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ON CONFLUENT MAPPINGS AND ESSENTIAL MAPPINGS—A SURVEY

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1. **Introduction.** In 1935, Eilenberg showed in [13] that if $g: X \rightarrow Y$ is a monotone mapping or an open mapping of compact metric spaces and if $f: Y \rightarrow S^1$ is a mapping such that $f \circ g$ is homotopic to a constant mapping, then f is homotopic to a constant mapping.

In 1964, Charatonik introduced in [9] the class of confluent mappings. These mappings are generalizations of both the class of monotone mappings and the class of open mappings. In 1966, Lelek extended in [44] Eilenberg's theorem to the class of confluent mappings. In §3 of this paper we present the various generalizations of Lelek's result to compact Hausdorff spaces and to semi-confluent mappings. We would like to emphasize these generalizations concern mappings into any one-dimensional connected ANR instead of into S^1. We also prove that semiconfluent mappings onto one-dimensional connected ANRs are essential.

In 1934, Mazurkiewicz showed in [63] that AH-essential mappings of compact metric spaces onto the 2-cell I^2 are weakly confluent, and he stated that his result extends to arbitrary dimensions. In §4 we give a proof of this result for AH-essential mappings of compact Hausdorff spaces onto connected manifolds M. We also show that for spaces X which are contractible with respect to S^{n-1} a mapping $f: X \rightarrow I^n$ is AH-essential if and only if the restriction of f to the preimage of the boundary of I^n is essential.

Eilenberg's theorem implies that monotone mappings and open mappings preserve metric continua with trivial first cohomology group. A very considerable literature has grown up concerning classes of continua which are preserved by these and related mappings. In §6 there is given a brief survey of this literature. Characterizations are given of the pseudo-confluent images of the arc and of dendrites.

In §5 reults of Lokuciewski and Holsztyński concerning the fixed point property of continua which are inverse limits of n-cells with essential bonding mappings are discussed. It is proved that cones over continua

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which are inverse limits of n-cells (resp. n-spheres) with AH-essential (resp. essential) bonding mappings have the fixed point property.

2. **Preliminaries.** By a *mapping* we mean a continuous function. By a *continuum* we mean a connected, compact, Hausdorff space. A mapping $f: X \to Y$ of a compact Hausdorff space X onto a Hausdorff space Y is said to be (m) *monotone*, (c) *confluent* [**9**], (s) *semi-confluent* [**55**], (w) *weakly confluent* [**47**] and (p) *pseudo-confluent* [**53**] if for each continuum C in Y

(m) $f^{-1}(C)$ is connected,

(c) each component of $f^{-1}(C)$ is mapped onto C,

(s) for each pair of components K and L of $f^{-1}(C)$ either $f(K) \subset f(L)$ or $f(L) \subset f(K)$,

(w) some component of $f^{-1}(C)$ is mapped onto C,

(p) some component of $f^{-1}(C)$ is mapped onto C if C is irreducible.

A mapping $f: X \to Y$ is *open* if $f(U)$ is open in Y for each U open in X. Whyburn proved in [**88**, p. 148] that open mappings are confluent. Clearly, monotone mappings are confluent, confluent mappings are semi-confluent, semi-confluent mappings are weakly confluent and weakly confluent mappings are pseudo-confluent. It is easy to see that none of the above implications can be reversed.

A mapping $f: X \to Y$ is said to be *essential* if f is not homotopic to a constant mapping. We write f non ~ 1. If $f: X \to Y$ is homotopic to a constant mapping, we write $f \sim 1$. If $f: X \to Y$ is such that f non ~ 1 but $f|B \sim 1$ for each proper closed subset B of X, write f irr non ~ 1.

If X is a compact Hausdorff space and G is an Abelian group, we let $H^n(X; G)$ denote the nth Čech cohomology group based on arbitrary open coverings and with coefficients in the group G. We let Z denote the group of integers. A continuum X is said to be *acyclic* if $H^n(X; \mathbf{Z}) = 0$ for each $n \geqq 1$.

If P is a collection of polyhedra and X is a compact Hausdorff space, we say that X is *P-like* if X is the inverse limit (see [**15**, p. 215]) of an inverse system $\{X_\alpha, \pi_\alpha^\lambda, \Lambda\}$ where each X_α is a member of the collection P of polyhedra, Λ is a directed set, for each $\lambda \geqq \alpha$ in Λ $\pi_\alpha^\lambda: X_\lambda \to X_\alpha$ is a mapping such that $\pi_\alpha^\alpha: X_\alpha \to X_\alpha$ is the identity and $\pi_\alpha^\lambda = \pi_\alpha^\beta \circ \pi_\beta^\lambda$ for $\alpha \leqq \beta \leqq \lambda$ in Λ.

A *graph* is a finite one-dimensional connected polyhedron. A *tree* is a simply connected graph. A *dendrite* is a locally connected metric continuum which contains no simple closed curve. By an ANR we mean a metric absolute neighbourhood retract [**6**]. We let $]a, b[$ denote the open interval from a to b.

A continuum X is *unicoherent* if $X \neq P \cup Q$ where P and Q are subcontinua of X such that $P \cap Q$ is not connected. A continuum is said to be *decomposable* if it can be written as the union of two proper subcontinua, otherwise, it is said to be *indecomposable.* A continuum is said to be *hereditarily unicoherent* (resp. *hereditarily decomposable*) if every subcontinuum is unicoherent (resp. decomposable).

If P is a collection of polyhedra, X is P-like and every member of P is a tree (resp. arc, resp. homeomorphic to S^n), then X is said to be tree-like (resp. arc-like, resp. S^n-like). It is obvious that arc-like continua are tree-like and tree-like continua are one-dimensional.

If A is a subset of a topological space X, then $\mathrm{Cl}(A)$, $\mathrm{Bd}(A)$ and $\mathrm{Int}(A)$ denote the closure of A in X, the boundary of A in X, and the interior of A in X, respectively.

An extensive study of the spaces of confluent and related mappings of compacta has been carried out by several authors. The reader may consult **[42]**, **[43]**, **[58]** and **[69]** for conditions under which the space of mappings in (m), (c), (s), (w) and (p) are closed or complete in the space Y^X.

3. **Confluent mappings and cohomology.** In this section we present the various generalizations of Lelek's theorem **[44]**. Recently, the following result was given in **[27]** answering a question of Pasynkov.

THEOREM 3.1. (**[27**, 5.2]). *Let g: $X \to Y$ be a confluent mapping of a compact Hausdorff space X onto a Hausdorff space Y, and let f: $Y \to G$ be a mapping of Y into a graph G. Then $f \circ g \sim 1$ implies $f \sim 1$.*

Using Theorem 3.1 we can show the following more general result.

THEOREM 3.2. *Let g: $X \to Y$ be a confluent mapping of a compact Hausdorff space X onto a Hausdorff space Y. and let f: $X \to M$ be a mapping of Y into a one-dimensional connected* ANR *M. Then $f \circ g \sim 1$ implies $f \sim 1$.*

PROOF. Since M is a one-dimensional connected ANR, it contains at most finitely many simple closed curves. Hence, there exists a graph $G \subset M$ with fundamental group $\pi_1(G) = \pi_1(M)$. Then, there exists a monotone retraction r: $M \to G$ of M onto G.

Since $f \circ g \sim 1$ we have that $r \circ f \circ g \sim 1$. By Theorem 3.1, since g is confluent and $r \circ f$: $Y \to G$ is a mapping such that $(r \circ f) \circ g \sim 1$, we have that $r \circ f \sim 1$. But the condition $\pi_1(G) = \pi_1(M)$ implies that $r \circ f$ considered as a mapping of Y into M is homotopic to f. Thus $f \sim 1$.

A space X is said to be *contractible with respect to a space Y* provided that every mapping of X into Y is homotopic to a constant. An immediate consequence of Theorem 3.2 is that contractibility of a compact Hausdorff space with respect to any one-dimensional connected ANR is preserved by confluent mappings. It follows from Theorem 3.1 and **[12**, Theorem 8.1] that if g: $X \to Y$ is a confluent onto mapping of compact Hausdorff spaces, then the induced mapping g^*: $H^1(Y; \mathbf{Z}) \to H^1(X; \mathbf{Z})$

is a monomorphism (compare with [**44**, p. 230]). An example given in [**44**, p. 233] shows that the above statement fails for groups $H^1(X; G)$ if G is different from $\mathbf{Z}$ and also for $H^n(X; \mathbf{Z})$, $n > 1$. Another immediate corollary of Theorem 3.2 is that confluent mappings of compact Hausdorff spaces onto nonsimply connected one-dimensional connected ANRs are essential.

