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A GOOD λ INEQUALITY FOR DOUBLE LAYER POTENTIALS OF SURFACES THAT ARE NOT LIPSCHITZ

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Introduction. In this paper we prove a good-λ inequality for the double layer potential operators. These have the form

$$C_\varepsilon^j(A,f)(x) = \int_{|x-y|>\varepsilon} \frac{(x_j - y_j)f(y)}{(|x-y|^2 + (A(x)-A(y))^2)^{(n+1)/2}} dy,$$

where x, y are in $\mathbf{R}^n$. The corresponding Maximal Operator is

$$C_*^j(A,f)(x) = \sup_{\varepsilon>0} |c_\varepsilon^j(A,f)(x)|$$

The hypersurface $t = A(x)$ is not assumed to be Lipschitz. The Good-λ inequality that we will prove can be used to obtain weighted L^p estimates for the Double Layer Potential Operators as was done in the one dimensional case for the Cauchy Integral Operator in [**4**].

Statement and proof of the main result. Throughout this paper we will consider the real number p fixed and strictly larger than n. With such a p let

$$A^*(x) = ((|\text{grad}\,(A)|^p)^*(x))^{1/p} = M_p(|\text{grad}\,(A)|)(x),$$

where $(\ )^*$ denotes the maximal function and M_p is the p-maximal function. We are assuming that $|\text{grad}\,(A)|^p$ is locally integrable and that $A^*(x)$ is finite a.e.

With this notation we will prove

THEOREM 1. *There exists a constant k such that, for all positive ε, one can find a constant C_ε such that the following holds:*

$$|\{x : C_*^j(A,f)(x) > (1+\varepsilon)\lambda \ \&\ (1+A^*(x))^k f^*(x) \le \lambda/C_\varepsilon\}| < 0.9|\{x : C_*^j(A,f)(x)\rangle\lambda\}|,$$

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where f^* is the Hardy-Littlewood Maximal function of f.

PROOF. We take a Whitney decomposition of the open set Q_λ (see [**5**, p. 167]):

$$Q_\lambda = \{x : C_*^j(A, f)(x) > \lambda\} = \cup\, Q_i.$$

The Q_i are Whitney cubes, i.e., they are closed, they have sides parallel to the axes, pairwise disjoint interiors, and distances to the complement of Q_λ comparable to their diameters:

$$\text{diam}\,(Q_i) \leq \text{dist}\,(Q_i, Q_\lambda^c) \leq 4\ \text{diam}\,(Q_i),$$

where diam (Q_i) = diameter of the cube Q_i.

Since the interiors are disjoint, it suffices to prove the theorem for each one of the Whitney cubes. Moreover, it suffices to consider only those that satisfy

$$|\{x \text{ in } Q_i : (1 + A^*(x))^k f^*(x) \leq \lambda/C_\varepsilon\}| \geq 0.9|Q_i|,$$

since otherwise there is nothing to prove. Therefore, we need to prove

$$|\{x \text{ in } Q_i : C_*^j(A, f)(x) > (1+\varepsilon)\lambda \ \&\ (1 + A^*(x))^k f^*(x) \leq \lambda/C_\varepsilon\}| < 0.9|Q_i|,$$

where Q_i is such that

$$|\{x \text{ in } Q_i : (1 + A^*(x))^k f^*(x) > \lambda/C_\varepsilon\}| \leq 0.1|Q_i|.$$

From the definition of the Whitney cubes it also follows that

$$C_*^j(A, f)(u)| \leq \lambda,$$

where $u = u(i)$ is some point in the cube Q^* which is centered around the same point as Q_i (with sides parallel to the axes), and with side $10\sqrt{n}$ times larger than the side of Q_i.

