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PSEUDOCOMPACTNESS VIA GRAPHS AND PROJECTIONS

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1. **Introduction.** In this article, we give characterizations of pseudocompact spaces which are not necessarily completely regular 1) in terms of graphs of functions into the space, and 2) in terms of projections; both of these — graphs and projections — are utilized in conjunction with the class $\mathscr{M}$ of zero-dimensional metric spaces to effect the characterizations. These characterizations parallel characterizations which have been found to be useful for compact spaces and characterizations which have been given recently for compactness generalizations **[9]**–**[11]**, **[13]**. Pseudocompact spaces which are not necessarily completely regular have been studied in **[17]**. The author has recently studied such spaces in **[14]**. Completely regular pseudocompact spaces have been extensively studied in **[1]**, **[3]**–**[7]**, **[12]**, **[16]** and more recently in **[8]** and in **[18]**.

2. **Preliminaries.** The closure of a subset K of a topological space will be denoted by cl(K). If $\psi, \lambda: X \rightarrow Y$ are functions, $\{x \in X: \psi(x) = \lambda(x)\}$ will be denoted by $\mathscr{E}(\psi, \lambda, X, Y)$. We will denote the class of continuous real-valued functions on X by $\mathscr{C}(X)$.

DEFINITION 2.1. A point x in a space X is in the *closure of a s ubset K of the space mod* $g \in \mathscr{C}(X)$ ($x \in \mathrm{cl}(K)$ (mod g)) if $K \cap g^{-1}(H) \neq \emptyset$ for each H open about $g(x)$. K is *closed mod* $g \in \mathscr{C}(X)$ (K is closed (mod g)) if cl(K) (mod g) $\subset K$; x is in the *adherence of a filterbase Ω on X mod* $g \in \mathscr{C}(X)$ ($x \in$ ad Ω mod g)) if $x \in \mathrm{cl}(F)$ (mod g) for each $F \in \Omega$.

DEFINITION 2.3. If X and Y are spaces, a point $(x, y) \in X \times Y$ *is in the closure of* $K \subset X \times Y$ *mod* $(f, g) \in \mathscr{C}(X) \times \mathscr{C}(Y)$ ($(x, y) \in \mathrm{cl}(K)$ (mod(f, g))) if $K \cap (f^{-1}(H) \times g^{-1}(M)) \neq \emptyset$ for all H, M open about $f(x)$ and $g(y)$, respectively. K is *closed mod* $(f, g) \in \mathscr{C}(X) \times \mathscr{C}(Y)$ if cl(K) (mod(f, g)) $\subset K$; we way that (x, y) is in the *first-coordinate-closure of K mod* $f \in \mathscr{C}(X)$ ($(x, y) \in (1)$-cl(K) (mod(f)) if $K \cap (f^{-1}(H) \times W) \neq \emptyset$ for all H open about $f(x)$ and W open in Y about y. *K is first-coordinate closed mod* $f \in \mathscr{C}(X)$ (K is (1)-*closed* (mod f)) if (1)-cl(K) (mod f) $\subset K$.

Our first theorem gives two equivalent statements of pseudocompactness. Equivalence (c) will be used frequently in the sequel.

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THEOREM 2.4. *The following statements are equivalent for a space X:*
(a) *X is pseudocompact;*
(b) *Each filterbase Ω on X has nonempty adherence* mod *g for each $g \in \mathscr{C}(X)$.*
(c) *Each countable filterbase Ω on X has nonempty adherence* mod *g for each $g \in \mathscr{C}(X)$.*

PROOF. It is obvious that (b) $\rightarrow$ (c). That (a) $\rightarrow$ (b) is seen from the fact that, for each $g \in \mathscr{C}(X)$, $g(X)$ is compact (Theorem 2.3 (vi) of **[17]**); so, for any filterbase Ω on X, $g(\Omega)$ has an adherent point $g(x)$. We see easily that $x \in \operatorname{ad} \Omega \pmod g$. To establish (c) $\rightarrow$ (a) requires merely that we note that, under (c), $F_n = X - g^{-1}(-n, n)$, $n = 1, 2, 3, \ldots$, cannot be a filterbase on X for any $g \in \mathscr{C}(X)$. The proof is complete.