LEMMA 3.3. *Let f: $X \to S^1$ be a semi-confluent mapping of a compact Hausdorff space X onto S^1. Then f is essential.*

PROOF. Suppose, on the contrary, that f is inessential. Then there exists a mapping ϕ: $X \to R^1$ where R^1 is the real line such that $f(x) = e^{2\pi i\phi(x)}$ for each $x \in X$. Let a and b be two real numbers with $a < b$ such that $\phi(X) = [a, b]$. Then $b - a \geqq 1$. We need to consider only the following two cases.

Case 1. There exist integers n and m such that

$$n < a \leqq n + 1/2 < m \leqq b < m + 1/2.$$

Let A be a component of $\phi^{-1}([n, n + 1/2]) = \phi^{-1}([a, n + 1/2])$ which is mapped by ϕ onto $[a, n + 1/2]$, and let B be a component of $\phi^{-1}([m, m + 1/2]) = \phi^{-1}([m, b])$ which is mapped by ϕ onto $[m, b]$. This is possible since every mapping onto an arc is weakly confluent (see [**75**]). We denote points of S^1 be means of polar coordinates. Let K be the following subset of S^1:

$$K = \{(1, \theta) \mid 0 \leqq \theta \leqq \pi\}.$$

Then A and B are components of $f^{-1}(K)$, since ϕ is continuous and

$$\phi(f^{-1}(K)) \subset \bigcup_{r \in z} [r, r + 1/2],$$

where Z denotes the set of integers. Since $(1, \pi) \in f(A)$ and $(1, \pi) \notin f(B)$, we obtain that $f(A) \not\subset f(B)$. Since $(1, 0) \in f(B)$ and $(1, 0) \notin f(A)$, we obtain that $f(B) \not\subset f(A)$. This contradicts the semi-confluence of f.

Case 2. There exist integers n and m such that

$$n < a \leqq n + 1/2 < m + 1/2 \leqq b < m + 1.$$

Let $c \in R$ be a number such that $b < c < m + 1$. Since ϕ is weakly confluent, there is a component A of $\phi^{-1}([n, n + c - m]) = \phi^{-1}([a, n + c - m])$ which is mapped by ϕ onto $[a, n + c - m]$, and a component B of $\phi^{-1}([m, c]) = \phi^{-1}([m, b])$ which is mapped by ϕ onto $[m, b]$. Let K be the continuum $K = f(A \cup B)$. Then, as in Case 1, A and B are components of $f^{-1}(K)$. Since $(1, 0) \in f(B)$ and $(1, 0) \notin f(A)$, we obtain that $f(B) \not\subset f(A)$. Since $e^{2\pi ic} \in f(A)$ and $e^{2\pi ic} \notin f(B)$, we obtain that $f(A) \not\subset f(B)$. This contradicts the semi-confluence of f, and the proof of the theorem is complete.

It is not true, though, that even monotone mappings of continua onto S^n for $n > 1$ are essential.

EXAMPLE 3.4. Let B^n denote the n-dimensional disc ($n \geqq 2$) and let S^{n-1} be its boundary. Let $f\colon B^n \to B^n/S^{n-1} \cong S^n$ be the quotient map. Then f is monotone, but f is not essential.

THEOREM 3.5. *Let $f\colon X \to Y$ be a semi-confluent mapping of a compact Hausdorff space X onto a non-simply connected one-dimensional connected* ANR. *Then f is essential.*

PROOF. As in the proof of Theorem 3.2, there exists a graph G with $\pi_1(G) = \pi_1(Y)$ and a monotone retraction $r\colon Y \to G$ of Y onto G We now show that G admits a monotone mapping onto S^1. For this let C be a simple closed curve in G and let $[a, b]$ be an arc with endpoints a and b in C such that $[a, b]\backslash\{a, b\}$ is open in G. Let $g\colon G \to S^1$ be a mapping which maps $[a, b]\backslash\{a, b\}$ homeomorphically onto $S^1\backslash\{(1, 0)\}$, and such that $g(a) = g(b) = g(G\backslash[a, b]) = (1, 0)$. Then g is monotone, and hence, g is confluent. By [55, p. 254], $g \circ r \circ f\colon X \to S^1$ is a semi-confluent mapping of X onto S^1. By Lemma 3.3, $g \circ r \circ f$ is essential, and hence, $r \circ f$ is essential Since $r \circ f$ can be considered to be a mapping of X into Y, it is obvious that $r \circ f \sim f$. Thus, f is essential.

Let $f\colon S^1 \to S^1$ be an onto mapping. Consider, the universal covering $p\colon R^1 \to S^1$, where R^1 is the real line and $p(X) = e^{2\pi i x}$ for each $x \in R^1$. Since the mapping $f \circ p\colon R^1 \to S^1$ is inessential, there exists a mapping $g\colon R^1 \to R^1$ such that $p \circ g = f \circ p$ [82, p. 67]. We say that f is a *wrapping function* if and only if g is a monotone mapping.

THEOREM 3.6. *Let $f\colon S^1 \to S^1$ be a mapping of S^1 onto S^1. Then the following are equivalent*:

(a) *f is confluent*;

(b) *f is semi-confluent*;

(c) *f is a wrapping function.*

PROOF. $(a) \Rightarrow (b)$. This is obvious.

$(b) \Rightarrow (c)$. Let $f\colon S^1 \to S^1$ be a semi-confluent mapping. By [55, Theorem 3.9] there exists a monotone mapping $f_1\colon S^1 \to Y$ of S^1 onto a continuum Y and a semi-confluent mapping $f_2\colon Y \to S^1$ such that $f = f_2 \circ f_1$ and such that f_2 is a light mapping (i.e., $f_2^{-1}(t)$ is zero-dimensional for each $t \in S^1$). By [88, p. 165] Y is homeomorphic to S^1. It is clear that if f_2 is a wrapping function, then f is also a wrapping function. We suppose, therefore, that f is a light mapping.

Let $p\colon R^1 \to S^1$ be the universal covering of S^1 (i.e., $p(x) = e^{2\pi i x}$ for each $X \in R^1$) and let $g\colon [0, 1] \to R^1$ be a mapping such the $p \circ g = f \circ p|[0, 1]$ (see [82, p. 67]. We shall prove that g is monotone. If g fails to be monotone, then we may suppose, without loss of generality, that there exists t_0 in the open interval $]0, 1[$ such that g has a local minimum at t_0. Then one of the following two cases holds.

Case (1). There exists $s_0 \in]0, t_0[\cup]t_0, 1[$ such that g has a local maximum at s_0.

Case (2). g has local maxima at both 0 and 1.

Since f is a light mapping, g is also light. It is now easy to show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that diameter $(C) < \varepsilon$ for each component C of $g^{-1}([g(t), g(t) + \delta])$ and for each $t \in [0. 1]$.

Let us suppose Case (1) holds. Let $0 < \delta < 1/2$ be such that

$$p([g(t_0), g(t_0) + \delta]) \cap p([g(s_0) - \delta, g(s_0)]) = \emptyset$$

if $p(g(s_0)) \neq p(g(t_0))$ and such that if $[a, b]$ (resp. $[c, d]$) is the component of $g^{-1}([g(t_0), g(t_0) + \delta])$ (resp. $g^{-1}([g(s_0) - \delta, g(s_0)])$) which contains t_0 (resp. s_0), then $0, 1 \notin [a, b] \cup [c, d]$ and

$$(*) \qquad g(a) = g(b) = g(t_0) + \delta \text{ and } g(c) = g(d) = g(s_0) - \delta.$$

The condition $(*)$ follows from the previous paragraph and from the fact that g has local extrema at t_0 and s_0. Notice that $g([a, b]) = [g(t_0), g(t_0) + \delta])$ and $g([c, d]) = [g(s_0) - \delta, g(s_0)]$ by $(*)$.

Let K be the arc in S^1 with endpoints $p(g(s_0) - \delta)$ and $p(g(t_0) + \delta)$ and which contains in its interior the points $p(g(s_0))$ and $p(g(t_0))$. It follows from condition $(*)$ that $[a, b]$ and $[c, d]$ are components of $p^{-1}f^{-1}(K)$. Notice that $f \circ p([a, b]) = p \circ g([a, b])$ is the arc in K with endpoints $p(g(t_0))$ and $p(g(t_0) + \delta)$. Similarly, $f \circ p([c, d])$ is the arc in K with endpoints $p(g(s_0))$ and $p(g(s_0) - \delta)$. Hence $f \circ p([a, b]) \not\subset f \circ p([c, d])$ and $f \circ p([c, d]) \not\subset f \circ p([a, b])$, which contradicts the semi-confluence of f. Hence in Case 1 g is monotone.

Case 2 can be handled in an analogous way by replacing $p(s_0)$ by $p(1) = p(0)$

$(c) \Rightarrow (a)$. It is clear that if f: $S^1 \to S^1$ is a wrapping function and the degree of f is n, then for each arc C in S^1, $f^{-1}(C)$ consists of exactly n components and each of these components is mapped onto C.

