$ and $l \in D_{i+1}$. Assume, on the contrary, that there exist $i \in \mathbf{N}$, $k \in D_i$ and $l \in D_{i+1}$, with $w(\sigma(k)) < w(\sigma(l))$. Define $\sigma_1 : \mathbf{N} \to \mathbf{N}$ by $\sigma_1(l) = \sigma(k)$, $\sigma_1(k) = \sigma(l)$ and $\sigma_1(n) = \sigma(n)$ for $n \notin \{k, l\}$. Since $v(k) > v(l)$ and $w(\sigma(k)) < w(\sigma(l))$,

$$\|v\|_{w,1} = \sum_{j=1}^{\infty} v(j) w(\sigma(j)) = f(v) < \sum_{j=1}^{\infty} v(j) w(\sigma_1(j)),$$

which is a contradiction. Now, applying the induction argument, we show that

$$\sum_{k\in D_i} w(\sigma(k)) = \sum_{k\in D_i} w(k)$$

for any $i \in \mathbf{N}$.

Let $Z_0 = \varnothing$,

$$Z_1 = \{j \in \mathbf{N} : w(j) = w(1)\},$$

$m_1 = \max Z_1$ and for $i \geq 2$,

$$Z_i = \{j \in \mathbf{N} : w(j) = w(m_{i-1} + 1)\},$$

where $m_{i-1} = \max Z_{i-1}$. Also define for $u \in \mathbf{N}$,

$$J_u = \left\{ j \in \mathbf{N} : Z_j \subset \bigcup_{i=1}^{u} D_i \right\} \cup \{0\}$$

and

$$j_u = \max \{j : j \in J_u\} + 1.$$

If $n_1 \leq m_1$, then $\sigma(D_1) \subset Z_1$, since $f(v) = \|v\|_{w,1}$. Consequently, since $\operatorname{card}(\sigma(D_1)) = n_1$,

$$\sum_{i\in D_1} w(i) = n_1 w(1) = \sum_{i\in D_1} w(\sigma(i)).$$

Also,

$$\sigma(D_1) = \bigcup_{j\in J_1} Z_j \cup (Z_{j_1} \cap \sigma(D_1)).$$

If $m_1 < n_1$, then

$$D_1 = \bigcup_{j\in J_1} Z_j \cup (Z_{j_1} \cap D_1).$$

Since $f(v) = \|v\|_{w,1}$, $Z_j \subset \sigma(D_1)$, for any $j \in J_1$ and

$$\sigma(D_1) \subset \bigcup_{j\in J_1} Z_j \cup Z_{j_1}.$$

Hence,

$$\sigma(D_1) = \bigcup_{j\in J_1} Z_j \cup (Z_{j_1} \cap \sigma(D_1)).$$

Set for $i \in J_1$, $c_i = \operatorname{card} Z_i$ and $d_1 = \operatorname{card}(Z_{j_1} \cap D_1)$. Note that $d_1 = \operatorname{card}(Z_{j_1} \cap \sigma(D_1))$. Consequently,

$$\sum_{k \in D_1} w(\sigma(k)) = \sum_{k \in \sigma(D_1)} w(k) = \sum_{j \in J_1} c_j w(m_j) + d_1 w(m_{j_1}) = \sum_{k \in D_1} w(k),$$

which ends the proof of the first step of the induction argument.