In stating our next two theorems — which may be readily established — we recall that a point x in a space is in the *θ-closure of subset K* ($x \in \theta\text{-cl}(K)$) of the space if each V open about x satisfies $K \cap \operatorname{cl}(V) \neq \emptyset$; and a subset K of a space is θ-closed if $\theta\text{-cl}(K) \subset K$, see **[18]**. We further recall from **[18]** that a point x in a space is in the *θ-adherence of a filterbase Ω* on the space ($x \in \theta\text{-ad}\, \Omega$) if $x \in \theta\text{-cl}(F)$ for each $F \in \Omega$.

THEOREM 2.5. *The following statements hold for a topological space X, subsets K and M of X, and $f \in \mathscr{C}(X)$:*
(a) *X and $\emptyset$ are closed* (mod f).
(b) $K \subset \operatorname{cl}(K) \subset \theta\text{-cl}(K) \subset \operatorname{cl}(K) \pmod f$.
(c) *Closed* (mod f) *subsets are θ-closed.*
(d) *K satisfies* $\operatorname{cl}(K) \pmod f$ *is closed* (mod f).
(e) *The intersection of any collection and union of any finite collection of closed* (mod f) *subsets are closed* (mod f).
(f) *If $K \subset M$, then* $\operatorname{cl}(K) \pmod f \subset \operatorname{cl}(M) \pmod f$.
(g) $\theta\text{-ad}\, \Omega \subset \operatorname{ad} \Omega \pmod f$ *for each filterbase Ω on X.*

THEOREM 2.6. *The following properties hold for topological spaces X and Y subsets P and Q of $X \times Y$, and $(f, g) \in \mathscr{C}(X) \times \mathscr{C}(Y)$:*
(a) *$X \times Y$ and $\emptyset$ are closed* (mod(f, g)).
(b) $P \subset \operatorname{cl}(P) \subset (1)\text{-cl}(P) \pmod f \subset \operatorname{cl}(P) \pmod{(f, g)}$.
(c) *All* (1)-*closed* (mod f) *subsets are closed.*
(d) *All closed* (mod(f, g)) *subsets are* (1)-*closed* (mod f).
(e) $(1)\text{-cl}(P) \pmod f$ *is* (1)-*closed* (mod f) *and* $\operatorname{cl}(P) \pmod{(f, g)}$ *is closed* (mod(f, g)).
(f) *The intersection of any collection and union of any finite collection of closed* (mod(f, g)) *subsets is closed* (mod(f, g)).
(g) *Same as* (f) *except that "closed* (mod(f, g))" *is replaced by* "(1)-*closed* (mod f)".
(h) *If $P \subset Q$, then* $\operatorname{cl}(P) \pmod{(f, g)} \subset \operatorname{cl}(Q) \pmod{(f, g)}$ *and* $(1)\text{-cl}(P) \pmod f \subset (1)\text{-cl}\,(Q) \pmod f$.

A function $\psi: X \to Y$ is *weakly-continuous* if for each $x \in X$ and W open about $\psi(x)$, there is a V open about x satisfying $\psi(V) \subset \mathrm{cl}(W)$, see **[15]**. We recall that a function $\psi: X \to Y$ is continuous if and only if $\psi(\mathrm{ad}\,\Omega) \subset$ *ad* $\psi(\Omega)$ for each filterbase Ω on X and we state the following theorem without proof.

THEOREM 2.7. *A function $\psi: X \to Y$ is weakly-continuous if and only if $\psi(\mathrm{ad}\,\Omega) \subset \theta\text{-ad}\,\psi(\Omega)$ for each filterbase Ω* on X.

DEFINITION 2.8. A function $\psi: X \to Y$ is *weakly-continuous* mod $g \in \mathscr{C}(Y)$ (weakly-continuous (mod g)) if $\psi(\mathrm{ad}\ \Omega) \subset \mathrm{ad}\ \psi(\Omega)$ (mod g) for each filterbase Ω on X.