A mapping f: $X \to Y$ of a compact Hausdorff space X onto a Hausdorff space Y is said to be *quasi-interior* **[91]** provided that for each $y \in Y$ and each neighbourhood U of a component C of $f^{-1}(y)$, we have $y \in \operatorname{Int} f(U)$. It is known (see **[52]**) that the confluent mappings onto locally connected continua are exactly the quasi-interior mappings. It is also known (see **[91]**) that the quasi-interior mappings are exactly the compositions of monotone mappings followed by light open mappings.

COROLLARY 3.7. *A mapping f: $S^1 \to S^1$ of S^1 onto S^1 is semi-confluent* *if and only if f is quasi-interior. Moreover, if f is semi-confluent and light, then f is open.*

Grace and Vought proved in [**18**] that semi-confluent images of tree-like metric continua are hereditarily unicoherent. The following theorem extends their result.

THEOREM 3.8. ([**27**, 4.1]). *Let X be a continuum with $H^1(X; \mathbf{Z}) = 0$, and let $f: X \to Y$ be a semi-confluent mapping of X onto a Hausdorff space Y. Then Y is unicoherent.*

The following result is a partial extension of Theorem 3.1 and Lelek's theorem [**44**] to semi-confluent mappings.

THEOREM 3.9. ([**27**, 4.2]). *Let $g: X \to Y$ be a semiconfluent mapping of a compact Hausdorff space X onto a hereditarily unicoherent Hausdorff space Y, and let $f: Y \to G$ be a mapping of Y onto a graph G such that $f \circ g \sim 1$. Then $f \sim 1$. Hence, if X is contractible with respect to G, then Y is also contractible with respect to G. Furthermore, $g^*: H^1(Y; \mathbf{Z}) \to H^1(X; \mathbf{Z})$ is a monomorphism.*

By using a proof identical with the one of Theorem 3.2, Theorem 3.9 can be generalized to the following theorem.

THEOREM 3.10. *Let $g: X \to Y$ be a semi-confluent mapping of a compact Hausdorff space X onto a hereditarily unicoherent Hausdorff space Y, and let $f: Y \to M$ be a mapping of Y into a one-dimensional connected* ANR *M such that $g \circ f \sim 1$. Then $f \sim 1$ Hence, if X is contractible with respect to M, then Y is also contractible with respect to M.*

QUESTION 1. Can the hypothesis in Theorem 3.10 that Y is hereditarily unicoherent be dropped?

4. **AH-essential mappings.** In this section by a a space we mean a topological space. Let I^n be the n-dimensional cube. Then by S^{n-1} we denote the boundary of $I^n (n \geqq 1)$. Let R^n denote the Euclidean n-space.

Let $f, g: X \to Y$ be mappings from a space X to a space Y and let $A \subset Y$. We say f is *homotopic to g relative to A* provided there exists a homotpoy $H: X \times [0, 1] \to Y$ such that $H(x, 0) = g(x)$ and $H(x, 1) = f(x)$ for each $x \in X$ and $H(x, t) = f(x)$ for each $x \in f^{-1}(A)$ and each $t \in [0, 1]$. Let M be a connected manifold with (possibly empty) boundary $\partial(M)$. A mapping $f: X \to M$ of a topological space X onto M is said to be *essential in the sense of Aleksandrov-Hopf*, written *AH-essential*, provided that if $g: X \to M$ is a mapping which is homotopic to f relative to $\partial(M)$, then $g(X) = M$. Otherwise, f is said to be *AH-inessential.* In the above definition we follow Krasinkiewicz [**34**]. If $M = S^n$, then $f: X \to S^n$ is AH-inessential if and only if f is homotopic to a constant mapping. If $M = I^n$, then $f: X \to I^n$

is AH-inessential if and only if there is a mapping $g\colon X \to I^n$ such that $f(x) = g(x)$ for each $x \in f^{-1}(S^{n-1})$) and such that $g(X) \neq I^n$. AH-essential mappings into I^n were first defined and used by P.S. Aleksandrov in 1932 (see **[1]** and **[2**, page 180]), when he characterized the dimension of subsets of Euclidean spaces in terms of AH-essential mappings. This characterization was extended in 1956 (see **[81]**) by Smirnov to the class of Tychonov spaces. This characterization in its most general form, was given by Morita in **[67**, page 43]. In order to state Morita's theorem we need the following definition.

DEFINITION. Let X be a space. A covering $\mathscr{U}$ is called a *normal covering* if there is a sequence of open coverings $\mathscr{U}_1, \mathscr{U}_2, \ldots$ such that $\mathscr{U}_1$-star refines $\mathscr{U}$ and $\mathscr{U}_{i+1}$-star refines $\mathscr{U}_i$ for $i = 1, 2, \ldots$.

In **[67]** Morita defines the covering dimension of X, denoted by dim X, to be the least integer n such that every finite normal open covering of X admits a finite normal open covering of order $\leqq n + 1$ as its refinement. In case X is a normal space, dim X as defined above coincides with the covering dimension in the usual sense.

THEOREM 4.1. (Morita **[67]**. *A space X has* dim $\geqq n$ *if and only if there exists an AH-essential mapping of X onto I^n.*

It is an easy exercise to prove that the identity mapping on I^n is *AH*-essential ($n \geqq 1$). The following example shows that there exist mappings of n-dimensional spaces onto I^n which are not *AH*-essential.

EXAMPLE 4.2. Let $X = \{(\rho, \theta) | 1/2 \leqq \rho \leqq 1, 0 \leqq \theta \leqq 2\pi\}$, where (ρ, θ) denotes a point of the plane in polar coordinates and let $Y = \{(\rho, \theta) | 0 \leqq \rho \leqq 1, 0 \leqq \theta \leqq 2\pi\}$. Notice that Y is homeomorphic to I^2, and define a mapping $f\colon X \to Y$ as follows. For each θ, $f(1/2, \theta) = (0, 0)$, $f(1, \theta) = (1, \theta)$ and f maps the convex arc $\{(\rho, \theta) | 1/2 \leqq \rho \leqq 1\}$ linearly onto the convex arc $\{(\rho, \theta) | 0 \leqq \rho \leqq 1\}$. Then $f | f^{-1}(S^1)$ is the identity, and as such it is essential, but f is not *AH*-essential. To see this simply define a mapping $g\colon X \to Y$ by putting $g(\rho, \theta) = (1, \theta)$ for each θ. Then $g | f^{-1}(S^1) = f | f^{-1}(S^1)$, but $g(X) \neq Y$.

In **[63]** Mazurkiewicz used AH-essential mappings in order to show that each compact metric space of dimension n with $2 \leqq n < \infty$ contains an indecomposable continuum. He proved this by showing that every AH-essential mapping of finite-dimensional compact metric space onto I^n is weakly confluent. In **[27]** Mazurkiewicz's result was generalized in the setting of mappings of compact Hausdorff spaces onto I^2. In **[17]** Feuerbacher proved the following theorem for the case $M = S^1$.

THEOREM 4.3. *If $f\colon X \to M$ is an AH-essential mapping of a compact Hausdorff space X onto a connected manifold M, then f is weakly confluent.*

Before we give the proof of Theorem 4.3, we need some auxiliary results.

PROPOSITION 4.4. *Let X be a normal space, let Y be an ANR, $f: X \to Y$ a mapping of X into Y, and A a G_δ-subset of X such that $f|A \sim 1$. Then there exists an open subset U of X containing A and such that $f|U \sim 1$.*

PROOF. Let $g: A \to Y$ be a constant mapping of A into Y and let $g(x) = b$ for each $x \in A$. Assume that $f|A \sim g$. Since $f|A$ admits an extension f over X, by the homotopy extension theorem [**68**, Theorem 7], there exists an extension $g_1: X \to Y$ of g such that $f \sim g_1$. Since $g_1|A = g$ is a constant mapping and since Y is an ANR, there exists a small neighbourhood U of A in X such that $g_1(U)$ is contractible. Thus, we have that $g_1|U \sim 1$, and hence, $f|U \sim g_1|U \sim 1$.

Proposition 4.4 generalizes [**27**, Proposition 2.1] as well as a result of Eilenberg in [**14**, p. 65].

The proof of Theorem 4.3 is based on that of Mazurkiewics [**63**]. Some of the proofs in the next series of lemmas are very similar to those given in [**27**, §3].

PROPOSITION 4.5. *Given a compact Hausdorff space X and a mapping $f: X \to Y$ of X into a connected ANR Y, in order for f to be essential it is necessary and sufficient that there exists a component X' of X such that $f|X'$ is essential.*

The proof of Proposition 4.5 is identical with the proof of [**13**, Lemma, p. 164] and as such it is omitted.