To continue the proof by the induction argument, assume that

$$\sum_{k \in D_i} w(\sigma(k)) = \sum_{k \in D_i} w(k) \tag{3.2}$$

for $i \leq u-1$ and

$$\sigma\Big(\bigcup_{i=1}^{u-1} D_i\Big) = \bigcup_{j \in J_{u-1}} Z_j \cup \Big(Z_{j_{u-1}} \cap \sigma\Big(\bigcup_{i=1}^{u-1} D_i\Big)\Big). \tag{3.3}$$

We will show that

$$\sum_{k \in D_i} w(\sigma(k)) = \sum_{k \in D_i} w(k) \tag{3.4}$$

for $i \leq u$ and

$$\sigma\Big(\bigcup_{i=1}^{u} D_i\Big) = \bigcup_{j \in J_u} Z_j \cup \Big(Z_{j_u} \cap \sigma\Big(\bigcup_{i=1}^{u} D_i\Big)\Big). \tag{3.5}$$

By (3.3)

$$\sigma(D_u) \subset \bigcup_{j \geq j_{u-1}} Z_j.$$

Since $w(\sigma(k)) \geq w(\sigma(l))$ for any $i \in \mathbf{N}$, and any $k \in D_i$, $l \in D_{i+1}$,

$$\sigma(D_u) \subset \bigcup_{j=j_{u-1}}^{j_u} Z_j.$$

If $J_{u-1} = J_u$, then

$$\sigma\Big(\bigcup_{i=1}^{u} D_i\Big) = \bigcup_{j\in J_{u-1}} Z_j \cup \Big(Z_{j_{u-1}} \cap \sigma\Big(\bigcup_{i=1}^{u-1} D_i\Big)\Big) \cup (Z_{j_{u-1}} \cap \sigma(D_u))$$
$$= \bigcup_{j\in J_u} Z_j \cup \Big(Z_{j_u} \cap \sigma\Big(\bigcup_{i=1}^{u} D_i\Big)\Big).$$

If $J_{u-1} \neq J_u$, then $J_u = J_{u-1} \cup A_u$, where

$$A_u = \{j \geq j_{u-1} : j \in J_u\}.$$

Since $\|v\|_{w,1} = f(v)$, by (3.3),

$$Z_j \subset \sigma(D_{u-1} \cup D_u),$$

for any $j \in A_u$. Consequently,

$$\sigma\Big(\bigcup_{i=1}^{u} D_i\Big) = \sigma\Big(\bigcup_{i=1}^{u} D_i\Big) \cap \bigcup_{j=1}^{j_u} Z_j = \sigma\Big(\bigcup_{i=1}^{u} D_i\Big) \cap \Big(\bigcup_{j\in J_u} Z_j \cup Z_{j_u}\Big)$$
$$= \bigcup_{j\in J_u} Z_j \cup \Big(Z_{j_u} \cap \sigma\Big(\bigcup_{i=1}^{u} D_i\Big)\Big),$$

which shows (3.5).

Now we prove (3.4). If $J_{u-1} = J_u$, then $\sigma(D_u) \subset Z_{j_{u-1}}$ and $D_u \subset Z_{j_{u-1}}$. Hence,

$$\sum_{k\in D_u} w(k) = \mathrm{card}\,(D_u) w(m_{j_{u-1}}) = (n_u - n_{u-1}) w(m_{j_{u-1}})$$
$$= \sum_{k\in\sigma(D_u)} w(k) = \sum_{k\in D_u} w(\sigma(k)),$$

which shows our claim. If $J_u \neq J_{u-1}$, by (3.3) and (3.5),

$$\sigma(D_u) = (\sigma(D_u) \cap Z_{j_{u-1}}) \cup \bigcup_{j\in J_u\setminus(J_{u-1}\cup\{j_{u-1}\})} Z_j \cup (Z_{j_u} \cap \sigma(D_u)). \tag{3.6}$$

Also,

$$D_u = (D_u \cap Z_{j_{u-1}}) \cup \bigcup_{j \in J_u \setminus (J_{u-1} \cup \{j_{u-1}\})} Z_j \cup (Z_{j_u} \cap D_u).$$

Since $J_u \neq J_{u-1}$, $Z_{j_{u-1}} \subset \cup_{i=1}^{u} D_i$ and, by (3.5), $Z_{j_{u-1}} \subset \sigma(\cup_{i=1}^{u} D_i)$. By (3.3),

$$\operatorname{card}\left(Z_{j_{u-1}} \cap \sigma\left(\bigcup_{i=1}^{u-1} D_i\right)\right) = \operatorname{card}\left(Z_{j_{u-1}} \cap \bigcup_{i=1}^{u-1} D_i\right).$$