THEOREM 2.9. *The following statements are equivalent for spaces X, Y and functions $\psi: X \to Y$ and $g \in \mathscr{C}(Y)$:*
(a) *The function ψ is weakly-continuous* (mod g).
(b) *The function ψ satisfies $\psi(\mathrm{cl}(K)) \subset \mathrm{cl}(\psi(K))$* (mod g) *for each $K \subset X$.*
(c) *The composition $g \circ \psi$ is continuous.*

PROOF. It is obvious that (a) $\to$ (b).
For (b) $\to$ (c), Let $x \in X$ and let H be open about $g \circ \psi(x)$. Then $g^{-1}(H)$ is open about $\psi(x)$ and $\psi(x) \notin \mathrm{cl}(\psi(X) - g^{-1}(H))$ (mod g). Hence $\psi(x) \notin \mathrm{cl}(\psi(X - (g \circ \psi)^{-1}(H)))$ (mod g) and from (b) we see that $\psi(x) \notin \psi(\mathrm{cl}\ (X - (g \circ \psi)^{-1}(H)))$. So $x \notin \mathrm{cl}(X - g \circ \psi)^{-1}(H)))$ and there is a V open about x in X with $g \circ \psi(V) \subset H$.

For (c) (a), let Ω be a filterbase on X. Then $g \circ \psi(\mathrm{ad}\,\Omega) \subset \mathrm{ad}\ g \circ \psi(\Omega)$; and for each $x \in \mathrm{ad}\ \Omega$, each H open about $g \circ \psi(x)$, and each $F \in \Omega$ we have $g \circ \psi(F) \cap H \neq \varnothing$. Consequently, $\psi(F) \cap g^{-1}(H) \neq \varnothing$ and $\psi(x) \in \mathrm{ad}\ \psi(\Omega)$(mod g).

In **[10]**, a function $\psi: X \to Y$ is said to have a *strongly-closed graph* if for each $(x, y) \in (X \times Y) - \mathscr{G}(\psi)$, there is a V open about x and a U open about y satisfying $(V \times \mathrm{cl}(U)) \cap \mathscr{G}(\psi) = \varnothing$. We state the following theorem without proof. The characterization in this theorem motivates definition 2.11.

THEOREM 2.10. *A function ψ from a space X to a space Y has a strongly-closed graph if and only if $\psi(x)$ is θ-closed in Y and θ-*ad $\psi(\Omega) \cup \{\psi(x)\} = \{\psi(x)\}$ *for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to x$.*

DEFINITION 2.11. A function $\psi: X \to Y$ has a *strongly-subclosed graph* mod $g \in \mathscr{C}(Y)$ if for each $x \in X$, we have ad $\psi(\Omega)$ (mod g) $\cup\ \{\psi(x)\} = \{\psi(x)\}$ for each filterbase Ω on $X - \{x\}$ with $\Omega \to x$.

It is not difficult to see from Theorem 2.5 (b) that if a function ψ maps to a Hausdorff space Y and $\mathscr{G}(\psi)$ is strongly-subclosed (mod g) for some $g \in \mathscr{C}(Y)$, then $\mathscr{G}(\psi)$ is strongly-closed.

DEFINITION 2.12. Let X be a set, let $x_0 \in X$, and let Ω be a filterbase on X; $\{A \subset X: x_0 \in X - A$ or $F \cup \{x_0\} \subset A$ for some $F \in \Omega\}$ is a topology on X which is called the *topology associated with* x_0 *and* Ω. X equipped with this topology is called the *space associated with* x_0 *and* Ω, and will be denoted here by $X(x_0, \Omega)$.

The following readily established theorem is used frequently throughout the remainder of the paper.

THEOREM 2.13. *Let X be a set, let $x_0 \in X$ and let Ω be a countable filterbase on X which has empty intersection on $X - \{x_0\}$. Then $X(x_0, \Omega)$ is in close $\mathscr{M}$.*

3. **Characterizations of pseudocompact spaces via graphs.** In **[10]**, it is proved that a Hausdorff space Y is H-closed if and only if for each space X in a class of spaces containing as a subclass the Hausdorff, completely normal, fully normal spaces, such $\psi: X \to Y$ with a strongly-closed graph is weakly-continuous. The first of our main theorems is an analogue of this result for pseudocompact spaces.

THEOREM 3.1. *A space Y is pseudocompact if and only if for each space X in $\mathscr{M}$ and $g \in \mathscr{C}(Y)$, each bijection $\psi: X \to Y$ with a strongly-subclosed* (mod g) *graph is weakly-continuous* (mod g).