PROPOSITION 4.6. *If $f: X \to Y$ is a mapping of a compact Hausdorff space X into a connected ANR Y such that f* non ~ 1, *then X contains a subcontinuum C such that $f|C$* irr non ~ 1.

PROOF. Let H be the collection of all subcontinua K of X for which $f|K$ non ~ 1. By Proposition 4.5, $H \neq \emptyset$. Consider that H is partially ordered by inclusion, and let H' be a chain in H. We claim that $\bigcap H'$ is an element of H. Suppose, on the contrary, that $\bigcap H' \notin H$. Then $f|\bigcap H' \sim 1$, and, by Proposition 4.4, there exists an open subset U of X containing $\bigcap H'$ and such that $f|U \sim 1$. Let $K \in H'$ such that $K \subset U$. Then we have that $f|K \sim 1$. This contradiction proves the claim. By the Zorn-Kuratowski Lemma, there exists a minimal element C of H. Then we have that $f|C$ irr non ~ 1.

LEMMA 4.7. *If $f: X \to I^n$ is an AH-essential mapping of a normal space X onto I^n, then $f_1 = f|f^{-1}(S^{n-1})$ is essential.*

PROOF. Suppose, on the contrary, that f_1 is homotopic to a constant mapping g. Since g admits an extension g_1 over X such that g_1 is constant, by [**68**, Theorem 7], f_1 admits an extension f_2 from X into S^{n-1} which is homotopic to g_1. Then f_2 coincides with f on $f^{-1}(S^{n-1})$, but $f_2(X) \subset S^{n-1} \neq I^n$. This contradicts the hypothesis that f is AH-essential and the lemma is proved.

LEMMA 4.8. *If $f: X \to I^n$ is an AH-essential mapping of a compact Hausdorff space X onto I^n, then $A_1 = f^{-1}(S^{n-1})$ has a component K such that $f|K: K \to S^{n-1}$ is an essential mapping.*

PROOF. By Lemma 4.7, $f|A_1$ is essential. The lemma follows by Proposition 4.5.

LEMMA 4.9. *If M is a connected n-manifold, $f: X \to M$ is an AH-essential mapping of a topological space X onto M, $J \subset M$ is an $(n - 1)$-sphere in M which is the boundary of an open n-cell H with $H \subset M\backslash\partial(M)$, then $f|f^{-1}[\mathrm{Cl}(H)]$ is an AH-essential mapping of $f^{-1}[\mathrm{Cl}(H)]$ onto $\mathrm{Cl}(H) = H \cup J$.*

PROOF. Suppose, on the contrary, that $f|f^{-1}[\mathrm{Cl}(H)]$ is not an AH-essential mapping. Then there exists a mapping $g: f^{-1}[\mathrm{Cl}(H)] \to \mathrm{Cl}(H)$ such that $g|f^{-1}(J) = f|f^{-1}(J)$ and $g(f^{-1}[\mathrm{Cl}(H)]) \neq \mathrm{Cl}(H)$. Define a function $g_1: X \to M$ by setting

$$g_1(x) = \begin{cases} g(x), & \text{if } x \in f^{-1}[\mathrm{Cl}(H)] \\ f(x), & \text{otherwise.} \end{cases}$$

It is clear that g_1 is continuous since $g|f^{-1}(J) = f|f^{-1}(J)$. We also have that $g_1|f^{-1}(S^{n-1}) = f|f^{-1}(S^{n-1})$ and that $g_1(X) \neq M$, since $g(f^{-1}[\mathrm{Cl}(H)]) \neq \mathrm{Cl}(H)$. This contradicts the fact that f is AH-essential, and the lemma is proved.

LEMMA 4.10. *If M is a connected n-manifold, $f: X \to M$ is an AH-essential mapping of a compact Hausdorff space X onto M and J is a copy of S^{n-1} in M which is the boundary of an open n-cell in $M\backslash\partial(M)$, then there exists a continuum K in X such that $f(K) = J$.*

PROOF OF THEOREM 4.3. Let K be a subcontinuum of M. Then we claim that K is the limit of a sequence $\{B_i\}_{i=1}^{\infty}$ of copies of S^{n-1} such that each B_i bounds an open n-cell in $M\backslash\partial(M)$. To see this, let $\{A_i\}_{i=1}^{\infty}$ be a sequence of polygonal arcs in $M\backslash\partial(M)$ such that $K = \mathrm{Lim}_{i\to\infty} A_i$. For each i let $C_i \subset S(A_i, 1/i)\backslash\partial(M)$ be a polyhedral n-cell such that $A_i \subset C_i$. Let B_i be the boundary of C_i. Then B_i is a copy of S^{n-1} and $K = \lim_{j\to\infty} B_i$.

By Lemma 4.10, for each i there exists a subcontinuum K_i of X such that $f(K_i) = B_i$. Let $\{K_{i_\alpha}\}_{\alpha\in A}$ be a convergent subnet of $\{K_i\}_{i=1}^{\infty}$. Then, if $L = \lim_{\alpha\in A} K_{i_\alpha}$ (see [**41**, pages 45 and 139]), we have L is a continuum and $f(L) = \lim_{\alpha\in A} f(K_{i_\alpha}) = \lim_{\alpha\in A} B_{i_\alpha} = K$.

In Lemma 4.7, it is proved that if $f: X \to I^n$ is an AH-essential mapping of a normal space X onto I^n, then $f|f^{-1}(S^{n-1})$ is essential. Example 4.2 shows that the converse of Lemma 4.7 is not true in general, although we shall prove in Theorem 4.11 that it is true if X is contractible with respect to S^{n-1}.

THEOREM 4.11. *Let X be a space which is contractible with respect to S^{n-1}, and let $f: X \to I^n$ be a mapping of X onto I^n such that $f|f^{-1}(S^{n-1})$ is essential. Then f is AH-essential.*

PROOF. Suppose, on the contrary, that f is not AH-essential. Then there exists a mapping $g: X \to I^n$ such that $g|f^{-1}(S^{n-1}) = f|f^{-1}(S^{n-1})$ and $g(X) \neq I^n$. Let $r: g(X) \to S^{n-1}$ be a retraction of $g(X)$ onto S^{n-1}. Then $r(g(a)) = f(a)$ for each $a \in f^{-1}(S^{n-1})$. Consider the mapping $r \circ g: X \to S^{n-1}$. Since X is contractible with respect to S^{n-1}, there exists a homotopy $F: X \times I \to S^{n-1}$ such that $F(x, 0) = r(g(x))$ and $F(x, 1) = b$ for some $b \in S^{n-1}$ and for each $x \in X$. Then $G = F|f^{-1}(S^{n-1}) \times I$ is a homotopy from $f^{-1}(S^{n-1}) \times I$ to S^{n-1} such that $G(a, 0) = r(g(a)) = f(a)$ and $G(a, 1) = b$ for each $a \in f^{-1}(S^{n-1})$· This shows that $f|f^{-1}(S^{n-1})$ is homotopic to a constant. This contradiction proves the theorem.

In Theorem 5.4, we prove that a mapping $f: X \to I^n$ is AH-essential if and only if it is universal in the sense of Holsztyński [**30**]. In view of this result, Theorem 4.12 together with Lemma 4.7 generalize Proposition 10 in [**30**]. One might hope to prove Theorem 4.3 for the mappings that are homotopically essential (i.e., are not homotopic to a constant mapping). There exist homotopically essential mappings of the circle S^1 onto the torus $S^1 \times S^1$ and onto the figure eight (i.e., one point union of two copies of S^1). Since the weakly confluent image of S^1 is atriodic, by 6.4. there exists no weakly confluent mapping of S^1 onto either $S^1 \times S^1$ or onto the figure eight.

The problem of mappings which preserve the dimension of spaces has been of continuing interest since Peano constructed in 1890 a mapping of I onto I^2. There is a very large literature on this problem but this is far from the scope of this paper. It is appropriate though to mention some results in this area. Anderson claimed in [**3**] and Wilson proved in [**93**] that there exist monotone and open mappings from the Menger universal curve (a one dimensional continuum) onto any Peano continuum such that the preimages of points are homeomorphic to the Menger universal curve. For a discussion on the matter and for a rather complete bibliography see [**92**] and [**93**].

It is worth noticing that the Menger universal curve is not acyclic. The following result is due to H. Cook (see [**53**] and [**27**]) and proves that even weaker than monotone and open mappings preserve one-dimensionality provided the domain is an acylcic continuum.

THEOREM 4.12. (Cook). *Let f: $X \to Y$ be a pseudo-confluent mapping of an acyclic one-dimensional continuum X onto a continuum Y. Then* dim $Y \leqq 1$.

R.L. Moore [**66**] and J.H. Roberts and N.E. Steenrod [**77**] proved that monotone images of 2-manifolds have dimension at most two. By using a recent result of Krasinkiewicz [**37**] one can prove the following more general result.

