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IDEALS DIFFERENTIAL UNDER HIGH ORDER DERIVATIONS

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ABSTRACT. In this paper we prove the following theorem: Let A be an R-algebra, S a multiplicatively closed set in A, $\mathcal{U}$ a subset of $\mathrm{Der}_R^{\infty}(A)$ and I an ideal of A. If I is $\mathcal{U}$-differential, then $S(I)$ is $\mathcal{U}$-differential as well. This implies that the nonembedded primary components of a differential ideal are differential. Nevertheless we give an example of an $\mathcal{U}$-differential ideal which has no $\mathcal{U}$-differential embedded primary component.

0. Introduction. Let A be a commutative noetherian ring, $\mathcal{U}$ a subset of all derivations of any order n from A into A and I an $\mathcal{U}$-differential ideal in A (i.e., $d(I) \subseteq I$ for all $d \in \mathcal{U}$). In this note we are concerned with the question of whether other ideals related to I, especially its primary components, are $\mathcal{U}$-differential. In his paper [**6**, Theorem 1] A. Seidenberg proved that if A is a noetherian algebra containing the rational numbers and I an ideal differential under a subset $\mathcal{U}$ of all derivations on A of order $n = 1$, then every $P \in \mathrm{Ass}\,(A/I)$ is differential and I can be written as an irredundant intersection $Q_1 \cap \cdots \cap Q_s$ of $\mathcal{U}$-differential primary ideals. Simple examples show that the elements of $\mathrm{Ass}\,(A/I)$ are not differential in general if $n > 1$ or even if $n > 1$ and $Q \subseteq A$. In this note we show that the nonembedded primary components of an $\mathcal{U}$-differential ideal I are always $\mathcal{U}$-differential but no embedded primary has to be $\mathcal{U}$-differential in general. To do so, we shall prove the following theorem: Let A be a commutative ring, S any multiplicatively closed set in A and $\mathcal{U}$ a subset of all derivations of any order n from A into A. If an ideal I of A is $\mathcal{U}$-differential, then its S-component $S(I)$ is $\mathcal{U}$-differential as well. Furthermore we give an example for an $\mathcal{U}$-differential ideal I where no embedded primary component is $\mathcal{U}$-differential (i.e., I cannot be written as an intersection of $\mathcal{U}$-differential primary ideals).

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1. Preliminaries. Throughout this paper we assume all rings are commutative and have an identity.

For a ring R and an R-algebra A and an A-module M we define a pairing $\phi : \mathrm{Hom}_R(A, M) \times A \to \mathrm{Hom}_R(A, M)$ by

$$\begin{aligned}(f, x) \to [f, x] &: A \to M\\ y &\to f(xy) - xf(y) - yf(x).\end{aligned}$$

DEFINITION. An element δ of $\mathrm{Hom}_R(A, M)$ is called an R-derivation of order 1, if $[\delta, x] \equiv 0$ for all $x \in A$ and a derivation of order $n > 1$, if $[\delta, x]$ is a derivation of order $n - 1$ for all $x \in A$.

For more information on high order derivations we refer the interested reader to [**5**] and [**1**].

We shall let $\mathrm{Der}_R^n(A)$ be the A-module of all R-linear derivations of order n from A into A, and $\mathrm{Der}_R^\infty(A) = \cup_{n=1}^\infty \mathrm{Der}_R^n(A)$.

DEFINITION. If $\mathcal{U}$ is a subset of $\mathrm{Der}_R^\infty(A)$ and I an ideal of A, then we shall say that I is $\mathcal{U}$-differential, if $\delta(I) \subseteq I$ for all $\delta \in \mathcal{U}$.

REMARK. If an ideal I is δ-differential, then I is also $[\delta, x]$-differential for any $x \in A$, since $[\delta, x](y) = \delta(xy) - yd(x) - xd(y) \in I$ for all $y \in I$. Therefore if I is $\mathcal{U}$-differential we always can replace the subset $\mathcal{U}$ by the submodule generated by $\mathcal{U}$ and all $[d, x]$ with $d \in \mathcal{U}$ and $x \in A$.