PROOF. Strong Necessity. Let Y be pseudocompact, let $g \in \mathscr{C}(Y)$, let X be any space, let Ω be any filterbase on X and let $\psi: X \to Y$ be any function with a strongly-subclosed (mod g) graph. If $y \in \psi(\text{ad } \Omega)$, choose $x \in \text{ad } \Omega$ with $\psi(x) = y$ and let Σ be an open set base at x. Let $\Omega^* = \{(V \cap F) - \{x\}: V \in \Sigma, F \in \Omega\}$. If Ω^* is not a filterbase on X, we have $V \cap F = \{x\}$ for some $V \in \Sigma$ and $F \in \Omega$; so, $x \in F$ for each $F \in \Omega$ and $y = \psi(x) \in \psi(F)$ for each $F \in \Omega$. Otherwise, Ω^* is a filterbase on $X - \{x\}$ with $\Omega^* \to x$. Furthermore, $\psi(\Omega^*)$ is a filterbase on Y and, consequently, $\psi(x) \in \text{ad } \psi(\Omega^*)$ (mod g) $\subset \text{ad } \psi(\Omega)$ (mod g).

Sufficiency. Let Ω be a countable filterbase on the space Y and suppose $\text{ad } \Omega(\text{mod } g) \cap (X - \{y_0\}) = \emptyset$ for some $y_0 \in Y$ and $g \in \mathscr{C}(Y)$. Let ψ: $Y(y_0, \Omega) \to Y$ be the identity function. We show that $\mathscr{G}(\psi)$ is strongly-subclosed (mod g). Let $y \in Y(y_0, \Omega)$ and let Ω^* be a filterbase on $Y(y_0, \Omega) - \{y\}$ such that $\Omega^* \to y$. Then $y = y_0$ and it follows that for each $F \in \Omega$, there is an $F^* \in \Omega^*$ with $F^* \subset F$. So, $\text{ad } \Omega^*(\text{mod } g) \subset \text{ad } \Omega(\text{mod } g)$. Thus, $\text{ad } \Omega^*(\text{mod } g) \cup \{y_0\} = \{y_0\}$ and $\mathscr{G}(\psi)$ is strongly-sub-closed (mod g). We now have that ψ is weakly-continuous (mod g); so, $y_0 \in \text{ad } \Omega = \psi(\text{ad } \Omega) \subset \text{ad } \psi(\Omega)$ (mod g) $= \text{ad } \Omega(\text{mod } g)$.

4. **Characterizations of pseudocompact spaces via projections.** In **[13]**, it is proved that a Hausdorff space X is H-closed if and only if the projection, $\pi_Y: X \times Y \to Y$, takes θ-closed subsets onto θ-closed subsets for every space Y in a class of spaces containing as a subclass the Hausdorff,

completely normal, fully normal spaces. In this section, we give similar characterizations of pseudocompact spaces.

THEOREM 4.1. *A space X is pseudocompact if and only if $\pi_Y: X \times Y \to Y$ maps* (1)*-closed* (mod f) *subsets onto closed subsets for each space Y in class $\mathcal{M}$ and $f \in \mathscr{C}(X)$.*

PROOF. Strong Necessity. Let X be a pseudocompact, let Y be any space, let $f \in \mathscr{C}(X)$, and let $K \subset X \times Y$ be (1)-closed (mod f). Let $y \in \mathrm{cl}(\pi_Y(K))$. Then $\Omega = \{\pi_X((X \times V) \cap K): V \text{ open about } y\}$ is a filterbase on X and ad $\Omega(\text{mod} f) \neq \varnothing$ by Theorem 2.4 (a). If $x \in \text{ad } \Omega(\text{mod} f)$, it is not difficult to show that $(x, y) \in (1)\text{-cl}(K) \text{ (mod } f) \subset K$; hence $y \in \pi_Y(K)$.