Consequently,

$$\operatorname{card}(Z_{j_{u-1}} \cap \sigma(D_u)) = \operatorname{card}(Z_{j_{u-1}} \cap D_u)$$

and, by (3.6),

$$\operatorname{card}(Z_{j_u} \cap \sigma(D_u)) = \operatorname{card}(Z_{j_u} \cap D_u).$$

Let $d_{u-1} = \operatorname{card}(Z_{j_{u-1}} \cap D_u)$, $c_j = \operatorname{card} Z_j$ for $j \in J_u \setminus (J_{u-1} \cup \{j_{u-1}\})$ and $d_u = \operatorname{card}(Z_{j_u} \cap D_u)$. Note that

$$\begin{aligned} \sum_{k \in D_u} w(\sigma(k)) = \sum_{k \in \sigma(D_u)} w(k) &= d_{u-1} w(m_{j_{u-1}}) \\ &+ \sum_{j \in J_u \setminus (J_{u-1} \cup \{j_{u-1}\})} c_j w(m_j) + d_u w(m_{j_u}) = \sum_{k \in D_u} w(k), \end{aligned}$$

which shows (3.4) for $i = u$. For $i \leq u-1$, (3.4) follows immediately from (3.2).

In order to prove the converse, note that

$$\begin{aligned} \|v\|_{w,1} &= \sum_{n=1}^{\infty} w(n) v(n) = \sum_{i \in \mathbf{N}} v(n_i) \sum_{j \in D_i} w(j) \\ &= \sum_{i \in \mathbf{N}} v(n_i) \sum_{j \in D_i} w(\sigma(j)) = \sum_{n=1}^{\infty} v(n) w(\sigma(n)). \end{aligned}$$

Thus, the proof is complete. $\square$

The next fact is an easy consequence of Lemma 3.7.

Lemma 3.8. *Let $v \in d(w,1)$ be such that* $\operatorname{card}(\operatorname{supp} v) = \infty$. *Let $E(v)$ and $D_i = D_i(v^*)$ be such as in Lemma 3.7. Letting $\pi : \mathbf{N} \to \operatorname{supp} v$ be a bijective mapping such that $|v(\pi(n))| = v^*(n)$ for any $n \in \mathbf{N}$, define for $i \in \mathbf{N}$,*

$$U_i = \pi(D_i).$$

Then $f \in E(v)$ if and only if $f \in \operatorname{ext} B_{d^*(w,1)}$ and, for any $i \in \mathbf{N}$,

$$\sum_{j \in D_i} w(j) = \sum_{j \in U_i} \operatorname{sign} v(j) f(j).$$

Proof. Let $f \in E(v)$. Note that

$$\begin{aligned}\sum_{j=1}^{\infty} v^*(j) w(j) = \|v^*\|_{w,1} = \|v\|_{w,1} = f(v) \\ &= \sum_{j \in \operatorname{supp} v} f(j) v(j) = \sum_{j \in \operatorname{supp} v} \operatorname{sign} v(j) f(j) |v(j)| \\ &= \sum_{j=1}^{\infty} \operatorname{sign} v(\pi(j)) f(\pi(j)) |v(\pi(j))| = \sum_{j=1}^{\infty} g(j) v^*(j),\end{aligned}$$

where $g(j) = \operatorname{sign} v(\pi(j)) f(\pi(j))$ for $j \in \mathbf{N}$. Set $g = (g(1), g(2), \dots)$. Since $f \in E(v)$, by Theorem 2.2, $f^* = w$. Since $\{|g(j)|\}_{j \in \mathbf{N}} \subset \{|f(j)|\}_{j \in \mathbf{N}}$, $g^*(j) \le f^*(j) = w(j)$ for any $j \in \mathbf{N}$. By the above calculations and the Hardy inequality,

$$\|v^*\|_{w,1} = \sum_{j=1}^{\infty} g(j) v^*(j) \le \sum_{j=1}^{\infty} g^*(j) v^*(j) \le \sum_{j=1}^{\infty} f^*(j) v^*(j) = \|v^*\|_{w,1}.$$

