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WILDNESS AND FLATNESS OF CODIMENSION ONE SPHERES HAVING DOUBLE TANGENT BALLS

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Introduction. A round n-dimensional ball B_p is said to be *tangent* to an $(n - 1)$-sphere Σ in E^n at a point $p \in \Sigma$ if $p \in B_p$ and $\Sigma \cap B_p \subset \text{Bd } B_p$. If $\text{Int } B \subset \text{Ext } \Sigma$, B_p is called an *exterior tangent ball* and if $\text{Int } B_p \subset \text{Int } \Sigma$, B_p is an *interior tangent ball.* When Σ has both an interior and an exterior tangent ball at p, Σ is said to have a *double tangent ball* at p. If Σ has a certain class of tangent ball for each point of a subset K of Σ, then Σ is said to have this class of *tangent balls over K.* A *uniform* collection of round is one in which every ball has the same radius.

One suspects that the subject of double tangent balls first arose as a rigidly geometric potential analogue to smoothness; if an $(n - 1)$-sphere Σ has double tangent balls at each point, then it would seem to be embedded with a geometrically nice kind of curvature. This would form a basis for a conjecture that, in this context, the double tangent balls property implies flatness. In response to a question by Bing **[2]** concerning this conjecture in 3-space, Bothe **[3]** and Loveland **[17]** independently proved that a 2-sphere in E^3 is flat if it has double tangent balls at each of its points. Griffith **[15]** had earlier produced an affirmative answer to Bing's question provided the collection of double tangent balls was known to be uniform. The situation when $n = 3$ is best summarized by the following theorem, which, although not explicitly stated in **[17]**, follows from the proof there. This generalization is also apparent from Cannon's subsequently developed *-taming set theory (see Corollary 6 of **[8]**).

THEOREM A. *If Σ is a 2-sphere in E^3 that is locally flat modulo a closed subset W of Σ and if Σ has double tangent balls over W, then Σ is flat.*

The examples from §1 show the impossibility of such a theorem for a codimension one sphere in E^n with $n > 3$; in fact, Theorem A does not generalize to $n > 3$ even with the added hypothesis that Σ has uniform double tangent balls over W. These examples stand as circumstantial evidence of the still unauthenticated possibility that an $(n - 1)$-sphere in E^n $(n < 3)$ with double tangent balls everywhere may fail to be flat.

Nevertheless there are interesting facts about higher dimensional

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spheres with various types of tangent balls. Most of the results in this paper are deduced as corollaries to Theorem 2.1 of §2, which deals with locally stable collections of tangent balls. A series of definitions pertaining to an $(n - 1)$-sphere Σ in E^n is needed to make this notion precise. A vector (a directed line segment) $v(p)$ from a point $p \in \Sigma$ to a point of Ext Σ (Int Σ) is called an *exterior* (interior)*vector* at p if its only intersection with Σ is p. An *exterior* (interior) *normal* to Σ at p is an exterior (interior) vector at p terminating at the center of an exterior (interior) tangent ball to Σ at p. The existence of a double tangent ball at p insures, of course, the uniqueness of the direction of both exterior and interior normals at p.

A collection B of exterior (interior) tangent balls (or normals to Σ) is said to be *stable over a subset* U of Σ if there is a point $p \in U$ and an exterior (interior) vector $v(p)$ at p such that for each $q \in U$ a ball B_q exists in B, tangent to Σ at q, whose associated normal $o(q)$ makes an acute angle with $v(p)$. Although $v(p)$ need not be a normal to Σ at p, there must be a normal $o(p)$ at p within $\pi/2$ of $v(p)$. The collection B *is locally stable over a subset* K of Σ if for each point $p \in K$ there exists an open subset U of Σ such that $p \in U$ and B is stable over U.

Theorem 2.1 states that Σ is locally flat at the points of an open set over which it has a stable collection of exterior (or interior) tangent balls. Consequently Σ is flat if it has a collection of exterior (interior) tangent balls which is locally stable over Σ (Corollary 2.2). Other consequences include a generalization of Griffith's work to all dimensions (Corollary 2.3), a reduction of the wildness of Σ to codimension two when Σ has exterior tangent balls at every point (Corollaries 2.4 and 2.5), and limitations on extending uniform double tangent balls from a closed set W to a larger subset of Σ (Corollary 2.6).

With the examples of §1 at hand, it is clear that additional hypotheses are needed in order to conclude that an $(n - 1)$-sphere Σ in E^n $(n > 3)$ is flat if Σ is locally flat modulo a closed subset X of Σ and Σ has double tangent balls over X. A modest attempt in this direction is presented in §3, where hypotheses are added which deal with the dimension of X and its embedding in Σ.

1. **An $(n - 1)$-sphere in $E^n (n > 3)$ with uniform double tangent balls over its wild set.** The promised example follows from the proposition below. Here we use B^n to denote the round n-cell in E^n of radius 1 centered at the origin and S^{n-1} to denote its boundary.