THEOREM 4.13. *If f: $M \to Y$ is a pseudo-confluent mapping of a continuum M, which is embeddable in a 2-manifold, onto a Hausdorff space Y, then* dim $Y \leqq 2$.

PROOF. Suppose, on the contrary, that dim $Y \geqq 3$. By Theorems 4.1 and 4.3, there exists a weakly confluent mapping $g : Y \to I^3$ of Y onto I^3. Then $g \circ f$: $M \to I^3$ is a pseudo-confluent mapping of M onto I^3 (see [**53**, 1.5]). Let D be a dyadic solenoid in I^3. Since D is irreducible, there exists a continuum K in M such that $f(K) = D$. By [**37**, 7.3], K is a movable continuum, and by [**37**, Theorem 6.2] D is movable, which is a contradiction (see for example [**35**, page 241]).

One might expect that such a theorem would be true in higher dimensions also, but this is not the case. Walsh [**86**] has proved (among other results) that every compact connected 3-manifold admits a monotone, open mapping onto every compact metric absolute retract.

5. **AH-essential mappings and the fixed-point property.** Let X and Y be topological spaces and let $\mathscr{U}$ be an open cover of X. A mapping f: $X \to Y$ is said to be *$\mathscr{U}$-mapping* [**30**] provided that for each $y \in Y$ there exists some $U \in U$ such that $f^{-1}(y) \subset U$. We say that a space X has the *fixed-point property* provided that for each mapping f: $X \to X$ there exists a point $x \in X$ such that $f(x) = x$.

In [**54**] Lokuciewski proved a fixed-point theorem for compact metric spaces by using AH-essential mappings. We observe that Lokuciewski's proofs generalize to arbitrary Hausdorff spaces. We wish to thank Professor Nadler for bringing to our attention reference [**54**] and for indicating to us the connection between AH-essential mappings and fixed-point theory.

LEMMA 5.1. (Lokuciewski [**54**]). *Let f and g be two mappings of a topological space X into I^n for some $n \geqq 1$. If g is an AH-essential mapping, then there exists a point $x \in X$ such that $f(x) = g(x)$.*

THEOREM 5.2. (Lokuciewski [**54**]). *If X is a Hausdorff space such that for each open cover $\mathscr{U}$ there exists a positive integer n and an AH-essential $\mathscr{U}$-mapping of X onto I^n, then X has the fixed-point property.*

Theorem 5.2 generalized Brouwer's fixed-point theorem and has various applications to fixed-point theorems. In [**34**] Krasinkiewicz used Theorem 5.2 in order to give a simple proof of the following theorem. We remark that the proof given in [**34**, 4.1] works for the non-metric case as well.

THEOREM 5.3. (Krasinkiewicz [**34**]). *If X is either an arc-like or a circle-like continuum, the hyperspace $C(X)$ of subcontinua of X has the fixed-point property.*

The same theorem was proved independently by Rogers [**78**] and for arc-like continua by Segal [**80**].

A mapping $f\colon X \to Y$ is said to be *universal* provided that for each mapping $g\colon X \to Y$ there exists some $x \in X$ for which $f(x) = g(x)$ [**30**]. Holsztyński proved in [**30**] that if $f\colon X \to Y$ is a universal mapping, then Y has the fixed-point property, and that f is onto.

The following result shows that relationship between universal mappings and AH-essential mappings, and proves that universal mappings are generalizations of AH-essential mappings. Nadler has evidently noticed this result independently.

THEOREM 5.4. *Let X be a topological space and let n be a positive integer. Then the mapping $f\colon X \to I^n$ is universal if and only if f is* AH-*essential.*

PROOF. Let $f\colon X \to I^n$ be an AH-essential mapping. It follows from Lemma 5.1 that f is universal.

Conversely, let $f\colon X \to I^n$ be a universal mapping and suppose, on the contrary, that f is not AH-essential. Then there exists a mapping $g\colon X \to I^n$ such that $g|f^{-1}(S^{n-1}) = f|f^{-1}(S^{n-1})$ and $g(X) \neq I^n$. Hence, there exists a retraction $r\colon g(X) \to S^{n-1}$ of $g(X)$ onto S^{n-1}. Then we have that $r(g(a)) = f(a)$ for each $a \in f^{-1}(S^{n-1})$. Consider the mapping $r \circ g\colon X \to S^{n-1}$ and define a mapping $h\colon X \to I^n$ by $h(x) = -r(g(x))$ where $-r(g(x))$ denotes the antipodal point of $r(g(x))$ on S^{n-1} for each $x \in X$. Then it is clear that for each $x \in X$ we have $f(x) \neq h(x)$, which contradicts the fact that f is universal.

The following result is obtained by Holsztyński in [**30**, Corollary 1].

THEOREM 5.5. (Holsztyński [**30**]). *Let $\{X_\alpha, f^\beta_\alpha, \Lambda\}$ be an inverse system of compact ANR's, where all the bonding mappings $f^\beta_\alpha\colon X_\beta \to X_\alpha$ $(\alpha \leqq \beta)$ are universal. Then the inverse limit of this inverse system has the fixed-point property.*

Using this result Holsztyński proved that the cone over a solenoid as well as the m-fold suspension of the cone over a solenoid have the fixed-point property. Holsztyński's result can be extended to S^n-like continua X with $H^n(X, G) \neq 0$ by using the next theorem.

Let X be a topological space and let I denote the unit interval [0, 1].

Then the *m-fold suspension of* X denoted by $S_m(X)$ is defined for each $m \geqq 0$ inductively as follows:

$$S_m(X) = \begin{cases} X & \text{, if } m = 0 \\ S_{m-1}(X) \times I/\{S_{m-1}(X) \times \{0\}\} \cup \{S_{m-1}(X) \times \{1\}\}, & \text{if } m > 0. \end{cases}$$

Let $f: X \to Y$ be a mapping of X onto Y. Define the mapping induced by the m-fold suspension.

$$S_m(f): S_m(X) \to S_m(Y)$$

by setting $S_0(f) = f$ and for $m > 0$

$$S_m(f)(x, t) = (S_{m-1}(f)(x), t)$$

for each $(x, t) \in S_{m-1}(X) \times]0, 1[$,

$$S_m(f)(\{S_{m-1}(X) \times \{1\}\}) = \{S_{m-1}(Y) \times \{1\}\},$$

and

$$S_m(f)(\{S_{m-1}(X) \times \{0\}\}) = \{S_{m-1}(Y) \times \{0\}\}.$$

THEOREM 5.6. *Let* $\chi = \varprojlim\{\chi_\alpha, f_\alpha^\beta, \Lambda\}$ *where each* χ_α *is an n-cell* $(n \geqq 1)$ *and the* f_α^β *is an AH-essential mapping. Then X, the cone over* χ, *and the m-fold suspension of X have the fixed-point property.*

PROOF. By Theorem 5.4, each f_α^β is universal, and hence by Theorem 5.5, χ has the fixed-point property.

We prove first that if $f: I^n \to I^n$ is a simplicial AH-essential, then $S_1(f)$: $S_1(I^n) = I^{n+1} \to S_1(I^n)$ is AH-essential. If $n > 1$, then by 4.7, $f|S^{n-1}$ is AH-essential. By [**82**, 8.5.11] $S_1(f|S^{n-1})$ is AH-essential. By 4.11, $S(f)$ is essential. If $n = 1$, then there is a component K of $S_1(f^{-1}\{0, 1\})$ such that K is the union of continua A and B such that $A \cap B = P \cup Q$ where P and Q are non-empty separated sets, $P \subset f^{-1}(0)$ and $Q \subset f^{-1}(1)$. Furthermore, $S_1(f)(B)$ is the bottom semicircle of S^1 and $S_1(f)(A)$ is the top semicircle of S^1. It is now easy to see that $S_1(f)|K$ is AH-essential. By 4.12, $S_1(f)$ is AH-essential.

By 6.9 of [**38**]. we may assume that each f_β^α is simplicial. By the above $S_m(f_\alpha^\beta)$ is AH-essential for each $\alpha \leqq \beta$. Notice that

$$S_m(X) = \varprojlim\{S_m(X_\alpha), S_m(f_\alpha^\beta), \Lambda\}.$$

Hence $S_m(X)$ has the fixed-point property by 5.4 and 5.5.

To show that the cone over X has the fixed-point property, Let $\bar{X} = X \times I/X \times \{1\}$ and for each α let $\bar{X}_\alpha = X_\alpha \times I/X_\alpha \times \{1\}$. For each α and β in Λ with $\alpha \leqq \beta$ define $\bar{f}_\alpha^\beta$: $\bar{X}_\beta \to \bar{X}_\alpha$ by setting $\bar{f}_\alpha^\beta(z, t) = (f_\alpha^\beta(z), t)$ for each $(z, t) \in X_\beta \times [0, 1]$ and setting $\bar{f}_\alpha^\beta(\{X_\beta \times \{1\}\}) = \{X_\alpha \times \{1\}\}$.