Consequently, $\|g\|_W = \|g^*\|_W = 1$ and

$$0 = \sum_{j=1}^{\infty} (w(j) - g^*(j)) v^*(j).$$

Since $v^*(j) > 0$ for any $j \in \mathbf{N}$, $g^*(j) = w(j)$ for any $j \in \mathbf{N}$. By Theorem 2.2, $g \in \operatorname{ext} B_{d^*(w,1)}$. Since $\|v^*\|_{w,1} = g(v^*)$, $g \in E(v^*)$. By Lemma 3.7 applied to v^* and g, for any $i \in \mathbf{N}$,

$$\sum_{j \in D_i} w(j) = \sum_{j \in D_i} g(j) = \sum_{j \in D_i} \operatorname{sign} v(\pi(j)) f(\pi(j)) = \sum_{j \in U_i} \operatorname{sign} v(j) f(j),$$

which shows our claim. To prove the converse, assume that $f \in \operatorname{ext} B_{d^*(w,1)}$ and for any $i \in \mathbf{N}$,

$$\sum_{j \in D_i} w(j) = \sum_{j \in U_i} \operatorname{sign} v(j) f(j).$$

Note that $|v(j)| = v^*(n_i)$ for $j \in U_i$, where n_i is a number defined in the proof of Lemma 3.7. Thus,

$$\begin{aligned} f(v) &= \sum_{j \in \operatorname{supp} v} f(j)v(j) = \sum_{i=1}^{\infty} \bigg(\sum_{j \in U_i} f(j)v(j) \bigg) \\ &= \sum_{i=1}^{\infty} \bigg(\sum_{j \in U_i} \operatorname{sign} v(j) f(j) |v(j)| \bigg) \\ &= \sum_{i=1}^{\infty} v^*(n_i) \bigg(\sum_{j \in U_i} \operatorname{sign} v(j) f(j) \bigg) = \sum_{i=1}^{\infty} v^*(n_i) \bigg(\sum_{j \in D_i} w(j) \bigg) \\ &= \sum_{j=1}^{\infty} v^*(j) w(j) = \|v^*\|_{w,1} = \|v\|_{w,1}. \end{aligned}$$

Hence, $f \in E(v)$, as required. □

Lemma 3.9. *Let $V \subset d(w,1)$ be a linear subspace. Set*

$$G_V = \{v \in V \setminus \{0\} : \text{ there exists a unique } f \in S_{V^*} : f(v) = \|v\|_{w,1}\}.$$

Assume $v \in G_V$ is such that card (supp v) $= \infty$*. Let, for any $k \in \mathbf{N}$,*

$$C_k = \{j \in \mathbf{N} : x(j) = x(k) \text{ for any } x \in V\}.$$

Let $D_i = D_i(v^*)$ and U_i be such as in Lemma 3.8. If $i \in \mathbf{N}$ is such that $w(k) > w(k+1)$ for some $k \in D_i$, then $U_i = C_k$.

Proof. First assume that $v = v^*$. In this case $\operatorname{supp} v = \mathbf{N}$, $\pi = id$ and $U_i = D_i$. Take $k \in D_i$, $k+1 \in D_i$ with $w(k) > w(k+1)$. We show that $C_k = D_i$. Indeed, inclusion $C_k \subset D_i$ is obvious by the definitions of C_k and D_i. In order to show the opposite inclusion, assume on the contrary that there exists $l \in D_i \setminus C_k$. If $l \geq k+1$, define for $x \in d(w,1)$

$$h_1(x) = \sum_{m=1}^{\infty} x(m)w(m),$$
$$h_2(x) = \sum_{m \neq l, k+1}^{\infty} x(m)w(m) + x(k)w(l) + x(l)w(k).$$