Sufficiency. Let Ω be a countable filterbase on X and let $f \in \mathscr{C}(X)$ with ad $\Omega(\text{mod } f) = \varnothing$; choose $y_0 \notin X$, let $Y = X \cup \{y_0\}$ and $K = \{(x, x): x \in X\}$. Then, $K \subset (1)\text{-cl}(K)(\text{mod} f)$, which is (1)-closed (mod f) in $X \times Y(y_0, \Omega)$. Thus, $y_0 \in \pi_Y((1)\text{-cl}(K)(\text{mod } f))$. Let $x \in X$ with $(x, y_0) \in (1)\text{-cl}(K)$ (mod f). For each H open about $f(x)$ and $F \in \Omega$, we must have $(f^{-1}(H) \times (F \cup \{y_0\})) \cap K \neq \varnothing$. So $f^{-1}(H) \cap F \neq \varnothing$; this is a contradiction.

THEOREM 4.2. *A space X is pseudocompact if and only if $\pi_Y: X \times Y \to Y$ maps closed* (mod(f, g)) *subsets onto closed* (mod g) *subsets for each space Y in class $\mathcal{M}$ and $(f, g) \in \mathscr{C}(X) \times \mathscr{C}(Y)$.*

PROOF. Strong Necessity. Let X be pseudocompact, let Y be any space, let $(f, g) \in \mathscr{C}(X) \times \mathscr{C}(Y)$, and let $K \subset X \times Y$ be closed (mod(f, g)). Let $y \in \mathrm{cl}(\pi_Y(K))(\text{mod } g)$. $\Omega = \{\pi_X((X \times g^{-1}(H)) \cap K): H \text{ is open about } g(y)\}$ is a filterbase on X. So ad $\Omega(\text{mod } f) \neq \varnothing$ by Theorem 2.4 (b). If $x \in \text{ad } \Omega(\text{mod} f)$, it follows that $(x, y) \in \mathrm{cl}(K)(\text{mod}(f, g))$; thus, $y \in \pi_Y(K)$.

Sufficiency. Let $\Omega = \{F(n)\}$ be a strictly decreasing countable filterbase on X and let $f \in \mathscr{C}(X)$ with ad $\Omega(\text{mod} f) = \varnothing$. Choose $y_0 \in X$, let $Y = X \cup \{y_0\}$ and $K = \{(x, x): x \in X\}$. Define f from $Y(y_0, \Omega)$ to the reals by $g(x) = 0$ for $x \in Y(y_0, \Omega) - (F(1) \cup \{y_0\})$, $g(y_0) = 1$ and $g(x) = 1 + 1/n$ for $x \in F(n) - F(n + 1)$. Then, $g \in \mathscr{C}(Y(y_0, \Omega))$. $K \subset \mathrm{cl}(K)(\text{mod}(f, g))$, which is closed (mod$(f, g)$) in $X \times Y(y_0, \Omega)$; so, $y_0 \in \pi_Y(\mathrm{cl}(K)(\text{mod}(f, g)))$. Let $x \in X$ with $(x, y_0) \in \mathrm{cl}(K)(\text{mod}(f, g))$. Let $F(m) \in \Omega$ and let H be open about $f(x)$; we have

$$g^{-1}(0, 1 + 1/m) = \{y_0\} \cup \bigcup_{n>m} (F(n) - F(n + 1))$$

and $(f^{-1}(H) \times g^{-1}(0, 1 + 1/m)) \cap K \neq \varnothing$. This implies that $f^{-1}(H) \cap F(m) \neq \varnothing$; this is a contradiction.

5. **Characterizations of spaces X for which filterbases Ω on X satisfying cardinality of ad Ω(mod g)(|ad Ω(mod g)|) $\leqq 1$ for some $g \in \mathscr{C}(X)$ are convergent.** In [11], minimal Hausdorff spaces have been characterized as precisely those Hausdorff spaces in which each filterbase Ω on the space

with $|\theta - \text{ad}\,\Omega| \leqq 1$ is convergent. In this section, we presentcharacterizations of spaces with the following property.

($\mathscr{T}$) *Each filterbase* Ω *on the space* X *with* $|\text{ad}\,\Omega(\text{mod}\ g)| \leqq 1$ *for some* $g \in \mathscr{C}(X)$ *is comvergent.*

Let $\mathscr{S}$ represent a class of spaces containing as a subclass the Hausdorff, completely normal, fully normal spaces.