DEFINITION. If S is a multiplicatively closed set in A and I an ideal, then the ideal $S(I) = \{x \in A |$ there exists an $s \in S$ such that $sx \in I\}$ is called the S-component of I.

2. S-components of differential ideals.

THEOREM. *Let A be an R-algebra, S a multiplicatively closed set in A, $\mathcal{U}$ a subset in $\mathrm{Der}_R^\infty(A)$ and I an ideal in A. If I is $\mathcal{U}$-differential, then $S(I)$ is $\mathcal{U}$-differential.*

PROOF. Let $y \in S(I)$ and $\delta \in \mathcal{U}$. We have to show $\delta(y) \in S(I)$.

We prove this by induction on the order of δ: $\delta = 1$: $\delta(sy) = s\delta(y) + y\delta(s) \in I$, hence $s^2\delta(y) \in I$ and so $\delta(y) \in S(I)$. $\delta = n+1$: $\delta(sy) = s\delta(y) + y\delta(s) + [\delta, s](y) \in I$. By the induction assumption we know $[\delta, s](y) \in S(I)$, therefore there exists an element t of S such that $t[\delta, s](y) \in I$, hence $ts^2\delta(y) \in I$ and we obtain again $\delta(y) \in S(I)$. □

COROLLARY 1. (W.C. Brown [**3**]). *Let D be a higher derivation on A, I a D-ideal of A, and y an arbitrary element of A, then $J = \cup_{n=0}^{\infty}(I : y^n)$ is a D-ideal of A.*

PROOF. Consider $S = \{y^n | n \in \mathbf{N}_0\}$, then $S(I) = J$. □

COROLLARY 2. *If S is a multiplicatively closed set in A, then* $\ker(\varphi : A \to A_S)$ *is always* $\mathrm{Der}_R^{\infty}(A)$*-differential.*

PROOF. Ker $(\varphi : A \to A_s) = S(0)$.

COROLLARY 3. *Let A be a ring, P a prime ideal, and I an ideal such that $P \not\supset I$, $P \cap I = 0$. Then P is a* $\mathrm{Der}_R^{\infty}(A)$*-differential.* (For derivations of order 1, see H. Matsumura [**4**].)

PROOF. Let $x \in I, x \notin P$. $S = \{x^n | n \in \mathbf{N}_0\}$, then $P = S(0)$. □

3. Primary components of differential ideals.

PROPOSITION. *Let A be an R-algebra and $\mathcal{U}$ a subset of* $\mathrm{Der}_R^{n}(A)$. *If I is an $\mathcal{U}$-differential ideal in A and Q a nonembedded primary component, then Q is $\mathcal{U}$-differential.*

PROOF. Let $P = \mathrm{rad}(Q)$ and $S = A \backslash P$. Then we have $Q = S(I)$ because P is nonembedded. □

COROLLARY 1. (W.C. Brown [**2**]). *If I is a $\mathcal{U}$-differential radical ideal and P an element of* $\mathrm{Ass}(A/I)$, *then P is $\mathcal{U}$-differential.*

COUNTEREXAMPLE FOR THE EMBEDDED PRIMARY COMPONENTS. We construct a δ-differential ideal I such that no embedded primary component Q of I is δ-differential.

Let k be a field of characteristic 0 and $A = k[X, Y]$ the polynomial ring in X and Y over k. We define a k-derivation δ of order 2 from A into A by $\delta(XY) = XY$, $\delta(X^2) = X^2$, $\delta(Y^2) = 2Y$, $\delta(X) = X$, $\delta(Y) = 1$. Let I be the ideal in A defined by $I = (X^2, XY)$. Possible primary decompositions of I are $I = (X) \cap (X,Y)^n, n \geq 2$. The ideal I has the following two properties:

1. I is δ-differential (respectively I is $\mathcal{U}$-differential where $\mathcal{U}$ is submodule of $\mathrm{Der}_k^2(A)$ generated by $\{\delta, [\delta, x] | x \in A\}$).