Note that $h_1(v) = h_2(v) = \|v\|_{w,1}$ and $\|h_i\|_W = 1$ for $i = 1, 2$. Since $l \notin C_k$ and $k \in C_k$, there exists a $z \in V$ such that $z(k) \neq z(l)$. We have

$$\begin{aligned} h_1(z) - h_2(z) &= z(k)(w(k) - w(l)) + z(l)(w(l) - w(k)) \\ &= (w(k) - w(l))(z(k) - z(l)). \end{aligned}$$

It follows that $h_1(z) \neq h_2(z)$ since $w(k) > w(k+1) \geq w(l)$ and $z(k) \neq z(l)$. Thus, $h_1 \neq h_2$ on V and so $v \notin G_V$; a contradiction. If $l < k$, consider $g_1, g_2 \in d^*(w,1) = (d(w,1))^*$ defined by

$$g_1(x) = \sum_{m=1}^{\infty} x(m)w(m),$$
$$g_2(x) = \sum_{m \neq l, K+1}^{\infty} x(m)w(m) + x(l)w(k+1) + x(k+1)w(l).$$

Note that $g_1(v) = g_2(v) = \|v\|_{w,1}$ and $\|g_i\|_W = 1$ for $i = 1, 2$. Since $l \notin C_k$ and by the above proof we have $k+1 \in C_k$, there exists $y \in V$ such that $y(l) \neq y(k+1)$. By the following equality

$$\begin{aligned} g_1(y) - g_2(y) &= y(l)(w(l) - w(k+1)) + y(k+1)(w(k+1) - w(l)) \\ &= (w(l) - w(k+1))(y(l) - y(k+1)), \end{aligned}$$

and in view of $w(l) \geq w(k) > w(k+1)$ and $y(l) \neq y(k+1)$, we have that $g_1(y) \neq g_2(y)$. Thus, $v \notin G_V$; a contradiction. Thus, the sets D_i and C_k coincide. If $v \neq v^*$, by similar reasoning we can show that $U_i = C_k$. $\square$

Now we are able to state the main result of this section.

Theorem 3.10. *Let $V \subset d(w,1)$, $V \neq \{0\}$ be a linear subspace. If V is an existence set, then V is one-complemented.*

Proof. Let
$$\operatorname{supp} V = \bigcup_{v \in V} \operatorname{supp} v.$$
First we assume that $\operatorname{card}(\operatorname{supp} V) = \infty$. For any $i \in \operatorname{supp} V$, define
$$C_{i,1} = \{j \in \mathbf{N}, j \neq i : x(i) = x(j) \text{ for any } x \in V\},$$
$$C_{i,2} = \{j \in \mathbf{N}, j \neq i : x(i) = -x(j) \text{ for any } x \in V\},$$
and
$$C_i = \{i\} \cup C_{i,1} \cup C_{i,2}.$$
Note that, for any $i, j \in \operatorname{supp} V$, $C_i = C_j$ or $C_i \cap C_j = \varnothing$. Since $d(w,1) \subset c_0$, C_i is a finite, nonempty set for any $i \in \operatorname{supp} V$, set $i_1 = \min \operatorname{supp} V$, $i_2 = \min\{\operatorname{supp} V \setminus C_{i_1}\}$ and $i_n = \min\{\operatorname{supp} V \setminus \cup_{j=1}^{n-1} C_{i_j}\}$. Note that $\operatorname{supp} V = \cup_{j=1}^{\infty} C_{i_j}$ and $C_{i_j} \cap C_{i_k} = \varnothing$ for $j \neq k$. Since, for any permutation $\sigma : \mathbf{N} \to \mathbf{N}$ and $\{\varepsilon(n)\}$ with $\varepsilon(n) = \pm 1$, the mapping $Tx = \{\varepsilon(n) x(\sigma(n))\}$ is a linear, surjective isometry of $d(w,1)$, in view of Lemma 3.4, we can assume without loss of generality that $C_{i_j,2} = \varnothing$ for any $j \in \mathbf{N}$. For simplicity we shall further denote the sets $\{C_{i_j}\}$ by $\{C_i\}$. Let X_C be the space considered in Lemma 3.5, generated by the sets C_i defined above. By Lemma 3.5, X_C is one-complemented in $d(w,1)$. By the construction of the sets C_i, and Lemma 3.4, we can assume that $V \subset X_C$ for modified sets C_i. Thus, in order to show that V is one-complemented in $d(w,1)$, it is enough to demonstrate that V is one-complemented in X_C. We will apply Theorem 3.3. Let
$$G_V = \{v \in V \setminus \{0\} : \text{ there exists a unique } f \in S_{V^*} : f(v) = \|v\|_{w,1}\},$$
and
$$G_{V,C} = \{v \in V \setminus \{0\} : \text{ there exists a unique } f \in S_{(X_C)^*} : f(v) = \|v\|_{w,1}\}.$$
We shall show that $G_V = G_{V,C}$. Note that, by the Hahn-Banach theorem, $G_{V,C} \subset G_V$. To prove the converse, assume that $v \in G_V$. We need to show that v is a smooth point in X_C.