THEOREM 5.1 *A space* Y *has property* $\mathscr{T}$ *if and only if for each space* X *in class* $\mathscr{S}$ *and* $g \in \mathscr{C}(Y)$, *each bijection* $\phi: X \to Y$ *with* $\mathscr{G}(\phi)$ *strongly-subclosed* (mod g) *is continuous.*

PROOF. Strong Necessity. Let Y have property $\mathscr{T}$, let X be any space, let $g \in \mathscr{C}(Y)$ and let $\phi: X \to Y$ be any function with $\mathscr{G}(\phi)$ strongly-subclosed (mod g). Let $x \in X$; if $\{x\}$ is open in X, ϕ is continuous at x. If not, then $\Omega = \{V - \{x\}: V \text{ open about } x\}$ is a filterbase on $X - \{x\}$ and $\Omega \to x$. So ad $\phi(\Omega)(\text{mod}\ g) = \emptyset$ or $\{\phi(x)\}$ since $\mathscr{G}(\phi)$ is strongly-subclosed (mod g). Thus, $\phi(\Omega) \to \phi(x)$ and, for each W open about $\phi(x)$, there is a V open about x with $\phi(V) \subset W$.

Sufficiency. Let Ω be a filterbase on Y; supose $y_0 \in Y$ and $g \in (\mathscr{C}(Y)$ satisfy ad $\phi(\Omega)$ (mod g) $\bigcup \{y_0\} = \{y_0\}$. If $\Omega \nrightarrow y_0$, there is an open set V_0 about y_0 with $\Omega = \{F \cap (Y - V_0): F \in \Omega\}$ a filterbase on Y. $Y(y_0, \Omega^*)$ is in class $\mathscr{S}$ and if $\phi: Y(y_0, \Omega^*) \to Y$ is the identity function, let Ω^{**} be a filterbase on $Y(Y_0, \Omega^*) - \{y\}$ with $\Omega^{**} \to y$: it follows that $y = y_0$, and that ad $\Omega^{**}(\text{mod}\ g) \subset$ ad $\Omega^*(\text{mod}\ g) \subset$ ad $\Omega(\text{mod}\ g)$. Therefore, $\mathscr{G}(\phi)$ is strongly-subclosed (mod g) and ϕ is continuous. Therefore, $\Omega \to y_0$ in Y since $\Omega \to y_0$ in $Y(y_0, \Omega^*)$. This is a contradiction.

DEFINITION 5.2. A function $\phi: X \to Y$ has a *subclosed graph* if ad $\phi(\Omega) \bigcup \{\phi(x)\} = \{\phi(x)\}$ for each $x \in X$ and filterbase Ω on $X - \{x\}$ with $\Omega \to x$.

We remark that a function with a closed graph has a subclosed graph, and that if Y is T_1 and $\phi: X \to Y$ has a subclosed graph, then ϕ has a closed graph.

THEOREM 5.3. *A space* Y *has property* $\mathscr{T}$ *if and only if for all spaces* X *in class* $\mathscr{S}$, $g \in \mathscr{C}(Y)$, *and bijections* $\lambda, \phi: X \to Y$ *with subclosed graph and strongly-subclosed* (mod g) *graph, respectively,* $\mathscr{E}(\phi, \lambda, X, Y)$ *is close in* X.

PROOF. Strong Necessity. Let Y have property $\mathscr{T}$, let $f \in \mathscr{C}(Y)$, let X be any space, and let $\lambda, \phi: X \to Y$ be any functions with $\mathscr{G}(\lambda)$ subclosed and $\mathscr{G}(\phi)$ strongly-subclosed (mod g); ϕ is continuous from the proof of the necessity in Theorem 5.1. Let $x \in \text{cl}(\mathscr{E}(\phi, \lambda, X, Y)) - \mathscr{E}(\phi, \lambda, X, Y)$. There is a filterbase Ω on $\mathscr{E}(\phi, \lambda, X, Y)$ with $\Omega \to x$. Since λ has a subclosed graph, and we have

$$\{\phi(x)\} \cup \{\lambda(x)\} \subset \phi(\text{ad } \Omega) \cup \{\lambda(x)\} \subset \text{ad } \phi(\Omega) \cup \{\lambda(x)\}$$
$$= \text{ad } \lambda(\Omega) \cup \{\lambda(x)\},$$

we obtain a contradiction.