Let $Q^\sim$ be a cube centered at the point u with side $20\sqrt{n}$ times the side of Q_i. We now perform a Whitney decomposition on $Q^\sim$, as follows:

$$\{x \text{ in } (Q^\sim)^0 : (1 + A*(x))^k f^*(x) > \lambda/C_\varepsilon\} = \cup J_m.$$

By assumption,

$$\sum_{J_m \subset Q_i} |J_m| < 0.1|Q_i|.$$

We let $J_m^\sim$ be a cube centered around the same point as J_m, but with side $2^{1/n}$ times larger than the side of J_m. Then $|J_m^\sim| = 2|J_m|$, and if

$$F_o = Q_i - \cup_{J_m^\sim \subset Q_i} (J_m^\sim)^o$$

then $|F_o| \geq 0.8|Q_i|$. Set $F = Q^\sim - \cup (J_m^\sim)^o$.

Let $A^\sim$ be the restriction of the function A to the set F. Let $A^\wedge = E_o(A^\sim)$ be the extension of $A^\sim$ to $\mathbf{R}_n$, i.e., E_o is the first extension operator described in [**5**], p. 171. On the set F the following holds:

$$(1 + A^*(x))^k f^*(x) \leq \lambda / C_\varepsilon.$$

Consequently, on this set

$$A^*(x) \leq \left| \frac{\lambda}{C_\varepsilon (1/|Q^\sim|) \int_{Q^\sim} |f|} \right|^{1/k} - 1 = v(Q_i).$$

Here we need to use the elementary estimate

$$\sup_{y \neq x} \frac{|A(y) - A(x)|}{|y - x|} \leq C A^*(x)$$

for some constant C. From this it follows that the Lipschitz norm of $A^\sim$ on F is dominated by $v(Q_i)$, and since the Extension Operator E_o is a continuous mapping of the respective Lipschitz spaces, we can conclude that the Lipschitz norm of $A^\wedge$ is also dominated by $v(Q_i)$.

We write $f = f_1 + f_2$, with $f_1 = f$ on $Q^\sim$, and $f_1 = 0$ elsewhere. The double layer potential operators for Lipschitz hypersurfaces are bound on L^2. (See [**1**; Theorem IX, p. 382].) Consequently (see [**2**; Theorem 20, p. 89]) we obtain the following weak type $(1, 1)$ estimate for $C_*^j(A^\wedge, f_1)$:

$$\begin{aligned} &|\{x \text{ in } Q^\sim : C_*^j(A^\wedge, f_1) > \lambda\varepsilon/5\}| \\ &\qquad \leq C(1 + v(Q_i))^k \Big(\int_{Q^\sim} |f_1| dx \Big) (5/\lambda\varepsilon) \\ &\qquad \leq 5C|Q_i|/\varepsilon\, C_\varepsilon < 0.1|Q_i|. \end{aligned}$$

The last inequality is obtained by taking $C_\varepsilon > 200C/\varepsilon$.

For x in the set F_o, we will consider the difference

$$h(x) = C^j(A, f_1)(x) - C^j(A^\wedge, f_1)(x).$$

The argument for $h_\varepsilon(x) = C^j_\varepsilon(A, f_1)(x) - C^j_\varepsilon(A^\wedge, f_1)(x)$ is the same. A simple calculation yields

$$|h(x)| \leq \int_{\cup J_m} |x-y|^{-n} G(x,y)|f(y)|dy$$

with

$$G(x,y) =$$
$$\left|\left(1 + \frac{(A(x) - A(y))^2}{|x-y|^2}\right)^{-(n+1)/2} - \left(1 + \frac{(A(x) - A^\wedge(y))^2}{|x-y|^2}\right)^{-(n+1)/2}\right|.$$

Note that, for x in F, $A(x) = A^\wedge(x)$. A simple application of the Mean Value Theorem on the function

$$g(t) = (1+t^2)^{-(n+1)/2}$$

combined with the boundedness of the derivative of that function, yields the estimate

$$G(x,y) \leq C\frac{|A(y) - A^\wedge(y)|}{|x-y|}.$$

Consequently, for $h(x)$, we obtain

$$|h(x)| \leq C \int_{\cup J_m} \frac{|A(y) - A^\wedge(y)|}{|x-y|^{n+1}} |f(y)|dy.$$