Notice that $\bar{X}_\alpha$ is homeomorphic to $S_1(X_\alpha)$ and f_α^β is homotopic to $S_1(f_\alpha^\beta)$ relative to $S^n = \partial(\bar{X}_\alpha)$. As above $\bar{X}$ has the fixed-point property.

In case $n = 1$ Theorem 5.6 may be restated as follows.

COROLLARY 5.7. *Let X be an arc-like continuum. Then X, the cone over X, and the m-fold suspension of X have the fixed-point property.*

That arc-like continua have the fixed-point property was first proved by Hamilton in [**29**]. We also state the following question which was asked by Knill in [**33**, page 36].

QUESTION 2. Does the cone over every tree-like continuum have the fixed-point property?

It has recently been proved by Bellamy [**4**] that there exists a tree-like continuum without the fixed point property.

In Theorem 5.4 we proved that universal mappings onto n-cells ($n \geqq 1$) are AH-essential, and in Theorem 4.3 we saw that AH-essential mappings are weakly confluent. The following question can be posed.

QUESTION 3. Let $f: X \to Y$ be a universal mapping, where X is a compact Hausdorff space. Is f weakly confluent?

The converse to Question 3 does not have to be true even if Y is a 2-cell and f is a monotone map (see Example 4.2).

6. **Classification of continua.** We start our discussion in this chapter with a very useful characterization of tree-like continua given in [**8**, p. 74–75]. As was remarked in [**27**], this result is also valid for non-metric tree-like continua. With the exception of 6.1 and 6.2 all the results in this section are for metric spaces.

THEOREM 6.1. (Case-Chamberlin [**8**]). *A continuum X is tree-like if and only if X is one-dimensional and each mapping of X into a graph is homotopic to a constant mapping.*

It follows from [**12**, Theorem 8.1] that a continuum is one-dimensional and acyclic if and only if it is contractible with respect to S^1. Thus, each tree-like continuum is one-dimensional and acyclic. A. Lelek proved in [**49**, 2.7] that confluent mappings preserve one-dimensional acyclic metric continua, and McLean proved in [**65**] that confluent mappings preserve tree-like metric continua. The following result generalizes the above mentioned results and is a consequence of Theorem 3.10 and Theorem 4.13.

THEOREM 6.2. ([**27**]). *Semi-confluent mappings preserve one-dimensional acyclic continua, tree-like continua, and continua which are contractible with respect to a particular one-dimensional connected ANR.*

Rosenholtz has proved in [**79**] that open mappings preserve arc-like continua, and Bing has proved (see [5, page 654]) that monotone mappings preserve arc-like continua.

QUESTION 4. (Lelek [**46**, p. 94]). Is the confluent image of an arc-like continuum always arc-like?

The same question was asked later by Maćkowiak but for semi-confluent mappings [**55**, 5.8]. A partial answer to Maćkowiak's question has been given in [**18**] where it is proved that semi-confluent mappings preserve hereditarily decomposable arc-like metric continua. In what follows all continua will be considered to be metric.

A continuum X is said to be a *triod* provided that X is the union of three proper subcontinua X_1, X_2 and X_3 of X such that $X_1 \cap X_2 = X_1 \cap X_3 = X_2 \cap X_3$ is connected and $X_i \not\subset X_j \cup X_k$ for each $i, j, k \in \{1, 2, 3\}$; $j \neq i \neq k$. The triod $X = X_1 \cup X_2 \cup X_3$ is said to be a *simple triod* provided that X_1, X_2 and X_3 are arcs which intersect only in a common end-point. A continuum is said to be *atriodic* provided it does not contain any triod. A continuum is said to be *rational* provided it admits a basis of open subsets whose boundaries are countable sets. A continuum is said to be *Suslinian* [**45**] provided it does not contain an uncountable collection of mutualy disjoint, non-degenerate subcontinua. It is easy to see that confluent mappings preserve atriodic continua (see [**61**, 5.19]). In [**11**] the following result was given.

THEOREM 6.3. (Cook and Lelek [**11**, 2.4]). *For a continuum X the following conditions are equivalent*:

(a) *X is atriodic and Suslinian*;

(b) *each weakly confluent image of X is atriodic and Suslinian*;

(c) *each weakly confluent image of X is atriodic*; *and*

(d) *no mapping of X onto a simple triod is weakly confluent.*

QUESTION 5 (Maćkowiak [**61**, 8.5]). Is the semi-confluent image of an atriodic continuum always atriodic?

The following result is a partial answer to this question.

THEOREM 6.4. (Grace and Vought [**18**]). *The semi-confluent image of an arc-like continuum is atriodic.*

It is known that weakly confluent mappings do not preserve either arc-like continua or tree-like continua, and pseudo-confluent mappings do not preserve atriodic continua. In fact, there exists a weakly confluent mapping of the unit interval onto the circle [**47**, p. 99], and a pseudo-confluent mapping of the unit interval onto a triod [**53**, p. 1343].

There exist several other classes of continua which are preserved by pseudo-confluent mappings. For example, continua in Class A, regular continua, finitely Suslinian continua, hereditarily locally connected continua and Suslinian continua are preserved by pseudo-confluent mappings (for the definitions and the results see [**53**]). The only singular class of continua in this classification [**49**, p. 55] is the class of rational continua. It is known that monotone mappings [**88**, p. 138] as well as open mappings [49, p. 57] preserve rational continua. Recently, both these results were generalized in [**19**, p. 127]. Finally, in [26] an example was given of a strongly confluent (and hence, confluent) mapping from a rational arc-like continuum onto a non-rational arc-like continuum, thus, proving that confluent mappings do not preserve rational continua. The first example though of a confluent mapping from a non-acyclic rational continuum onto a non-rational continuum was given in [**83**].

Let $T = T_1 \cup T_2 \cup T_3$ be a triod, where T_1, T_2 and T_3 are subcontinua of T whose intersection is connected and such that no one is contained in the union of the other two. Then the continuum $T_1 \cap T_2 \cap T_3$ is called the *branch-continuum* of T.

THEOREM 6.5. (Cook and Lelek [**11**, 3.2]). *Let f: $X \to Y$ be a weakly confluent mapping of a Suslinian continuum onto a Hausdorff space Y. If Y_0 is a branch-continuum in Y and $U \subset Y$ is an open set containing Y_0, then there exists a triod $T \subset X$ such that $f(T) \subset U$*

Consequently, if y_0 is a branch-point of a simple triod in Y, then there exists a sequence $T_1, T_2, \ldots$ of triods in X such that

$$\operatorname*{Lim}_{n\to\infty} f(T_n) = \{y_0\}.$$

In [**20**] the following characterization of confluent mappings was given, thus answering a question of Lelek and generalizing [**53**, 4.1].

THEOREM 6.6. ([**20**]). *Let f: $X \to Y$ be a mapping of a compact metric space X onto a hereditarily locally connected Hausdorff space Y. Then the following are equivalent*:

(a) *f is confluent*; *and*

(b) *f is strongly confluent, i.e., if K is a connected subset of Y, then every component of $f^{-1}(K)$ is mapped by f onto K.*

An example was also given in [**20**] to show that the assumption that Y is hereditarily locally connected cannot be dropped.

Epps proved in his thesis [**16**] the following result.

THEOREM 6.7. (Epps [**16**, Theorem 3]). *If G is a graph, then there is a finite tree T and a finite-to-one weakly confluent mapping from T onto G.*

Since finite trees contain at most finitely many distinct maximal arcs, the following result is an easy exercise.

PROPOSITION 6.8. *If T is a finite tree, then there exists an arc I and a pseudo-confluent mapping of I onto T.*

By combining 6.5 and [**16**, 4.8] we obtain that every graph is the pseudo-confluent image of an arc.

For some other results in this direction see also [**28**].

THEOREM 6.9. *Pseudo-confluent mappings preserve graphs.*

PROOF. Let f: $[0, 1] \to Y$ be a pseudo-confluent mapping. We prove first that each arc in Y has at most finitely many ramification points. Let A be an arc in Y with endpoints a and b. Suppose $\{a_1, a_2, \ldots\}$ is an infinite set of ramification points in A. Let A have its natural order with initial point a. We may suppose without loss of generality that $a_1 < a_2 < \ldots$. Let A_i denote the arc in A with endpoints a and a_i. For each i let B_i be an arc in Y such that $B_i \cap A = \{a_i\}$, a_i and b_i are the endpoints of B_i and $B_i \cap B_j = \emptyset$ for $i \neq j$.