Sufficiency. Let Ω be a filterbase on Y, let $y_0 \in Y$, and let $g \in \mathscr{C}(Y)$ with ad $\Omega(\text{mod } g) \cup \{y_0\} = \{y_0\}$. If $\Omega \nrightarrow y_0$, there is an open set V_0 about y_0 with $\Omega^* = \{F \cap (Y - V_0): F \in \Omega\}$ a filterbase on Y. Choose $x_0 \in Y - \{y_0\}$, let $\phi: Y(y_0, \Omega^*) \to Y$ be the identity function, and let $\lambda: Y(y_0, \Omega^*) \to Y$ be defined by $\lambda(x_0) = y_0$, $\lambda(y_0) = x_0$ and $\lambda(x) = x$ otherwise. We see easily that $\mathscr{E}(\phi, \lambda, Y(y_0, \Omega^*), Y) = Y - \{x_0, y_0\}$, which is not closed in $Y(y_0, \Omega^*)$. We show that $\mathscr{G}(\phi)$ is strongly-subclosed (mod g) and that λ has a subclosed graph; this will yield a contradiction.

(a) $\mathscr{G}(\phi)$ is strongly-subclosed (mod g). See proof of the sufficiency of Theorem 5.1.

(b) $\mathscr{G}(\lambda)$ is subclosed. Let $y \in Y$ and let Ω^{**} be a filterbase on $Y(y_0, \Omega^*)$ with $\Omega^{**} \to y$. Then, $y = y_0$ and for each $F \in \Omega$, there is an $F^{**} \in \Omega^{**}$ satisfying $F^{**} \subset F \cap (Y - V_0)$. We may assume, without loss, that Ω^{**} is a filterbase on $Y - \{x_0, y_0\}$. Thus, ad $\lambda(\Omega^{**}) =$ ad $\Omega^{**} \subset$ ad $\Omega^* = \varnothing$ and, consequently, λ has a subclosed graph.

THEOREM 5.4. *A space Y has property $\mathscr{T}$ if and only if for all spaces X in class $\mathscr{S}$, $g \in \mathscr{C}(Y)$, and functions (one of these a bijection) λ, $\phi: X \to Y$ with subclosed graph and strongly-subsclosed* (mod g) *graph, respectively, $\mathscr{E}(\phi, \lambda, X, Y) = X$ whenever $\mathscr{E}(\phi, \lambda, X, Y)$ is dense in X.*

PROOF. Strong Necessity. Let Y have property $\mathscr{T}$, let $g \in \mathscr{C}(Y)$, let X be any space, and let ϕ, $\lambda: X \to Y$ be any functions with $\mathscr{G}(\phi)$ subclosed and $\mathscr{G}(\lambda)$ strongly-subclosed (mod g). If $\mathscr{E}(\phi, \lambda, X, Y)$ is dense in X, then $\mathscr{E}(\phi, \lambda, X, Y) = X$ since $\mathscr{E}(\phi, \lambda, X, Y)$ is closed in X by Theorem 5.3.

Sufficiency. We follow the proof of the sufficiency of Theorem 5.3 to the point immediately preceding the definition of λ. We define λ by $\lambda(x) = x$ if $x \neq y_0$ and $\lambda(y_0) = x_0$. Arguments similar to those in the proof of the sufficiency of Theorem 5.3 show that $\mathscr{G}(\phi)$ and $\mathscr{G}(\lambda)$ are strongly-subclosed (mod g) and subclosed, respectively. $\mathscr{E}(\phi, \lambda, Y(y_0, \Omega), Y) = Y - \{y_0\}$ which is dense in $Y(y_0, \Omega)$. This is a contraditoon.

In closing this article, we note that the theorems obtained by replacing each $\mathscr{S}$ by $\mathscr{M}$ and property $\mathscr{T}$ by property $\mathscr{Q}$-stated below—in Theorems 5.1, 5.3 and 5.4 are valid.

($\mathscr{Q}$) *Each countable filterbase Ω on X with* $|\text{ad } \Omega(\text{mod } g)| \leqq 1$ *for some $g \in \mathscr{C}(x)$ is convergent.*

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