Let J^*_m be the cube with sides parallel to the axes, the same center as J_m, and side $10\sqrt{n}$ times the side of J_m. Since the J_m form a Whitney decomposition we can find a point u_m in J^*_m so that

$$(1 + A^*(u_m))^k f^*(u_m) \leq \lambda/C_\varepsilon.$$

This implies that $A^*(u_m) \leq v(Q_i)$. Consequently,

$$\begin{aligned} |A(y) - A^\wedge(y)| &\leq |A(y) - A(u_m)| + |A(u_m) - A^\wedge(y)| \\ &\leq C \operatorname{diam}(J_m) v(Q_i). \end{aligned}$$

We have used the fact that $A(u_m) = A^\wedge(u_m)$, and that the Lipschitz norm of $A^\wedge$ is dominated by $v(Q_i)$. Therefore,

$$\int_{F_o} |h(x)|dx \le Cv(Q_i)\sum_m \int_{F_o}\int_{J_m} \frac{\operatorname{diam}(J_m)}{|x-y|^{n+1}}|f(y)|dy\,dx$$

$$\le Cv(Q_i)\sum_m \int_{J_m} |f(y)|dy \le Cv(Q_i)\int_{Q^\sim} |f(y)|dy.$$

From this it follows that

$$\begin{aligned}|\{x \text{ in } F_o : |h(x)| > \lambda\varepsilon/5\}| &\le (5/\lambda\varepsilon)\int_{F_o} |h(x)|dx \\ &\le (5C/\lambda\varepsilon)v(Q_i)\int_{Q^\sim} |f(y)|dy \\ &\le (5C/\lambda\varepsilon)(1+v(Q_i))^k \int_{Q^\sim} |f(y)|dy \\ &\le (5C/\varepsilon C_\varepsilon)|Q_i| < 0.1|Q_i|.\end{aligned}$$

At this point we have selected $C_\varepsilon > 50C/\varepsilon$. By combining this estimate, with the weak-type estimate for $C_*^j(A^\wedge, f_1)$ that we have already proved, we obtain

$$\begin{aligned}&|\{x \text{ in } F_o : C_*^j(A, f_1)(x) > 2\varepsilon\lambda/5\}| \\ &= |\{x \text{ in } F_o : \sup_{\varepsilon>0} |C_\varepsilon^j(A, f_1)(x)| > 2\varepsilon\lambda/5\}| \\ &\le |\{x \text{ in } F_o : \sup_{\varepsilon>0} |C_*^j(A^\wedge, f_1)(x)| > \varepsilon\lambda/5\}| \\ &\quad + |\{x \text{ in } F_o : \sup_{\varepsilon>0} |C_*^j(A, f_1)(x) - C_\varepsilon^j(A^\wedge, f_1)(x)| > \varepsilon\lambda/5\}| \\ &= |\{ \text{ in } F_o : C_*^j(A^\wedge, f_1)(x) > \varepsilon\lambda/5\}| \\ &\quad + |\{x \text{ in } F_o : \sup_{\varepsilon>0} |h(x)| > \varepsilon\lambda/5\}| < 0.2|Q_i|.\end{aligned}$$

Since $|F_o| \ge 0.8|Q_i|$ it follows that

$$(*) \qquad |\{x \text{ in } Q_i : C_*^j(A, f_1)(x) < 2\varepsilon\lambda/5\}| \ge 0.6|Q_i|.$$

Equation $(*)$ allows us to control the part that involves f_1. We now need to know how to control the part involving f_2. We claim that the following is true:

$$(**) \qquad \text{for all } x \text{ in } F_o, C_*^j(A, f_2)(x) < \lambda + \varepsilon\lambda/5.$$

Assume (**) is true for a moment, so that we can complete the proof of the theorem. (*) and (**) imply that

$$|\{x \text{ in } Q_i : C^j_*(A,f)(x) < \lambda + 3\varepsilon\lambda/5\}| \geq 0.6|Q_i|,$$

and consequently

$$|\{x \text{ in } Q_i : C^j_*(A,f)(x) > (1+\varepsilon)\lambda\}| \leq 0.4|Q_i|.$$

This completes the proof of the theorem. □

Proof of Equation (**). Recall that $C^j_*(A,f)(u_m) \leq \lambda$, and that f_2 is supported outside $Q^\sim$. Therefore $C^j_*(A,f_2)(u_m) \leq \lambda$ and it suffices to obtain the estimate

$$\sup_{\varepsilon>0} |C^j_\varepsilon(A,f_2)(x) - C^j_\varepsilon(A,f_2)(u_m)| < \varepsilon\lambda/5$$

for all x in F_o.