First assume that $v = v^*$.

Note also that $\operatorname{card}(\operatorname{supp} v) = \infty$. Indeed, if we assume that $\operatorname{supp} v = \{1, \dots, n\}$, then in view of $\operatorname{card}(\operatorname{supp} V) = \infty$, there exist $j > n$ and $y \in V$ with $y(j) \neq 0$. Defining for $x \in d(w, 1)$,

$$f_1(x) = \sum_{m=1}^{\infty} x(m)w(m)$$

and

$$f_2(x) = f_1(x) - 2x(j)w(j),$$

we have $f_1(v) = f_2(v) = \|v\|_{w,1}$ and $|f_i(x)| \leq \|x\|_{w,1}$, $i = 1, 2$, by the Hardy inequality. Thus, $\|f_1|_V\| = \|f_2|_V\| = 1$. Since also $f_1(y) \neq f_2(y)$, so $v \notin G_V$; a contradiction. Thus $\operatorname{supp} v$ is infinite.

Let

$$E(v, C) = \{f \in \operatorname{ext} B_{(X_C)^*} : f(v) = \|v\|_{w,1}\}.$$

By Lemma 3.6 applied to v and X_C, $E(v, C) \neq \varnothing$. We shall show that $\operatorname{card} E(v, C) = 1$. Recall that

$$E(v) = \{f \in \operatorname{ext} B_{d^*(w,1)} : f(v) = \|v\|_{w,1}\}.$$

We have the following inclusion

$$E(v, C) \subset E(v)|_{X_C} = \{h|_{X_C} : h \in E(v)\}.$$

Indeed, let $g \in S_{V^*}$ be such that $g(v) = \|v\|_{w,1}$. Since $v \in G_V$, g is uniquely determined, $g \in \operatorname{ext} B_{V^*}$, and thus for any $f \in E(v, C)$, $f|_V = g$. Hence,

$$E(v, C) = \{f \in B_{(X_C)^*} : f|_V = g\}$$

and

$$E(v) = \{h \in B_{d^*(w,1)} : h|_V = g\}.$$

If $f \in E(v, C)$, then the set of all norm preserving extensions of f to $d(w, 1)$ is denoted by

$$G(f) = \{h \in B_{d^*(w,1)} : h|_{X_C} = f\}.$$

Since $G(f)$ is nonempty and weakly* compact, by the Krein-Milman theorem, $\operatorname{ext} G(f) \neq \varnothing$. It is clear that, for any $h \in \operatorname{ext} G(f)$, $h|_{X_C} = f$ and $h|_V = g$, which shows the required inclusion.