Let $[c_1, d_1]$ be an irreducible arc in $[0, 1]$ such that $f([c_1, d_1]) = A_1 \cup B_1$. Without loss of generality $f(c_1) = a$ and $f(d_1) = b_1$. Let $[c_2, d_2]$ be an irreducible arc in $[0, 1]$ such that $f([c_2, d_2]) = A_2 \cup B_2$. Then either $[c_2, d_2] \subset [0, c_1]$ or $[c_2, d_2] \subset [d_1, 1]$. Similarly, we can choose arcs $[c_i, d_i]$ in $[0, 1]$ such that $f([c_i, d_i]) = A_i \cup B_i$ and such that $[c_i, d_i] \cap [c_j, d_j]$ contains at most one point for each $i \neq j$ Since the arcs $A_i \cup B_i$ do not form a null sequence, it follows that the arcs $[c_i, d_i]$ do not form a null sequence. This is a contradiction since $[0, 1]$ does not contain an infinite collection of large pairwise disjoint arcs. This completes the proof of the fact that each arc in Y has at most finitely many ramification points. A similar argument can be used to show that there do not exist infinitely many arcs C_i in Y and a point $y \in Y$ such that $C_i \cap C_j = \{y\}$ for each $i \neq j$.

To prove that Y is a graph is suffices to prove that Y is locally a graph. Let $y \in Y$. Let $\{C_1, \ldots, C_n\}$ be a maximal family of arcs such that $C_i \cap C_j = \{y\}$ for each $i \neq j$. Let $\{a_1, \ldots, a_n\}$ be the ramification points of Y different from y which are contained in $C_1 \cup \ldots \cup C_n$. Since Y is locally connected it follows that the component of $(C_1 \cup \ldots \cup C_n) \backslash \{a_1, \ldots, a_n\}$ which contains y is a neighbourhood of y which is also a finite graph.

Since every graph is the pseudo-confluent image of $[0, 1]$ and since the composition of pseudo-confluent mappings is pseudo-confluent the theorem is proved.

REMARK 6.10. Theorem 6.9 resolves in the affirmative Problem 9.30 in [**61**], and was obtained by *A*. Lelek and the second author and independently by the first author in the spring of 1976. Theorem 6.9, also generalizes a previous result [**56**, (4.3)].

In [**84**] the second author characterized the weakly confluent images of dendrites. We would like to mention that the same proof with only the obvious modifications works to give the following results.

PROPOSITION 6.11. *Let X be a continuum. Let x_1 and x_2 be points of X and let ε be a positive number. If X is the pseudo-confluent image of a dendrite, then X does not contain a sequence $A_1, A_2, \ldots$ of distinct arcs with endpoints x_1 and x_2 such that the arcs A_i agree on the ε-neighbourhoods of both x_1 and x_2.*

THEOREM 6.12. *A locally connected continuum X is a pseudo-confluent image of some dendrite if and only if X satisfies the following two conditions:*
(i) *each true cyclic element of X is a finite graph; and*
(ii) *if $E_1, E_2, \ldots$ is a sequence of distinct true cyclic elements of X all of which lie in a cyclic chain C, then the sequence $E_1, E_2, \ldots$ has at most two cluster points. Each cluster point is an endpoint of the cyclic chain C. No cluster point of $E_1, E_2, \ldots$ lies in a true cyclic element of X.*

COROLLARY 6.13. *If X is a continuum the following are equivalent:*
(i) *X is the weakly confluent image of a dendrite; and*
(ii) *X is the pseudo-confluent image of a dendrite.*

7. Another classification of continua. In this chapter all spaces are metric. In 1967 H. Cook **[10]** proved that if X is a hereditarily indecomposable continuum, then every mapping of any continuum onto X is confluent. Later Lelek and Read **[52]** proved that if X is a continuum such that every mapping of any continuum onto X is confluent, then X is hereditarily indecomposable. This gave rise to an attempt at classifying continua in terms of the types of onto mappings that they admit.

A continuum X is said to be in Class (C) (resp. Class (S), Class (W), Class (P)) provided that every mapping of any continuum onto X is confluent (resp. semi-confluent, weakly confluent, pseudo-confluent).

THEOREM 7.1. (Cook **[10]** and Lelek and Read **[52]**). *The Class* (C) *is exactly the class of hereditarily indecomposable continua.*

The following theorem was proved by the authors in **[25]**.

THEOREM 7.2. ([**25**, Theorem 5.1]). *The Class* (S) *is exactly the class of hereditarily indecomposable continua.*

In **[75]** Read proved that arc-like continua are in Class (W) and in **[17]** Feuerbacher proved that non-planar circle-like continua are in Class (W). By Cook's result **[10]** hereditarily indecomposable continua are also in Class (W). In **[21]** Feuerbacher's result was generalized by proving that circle-like continua with no local separating subcontinua are in Class (W). Finally, in **[22]** the authors gave a general theorem which implies all the above mentioned results as well as the following theorem.

THEOREM 7.3. ([**22**, 5.6]). *Tree-like atriodic continua and the Case-Chamberlin continuum* [**8**] *are in Class* (W).

Finally in [**24**] and [**23**] the authors gave the following characterizations of Class (W) and Class (P), respectively.

THEOREM 7.4. ([**24**, 3.2]). *Let X be a continuum. Then the following are equivalent*:

(a) *X is in Class* (W);

(b) μ: $C(X) \to [0, 1]$ *is a Whitney map for $C(X)$, the hyperspace of subcontinua of X, and Λ is a subcontinuum of $\mu^{-1}(t)$ for some* $t \in [0, 1]$ *with* $\bigcup \Lambda = X$, *then* $\Lambda = \mu^{-1}(t)$; *and*

(c) *if Y is a continuum with $X \subset Y$ and $X_1, X_2, \ldots$ are subcontinua of Y with $X = \lim_{i\to\infty} X_i$, then for every subcontinuum A of X there exist continua $A_1, A_2, \ldots$ with $A_i \subset X_i$ for each i such that $A = \lim_{i\to\infty} A_i$.*

THEOREM 7.5. ([**23**, 5.2]). *Let X be a continuum. Then the following are equivalent*:

(a) *X is in Class* (P);

(b) *if* μ: $C(X) \to [0, 1]$ *is a Whitney map for $C(X)$ and Λ is a subcontinuum of $\mu^{-1}(t)$ for some* $t \in [0, 1]$ *with* $\bigcup \Lambda = X$, *then every irreducible subcontinuum A of X with $A \in \mu^{-1}(t)$ belongs to Λ; and*

(c) *if Y is continuum with $X \subset Y$ and $X_1, X_2, \ldots$ are subcontinua of Y with $X = \lim_{i\to\infty} X_i$, then for every irreducible subcontinuum A of X there exist continua $A_1, A_2, \ldots$ with $A_i \subset X_i$ for each i such that $A = \lim_{i\to\infty} A_i$.*

For an extensive discussion on the subject and for more results we refer the reader to [**25**].

Added in proof. 1) Question 1 has been answered in the affirmative (see [**85**]).

2) Nadler [**74**] has announced a negative solution to question 3.

REFERENCES

1. P. Alexandrov, *Dimensions theorie*, Math. Ann. **106** (1932), 161–238.

2. ———, *Combinatorial Topology*, I, Graylock Press, Baltimore, 1956.

3. R. D. Anderson, *A continuous curve admitting monotone open maps onto all locally connected metric continua*, Bull. Amer. Math. Soc. **62** (1956), 264–265.

4. D. P. Bellamy, *A tree-like continuum without the fixed-point property*, preprint.

5. R. H. Bing, *Snake-like continua*, Duke Math. J. **18** (1951), 653–654.

6. K. Borsuk, *Theory of Retracts*, Polish Scientific Publ., Warsaw, 1967.

7. ———, *Sur un continu acyclique qui se laisse transformer topologiquement en lui même sans points invariants*, Fund. Math. **24** (1935), 51–58.

8. J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pac. J. Math. **10** (1960), 73–84.

9. J. J. Charatonik, *Confluent mappings and unicoherence of continua*, Fund. Math.

56 (1964), 213–220.

10. H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. **60** (1967), 241–249.

11. ———, and A. Lelek, *Weakly confluent mappings and atriodic Suslinian curves*, Can. J. Math. **30** (1978), 32–44.

12. C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. **69** (1947), 200–242.

13. S. Eilenberg, *Sur les transformations d'espaces metriques en circonférence*, Fund. Math. **24** (1935), 160–176.

14. ———, *Transformation continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), 61–112.

15. ——— and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, Princeton, 1952.

16. B. B. Epps, *A classification of continua and weakly confluent mappings*, Colloq. Math. **36** (1976), 217–227.

17. G. A. Feuerbacher, *Weakly chainable circle-like continua*, Doctoral Dissertaion, Univ. of Houston, Houston, 1974.

18. E. E. Grace and E. J. Vought, *Semi-confluent and weakly confluent images of tree-like and atriodic continua*, Fund. Math., to appear.