Fix a cube Q_x (with sides parallel to the axes) centered at x, and let Q_u be a cube of the same size as Q_x centered at u. Then

$$\begin{aligned}
&\Big|\int_{R^n-Q_x} \frac{(x_j-y_j)f_2(y)}{|(x-y,A(x)-A(y))|^{n+1}}dy \\
&\quad - \int_{R^n-Q_u} \frac{(u_j-y_j)f_2(y)}{|(u-y,A(u)-A(y))|^{n+1}}dy\Big| \\
&\leq \Big|\int_{F_1}\Big(\frac{x_j-y_j}{|(x-y,A(x)-A(y))|^{n+1}} - \frac{u_j-y_j}{|(u-y,A(u)-A(y))|^{n+1}}\Big)f(y)dy\Big| \\
&\quad + \int_{F_2}\frac{|f_2(y)|}{|x-y|^n}dy + \int_{F_2}\frac{|f_2(y)|}{|u-y|^n}dy,
\end{aligned}$$

where $F_1 = R^n - Q_x \cup Q_u \cup Q^\sim$, and $F_2 = Q_x \Delta Q_u$.

Since f_2 is supported on the complement of $Q^\sim$, $|x-y|$ and $|u-y|$ are of the same order of magnitude, so the last two integrals are dominated by $f^*(x)$. To estimate the first integral we move the absolute value inside, and enlarge the domain of integration to obtain

$$\int_{\mathbf{R}^n-Q}\left|\frac{x_j-y_j}{|(x-y,A(x)-A(y))|^{n+1}} - \frac{u_j-y_j}{|(u-y,A(u)-A(y))|^{n+1}}\right||f(y)|dy.$$

Since

$$\left|\text{grad}\left(\frac{w_j}{|(w,t)|^{n+1}}\right)\right| \le C|w|^{-(n+1)}$$

the above integral is dominated by

$$C\int_{\mathbf{R}^n - Q^\sim} \frac{|(u-x, A(u)-A(x))|}{|x-y|^{n+1}}|f(y)|dy$$
$$\le C\int_{\mathbf{R}^n - Q^\sim} \frac{\text{diam}\,(Q_i)(1+A^*(x))}{|x-y|^{n+1}}|f(y)|dy:$$

To obtain the last inequality we again used the fact that the Lipschitz norm of A is dominated by the p-maximal function of its gradient. Since the last integral is dominated by $(1+A^*(x))f^*(x)$ we can now put all our estimates together:

$$\sup_{\varepsilon>0}|C^j_\varepsilon(A,f_2)(x) - C^j_\varepsilon(A,f_2)(u)| \le C(1+A^*(x))f^*(x)$$
$$\le C(1+A^*(x))^k f^*(x).$$

Since x is in F_o, the last expression does not exceed $\lambda C/C_\varepsilon < \lambda\varepsilon/5$ if we choose $C_\varepsilon > 5C/\varepsilon$. This completes the proof of the claim. □

COROLLARY. *For all j, and all $s>1$ the following weighted L^p estimate holds for the double layer potential operators on a surface $t = A(x)$:*

$$||C^j_*(A,f)||^s_s \le C_s\int_{\mathbf{R}^n}|f(y)|^s w(y)dy$$

where $w(y) = (((1+A^*(y))^{ks+1})^*)^{ks/(ks+1)}$.

PROOF. The weight w is of class A^1. The rest follows by a standard argument for getting L^p estimates from a Good-inequality, followed by an application of a weighted norm inequality. See [**3**].

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