2. No embedded primary component is δ-differential.

PROOF. 1. Since I is generated as a k-module by elements of the form $X^nY^m, n \in \mathbf{N}, m \in N_0$ and $n + m \geq 2$, we have to show $\delta(XY^m) \in I$ for all $n \in \mathbf{N}, m \in \mathbf{N}_0$. (Note that it is not enough to show $\delta(X^2)$ and $\delta(XY)$ are in I for derivations of order $n > 1$.) We make induction on $n + m$. $n + m = 2, 3 : \delta(X^2), \delta(XY)$, $\delta(XY^2) = X\delta(Y^2) + 2Y\delta(XY) - 2XY\delta(Y) - Y^2\delta(X)$ and $\delta(X^2Y) = Y\delta(X^2) + 2X\delta(XY) - 2XY\delta(X) - X^2\delta(Y)$ are elements of I by definition of δ.

Now we assume $\delta(X^nY^m) \in I$ for $n + m = s \geq 3$. We have to prove $\delta(X^nY^m) \in I$ for $n + m = s + 1 \geq 4$.

Case 1. $n = 1, 2$, (i.e., $m \geq 2$)

$$\begin{aligned}\delta(X^nY^m) &= \delta(XX^{n-1}Y^{m-1}Y) \\ &= X\delta(X^{n-1}Y^m) + X^{n-1}Y^{m-1}\delta(XY) + Y\delta(X^nY^{m-1}) \\ &\quad - X^nY^{m-1}\delta(Y) - X^{n-1}Y^m\delta(X) - XY\delta(X^{n-1}Y^{m-1})\end{aligned}$$

Case 2. $n > 2$.

$$\begin{aligned}\delta(X^nY^m) &= \delta(X^2 \cdot X^{n-2}Y^m) \\ &= X^2\delta(X^{n-2}Y^m) + X^{n-2}\delta(X^2Y^m) + Y^m\delta(X^n) \\ &\quad - X^n\delta(Y^m) - X^2Y^m\delta(X^{n-2}) - X^{n-2}Y^m\delta(X^2)\end{aligned}$$

In either case using the induction assumption and the power rule for derivation of order 2 (Y. Nakai [**5**]), we find that each monomial on the right hand side is in I, therefore $\delta(X^nY^m) \in I$.

2. Let Q be an embedded component of I, then $Q \supseteq m^n = (X, Y)^n$ for a suitable $n \in \mathbf{N}$. Therefore $Y^n \in Q$ for an $n \in \mathbf{N}$. Let m be the smallest number with $Y^m \in \mathbf{N}$, then

$$\delta(Y^m) = \begin{cases} 1 & \text{if } m = 1 \\ \binom{m}{2} Y^{m-2}\delta(Y^2) - m(m-2)Y^{m-1}\delta(Y) & \\ = \binom{m}{2} Y^{m-2} \cdot 2Y - m(m-2)Y^{m-1} & \\ = m \cdot Y^{m-1} & \text{if } m > 1 \end{cases}$$

hence $\delta(Y^m) \notin Q$. This shows Q is not δ-differential.

REMARK. Since $\delta(I) \subseteq I$, δ induces a second order derivation $\overline{\delta}$ on A/I. The ideal $(\overline{0})$ is of course $\operatorname{Der}_k^n(A)$-differential for all $n \in \mathbf{N}$ (resp. $\operatorname{Der}_k^\infty(A)$-differential) but no embedded primary component of $(\overline{0})$ is $\operatorname{Der}_k^n(A)$-differential for $n \geq 2$ (resp. $\operatorname{Der}_k^\infty(A)$-differential).

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