Now we claim that, for any $h \in E(v)$ and $x \in X_C$,

$$h(x) = \sum_{n=1}^{\infty} w(n)x(n).$$

In fact, by Lemma 3.7, $h = w \circ \sigma$, where the permutation σ is such that, for any $i \in \mathbf{N}$,

$$\sum_{j \in D_i} w(j) = \sum_{j \in D_i} w(\sigma(j))$$

and D_i are such as in Lemma 3.7. Therefore, it is enough to demonstrate that, for any $i \in \mathbf{N}$ and any $x \in X_C$,

$$\sum_{j \in D_i} x(j)w(j) = \sum_{j \in D_i} x(j)w(\sigma(j)).$$

Fix $i \in \mathbf{N}$. If $D_i \subset Z_j$ for some $j \in J_{i+1}$, where Z_j and J_i have been defined in the proof of Lemma 3.7, then for any $k \in D_i$,

$$w(k) = w(m_j) = w(\sigma(k)).$$

Hence,

$$\sum_{j \in D_i} x(j)w(j) = w(m_j) \sum_{j \in D_i} x(j) = \sum_{j \in D_i} x(j)w(\sigma(j)).$$

If $D_i \setminus Z_j \neq \varnothing$ for any $j \in \mathbf{N}$, then $w(k) > w(l)$ for some $k, l \in D_i$. By Lemma 3.9, $D_i = C_k$ for some $k \in \mathbf{N}$, and in view of Lemma 3.7 we get

$$\begin{aligned}\sum_{j \in D_i} x(j)w(j) &= \sum_{j \in C_k} x(j)w(j) = x(n_i) \sum_{j \in D_i} w(i) \\ &= x(n_i) \sum_{j \in D_i} w(\sigma(i)) = \sum_{j \in D_i} x(j)w(\sigma(j)),\end{aligned}$$

which shows our claim. Thus, $E(v)|_{X_C}$ consists of exactly one element and consequently $\operatorname{card} E(v, C) = 1$, since $E(v, C) \subset E(v)|_{X_C}$, and $E(v, C)$ is nonempty.

If $v \neq v^*$, then applying Lemma 3.8 instead of Lemma 3.7 we can show in an analogous way that $\operatorname{card} E(v, C) = 1$.

By Lemma 3.6, v is a smooth point in X_C and consequently $v \in G_{V,C}$. Thus, $G_V = G_{V,C}$. Since V is an existence set in $d(w, 1)$ and $V \subset X_C \subset d(w, 1)$, V is an existence set in X_C. Moreover, by separability of $d(w, 1)$ and by the Mazur theorem [**12**, Theorem 4.12], that the collection of smooth points in a separable Banach space X is dense in X, G_V is dense in V. Applying now Theorem 3.3 to V and X_C, there exists a norm-one projection $P \in P(X_C, V)$. In view of Lemma 3.5 we can also find a norm-one projection $Q \in P(d(w, 1), X_C)$. Hence, $R = P \circ Q$ is a norm-one projection from $d(w, 1)$ onto V. The proof is complete in the case when $\operatorname{supp} V$ is infinite.

If $\operatorname{supp} V$ is a finite set, by Lemma 3.4, we can assume that $\operatorname{supp} V = \{1, \dots, n\}$ for some $n \in \mathbf{N}$. In this case we can consider V as a subspace of $d^n(w, 1)$. Since V is an existence set in $d(w, 1)$, V is also an existence set in $d^n(w, 1)$. Reasoning as above we can show that V is one-complemented in $d^n(w, 1)$. Since the norm in $d(w, 1)$ is monotone, the mapping

$$Qx = (x(1), \dots x(n), 0, \dots)$$

is a norm-one projection from $d(w, 1)$ onto $d^n(w, 1)$. Hence, V is one-complemented in $d(w, 1)$, as required. $\square$

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