19. J. Grispolakis, *Confluent and related mappings defined by means of quasi-components*, Can. J. Math. **30** (1978), 112–132.

20. ———, *On a problem of Lelek concerning open mappings*, Colloq. Math., to appear.

21. ——— and S. B. Nadler, Jr., and E. D. Tymchatyn, *Some properties of hyperspaces with applications to continua theory*, Can. J. Math., to appear.

22. J. Grispolakis and E. D. Tymchatyn, *Continua which are images of weakly confluent mappings only*, submitted to Houston J. Math.

23. ———, *Continua which are images of weakly confluent mappings only*, II, submitted to Houston J. Math.

24. ———, *Weakly confluent mappings and the covering property of hyperspaces*, to appear in Proc. A.M.S

25. ———, *Continua which admit only certain classes of onto mappings*, Topology Proc. **3** (1978), submitted.

26. ———, *Confluent images of rational continua*, submitted to Houston J. Math.

27. ———, *Semi-confluent mappings and acyclicity*, to appear in Houston J. Math.

28. ———, *Homotopy classes of mappings into graphs*, submitted to Pac. J. Math.

29. O. H. Hamilton, *A fixed point theorem for pseudo-arc and certain other metric continua*, Proc. Amer. Math. Soc. **2** (1951), 173–174.

30. W. Holsztyński, *Universal mappings and fixed-point theorems*, Bull. Acad. Polon. Sci. **15** (1967), 433–438.

31. W. T. Ingram, *Concerning atriodic tree-like continua*, Fund. Math. to appear.

32. ———, *Atriodic tree-like continua and the span of mappings*, Topology Proceedings **1** (1976), 329–333.

33. R. J. Knill, *Cones, products and fixed points*, Fund. Math. **60** (1967), 35–46.

34. J. Krasinkiewicz, *On the hyperspaces of snake-like and circle-like continua*, Fund. Math. **83** (1974), 155–164.

35. ———, *Curves which are continuous images of tree-like continua are movable*, Fund. Math. **89** (1975), 233–260.

36. ———, *On one-point union of two circles*, Houston J. Math. **2** (1976), 91–95.

37. ———, *Continuous images of continua and* 1-*movability*, Fund. Math. **98** (1978), 141–164.

38. ———, *On a method of constructing* ANR *sets*: *An application of inverse limits*, Fund. Math. **92** (1976), 95–112.

39. ——— and S. B. Nadler, Jr., *Whitney properties*, Fund. Math. **98** (1978), 165–180.

40. K. Kuratowski, *Topology* I, Academic Press, New York, 1966.

41. ———, *Topology* II, Academic Press, New York, 1968.

42. ———, *On the completeness of the space of monotone mappings and some related problems*, Bull. Acad. Polon. Sci. **16** (1968), 283–285.

43. ——— and R. C. Lacher, *A theorem on the space of monotone mappings*, Bull. Acad. Polon. Sci. **17** (1969), 797–800.

44. A. Lelek, *On confluent mappings*, Colloq. Math. **15** (1966), 223–233.

45. ———, *On the topology of curves* II, Fund. Math. **70** (1971), 131–138.

46. ———, *Some problems concerning curves*, Colloq. Math. **23** (1971), 93–98.

47. ———, *A classification of mappings pertinent to curve theory*, Proc. Univ. Oklahoma Topology Conference, 1972, 97–103.

48. ———, *Report on weakly confluent mappings*, Proc. Virginia Polyt. Inst. State Univ. Topology Conference, 1973, 168–170.

49. ———, *Properties of mappings and continua theory*, Rocky Mountain J. of Math. **6** (1976), 47–59.

50. ——— *The* OM-*mappings of continua*, Bull. Acad. Polon. Sci. **25** (1977), 781–785.

51. ———, *Some union theorems for confluent mappings*, Colloq. Math. **31** (1974), 57–65.

52. ——— and D. R. Read, *Compositions of confluent mappings and some other classes of functions*, Colloq. Math. **29** (1974), 101–112.

53. A. Lelek and E. D. Tymchatyn, *Pseudo-confluent mappings and a classification of continua*, Can. J. Math. **27** (1975), 1336–1348.

54. O. B. Lokuciewski, *A theorem on fixed points*, Usp. Mat. Navk **12** 3(75) (1957), 171–172 (Russian).

55. T. Maćkowiak, *Semi-confluent mappings and their invariants*, Fund. Math. **79** (1973), 251–264.

56. ———, *Locally weakly confluent mappings on hereditarily locally connected continua*, Fund. Math. **88** (1975), 59–74.

57. ———, *The hereditary classes of mappings*, Fund. Math. **97** (1977), 123–150.

58. ———, *On the subsets of confluent and related mappings of the space Y^X*, Colloq. Math. **36** (1976), 69–80.

59. ———, *The product of confluent and locally confluent mappings*, Bull. Acad. Polon. Sci. **24** (1976), 183–185.

60. ———, *Confluent mappings and smoothness of dendroids*, Bull. Acad. Polon. Sci. **21** (1973), 719–725.

61. ———, *Continuous mappings on continua*, Dissertationes, Math., to appear.

62. ———, *Some examples of irreducible confluent mappings*, to appear.

63. S. Mazurkiewicz, *Sur l'existence des continues indécomposables*, Fund. Math. **25** (1935), 327–328.

64. L. F. McAuley, *Some fundamental theorems and problems related to monotone mappings*, Proc. of Conference on Monotone and Open Mappings at Binghamton, New York, 1971, 1–36.

65. T. B. McLean, *Confluent images of tree-like curves are tree-like*, Duke Math. J. **39** (1972), 465–473.

66. R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. **27** (1925), 416–428.

67. K. Morita, *Čech cohomology and covering dimensions for topological spaces*, Fund. Math. **87** (1975), 31–52.

68. ———, *On generalizations of Borsuk's homotopy extension theorem*, Fund. Math.

88 (1975), 1–6.

69. S. B. Nadler, Jr., *Concerning completeness of the space of confluent mappings*, Houston J. Math. **2** (1976), 561–580.

70. ———, *Confluent images of the sinusoidal curve*, Houston J. Math. **3** (1977), 515–519.

71. ———, *The metric confluent images of half-lines and lines*, Fund. Math., to appear.

72. ———, *A fixed point theorem for hyperspace suspensions*, preprint.

73. ———, *Hyperspaces of Sets*, Marcel-Dekker, New York, 1978.

74. ———, *Universal mappings and weakly confluent mappings*, Notices Amer. Math. Soc. (5) **26** (1979), *A*-445.

75. D. R. Read, *Confluent and related mappings*, Colloq. Math. **29** (1974), 233–239.

76. ———, *Irreducibly confluent mappings*, Colloq. Math. **34** (1975), 49–55.

77. J. H. Roberts and N. E. Steenrod, *Monotone transformations of* 2*-dimensional manifolds*, Ann. Math. **39** (1939), 851–862.

78. J. T. Rogers, *Hyperspaces of arc-like and circle-like continua*, Proc. Virginia Polyt. Inst. State Univ. Topology Conference, 1972, 231–235.

79. I. Rosenholtz, *Open maps of chainable continua*, Proc. Amer. Math. Soc. **42** (1974), 258–264.

80. J. Segal, *A fixed point theorem for the hyperspaces of a snake-like continuum*, Fund. Math. **50** (1962), 237–248.

81. Yu. M. Smirnov, *On the dimension of proximity spaces*, Mat. Sbornik **38** (1956), 238–302 (Russian).

82. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

83. E. D. Tymchatyn, *Some rational curves*, submitted to Rocky Mountain J. Math.

84. ———, *Weakly confluent mappings and a classification of continua*, Colloq. Math. **36** (1976), 229–233.

85. ———, *On absolutely essential mappings*, to appear.

86. J. J. Walsh, *Monotone and open mappings onto* ANR*'s*, Proc. Amer. Math. Soc. **60** (1976), 286–288.

87. P. A. White, *Regular convergence*, Bull. Amer. Math. Soc. **60** (1954), 431–443.

88. G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc., Providence, 1942.

89. ———, *Homotopy reductions of mappings into the circle*, Duke J. Math. **11** (1944), 35–42.

90. ———, *Interior mappings into the circle*, Duke J. Math. **11** (1944), 431–434.

91. ———, *Open mappings on locally compact spaces*, Memoirs of Amer. Math. Soc. **1** (1950).

92. D. C. Wilson, *Monotone mappings of manifolds onto cells*, Proc. of Conference on Monotone and Open Mappings at Binghamton, SUNY at Binghamton, New York, 1971, 37–54.

93. ———, *Open mappings of the universal curve onto continuous curves*, Trans. Amer. Math. Soc. **168** (1972), 497–515.

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