PROPOSITION 1.1. *Suppose Y is a Cantor set tamely embedded in S^{n-1} $(n > 3)$, U is an open subset of B^n containing $S^{n-1} - Y$, and Σ is an $(n - 1)$-sphere in E^n embedded locally flatly modulo a Cantor set X that is tame in E^n. Then there exists a homeomorphism h of E^n to itself such that $h(X) = Y$, $h(\Sigma) \subset U \cup Y$, and* Int $h(\Sigma) \supset$ Int $B^n - U$.

PROOF. Standard embedding techniques provide an $(n - 1)$-sphere S in E^n such that S contains X as a tamely embedded subset, S is locally flatly embedded modulo X, and Int S contains $\Sigma - X$. Because S is locally flat modulo the twice-tame Cantor set X, it bounds a flat n-cell F containing Σ [**16**]. Accordingly, there exists a homeomorphism h_1 of E^n to itself such that $h_1(F) = B^n$, which then can be adjusted further so that $h_1(X) = Y$.

Without loss of generality we assume that $(B^n - U) \cap \operatorname{Int} h_1(\Sigma)$ contains an open set V. Then V, in turn, contains an n-cell Q such that $B^n - \operatorname{Int} Q$ is an annulus; specifically, we name a homeomorphism f of $S^{n-1} \times I$ onto $B^n - \operatorname{Int} Q$ such that $f(y, 0) = y$ for each $y \in Y$.

Next we shall obtain an ambient homeomorphism h_2 of E^n fixing both Q and $E^n - B^n$ such that $f(Y \times I) \cap h_2h_1(\Sigma) = Y$. As a structural guide, we first build a new embedding g of $Y \times I$ in $f(S^{n-1} \times I)$ such that $g(y \times 0) = y$ and $g(y \times 1) = f(y \times 1)$ for each $y \in Y$ and that $g(Y \times I) \cap h_1(\Sigma) = Y$. By itself this is easy enough to do manually, since Int $h_1(\Sigma)$ is 0-ULC; some may prefer, however, to deform a thickened copy of Q through Int $h_1(\Sigma)$, fixing points of the unthickened Q, so that the boundary contains Y, in the manner used to construct S and F, and then to view the embedded $g(Y \times I)$ as the resulting (deformed) fibers from Y to $f(Y \times 1)$. In either case, one insists (or notes) that, in addition, $g(Y \times (0, 1))$ have codimension 3 (that 2-complexes in $f(S^{n-1} \times (0, 1))$ can be pushed off $g(Y \times (0, 1))$ with small ambient homeomorphisms). Since the two embeddings f and g of $Y \times I$ in $f(S^{n-1} \times I)$ are homotopic, rel$f(Y \times \operatorname{Bd} I)$, it follows from variations to work of Bryant [**5, 6**] or Stan'ko [**18**, Theorem 2], providing majorant rather than merely uniform control, that there exists a homeomorphism h_2 of E^n to itself, fixing points of $f(S^{n-1} \times I)$, such that $h_2g(Y \times I) = f(Y \times I)$. (Remark: with the codimension 3 hypothesis operative here, Bryant's work in [**6**] applies, not only for $n = 5$, but to $n = 4$ as well).

Finally, since $h_2h_1(\Sigma)$ misses $f(Y \times (0, 1])$ and $f(S^{n-1} \times 1)$, we can name a third homeomorphism h_3 of E^n fixing points of $f(Y \times I)$ and outside $f(S^{n-1} \times I)$ and pushing $h_2h_1(\Sigma)$ away from $f(S^{n-1} \times 1)$ so close to $f(S^{n-1} \times 0)$ that $h_3h_2h_1(\Sigma)$ lies in U. Then $h = h_3h_2h_1$ is the required homeomorphism.

COROLLARY 1.2. *For $n > 3$ there exists a wild $(n - 1)$-sphere in E^n with uniform double tangent balls over its wild set.*

PROOF. Let Y denote a Cantor set tamely embedded in S^{n-1}. For each $y \in Y$ let B_y denote the round ball of radius 1/2 centered midway between y and the origin; clearly B_y is tangent to S^{n-1} at y and $S^{n-1} \cap B_y = \{y\}$. Define U as $B^n - \bigcup_{y \in Y} B_y$.

It is known that E^n contains a wildly embedded $(n - 1)$-sphere Σ that is locally flat modulo a Cantor set X that is tame in E^n (cf. [**13**, Example

9.2]). By Proposition 1.1, Σ can be rearranged, via an ambient homeomorphism h, so that $h(\Sigma) \subset U \cup Y$ and $h(\Sigma)$ separates $E^n - B^n$ from $B^n - (U \cup Y)$, guaranteeing that $h(\Sigma)$ has double tangent balls of radius $1/2$ at each point of $h(X) = Y$.

2. **A codimension one sphere is flat if it has locally stable exterior tangent balls everywhere.** The six corollaries given in this section of the paper, the theorem in the section's title included, are deduced from Theorem 2.1.

THEOREM 2.1. *If U is an open subset of an $(n - 1)$-sphere Σ in E^n and B is a collection of exterior (interior) tangent balls which is stable over U, then Σ is locally bicollared at each point of U. Consequently Σ is locally flat at each point of U.*

PROOF. The hypothesis implies the existence of a point $p \in U$, an exterior vector $v(p)$ at p, a subcollection B' of B, and the set N of normals to Σ corresponding to the elements of B' such that

(1) U, N, and B' are equivalent sets under the correspondence
$$q \leftrightarrow o(q) \leftrightarrow B_q,$$

and

(2) for each $q \in U$, the angle between $v(p)$ and $o(q)$ is acute.

The direction of $v(p)$ will be regarded as the upward vertical, and $L(x)$ will denote the vertical line through the point x. Condition (2) above shows that, for each $q \in U$, there is an open interval $A(q)$ in $L(q) \cap \text{Ext } \Sigma$ lying above q and having q as its lower endpoint.

Let x be an arbitrary point of U, and let D be an $(n - 1)$-cell in U with x in its interior. For each $q \in D$, let $J(q)$ be the closed interval in $L(q)$ having q as its midpoint and having length $d(D, \Sigma - U)/2$. The union of $\{J(q) | q \in D\}$ is homeomorphic to the product of D with an interval and is the desired bicollar over D. This will be clear when it is shown that for each $q \in D$,

(a) $$J(q) \cap \Sigma = \{q\}$$

and

(b) $J(q)$ intersects both $\text{Int } \Sigma$ and $\text{Ext } \Sigma$.

Notice that $J(q) \cap \Sigma \subset U$.

To establish (a), let V be a tubular neighborhood of $J(q)$ such that $V \cap \Sigma \subset U$, and suppose a point s exists in $J(q) \cap U$ such that $s \neq q$. Choose disjoint open n-balls $V(s)$ and $V(q)$ in $V \cap \text{Ext } \Sigma$ centered at points of $A(s)$ and $A(q)$, respectively, and choose a point y of $\text{Int } \Sigma$ close enough to the uppermost point of $\{s, q\}$ to insure that $L(y)$ intersects both

$V(s)$ and $V(q)$. Then the point y of Int Σ lies above a component of $L(y) \cap$ Ext Σ, so there must be a highest point t of the compact set $\Sigma \cap L(y)$ below y. But $A(t)$ lies above t and in Ext Σ, so there must be points of $\Sigma \cap L(y)$ between t and y, contradicting the definition of t. Thus (a) is known.

Fact (b) can be proved in the same manner, for suppose $J(q) \subset \Sigma \cup$ Ext Σ for some q in D. By (a), the upper and lower open halves of $J(q)$ each intersects Ext Σ, so it is possible to choose disjoint open n-balls V_1 and V_2 above and below q such that $V_i \cap \Sigma \subset U$ for $i = 1, 2$. Then a point y is chosen in Int Σ so close to q that $L(y)$ intersects both V_1 and V_2. Now the same structure as in the previous paragraph has been set up, and the same contradiction exists.

Of course the local flatness conclusion is a consequence of Brown's work **[5]**.

Corollary 2.2. *If Σ is an $(n - 1)$-sphere in E^n and B is a collection of exterior (interior) tangent balls that is locally stable over Σ, then Σ is flat in E^n.*

Griffith **[15]** proved that a 2-sphere Σ in E^3 is flat Σ has a uniform set of double tangent balls. His technique was to prove Σ is locally spanned in both complementary domains because Burgess **[4]** showed this implies $E^3 - \Sigma$ is 1-ULC. The dependence upon the 1-ULC characterization of flatness restricts generalization of Griffith's proof to cases where $n \neq 4$ (see **[1**; **10**; **12]**). However, this restriction is not needed when the result is viewed as a corollary to Theorem 2.1.

Corollary 2.3. *If an $(n - 1)$-sphere Σ in E^n has uniform double tangent balls at each of its points, then Σ is flat.*

Proof. By hypothesis there is a positive number δ and two collections B_i and B_e of balls of radius δ such that for each $p \in \Sigma$ there exist unique elements $B_i(p)$ and $B_e(p)$ of B_i and B_e, respectively, which are tangent to Σ at p with $B_i(p) \cap \operatorname{Ext} \Sigma = \emptyset$ and $B_e(p) \cap \operatorname{Int} \Sigma = \emptyset$. Let $p \in \Sigma$ and choose an $(n - 1)$-cell D in Σ with $p \in \operatorname{Int} D$ so small that the exterior normals $o(p)$ and $o(q)$ make an acute angle for each $q \in D$. This is possible because $\{B_e(q_i)\}$ must converge to $B_e(p)$ where $\{q_i\}$ converges to p. Thus B_e is locally stable over D, and Σ is locally bicollared at p by Theorem 2.1.

Corollary 2.4. *If W is the set of points where an $(n - 1)$-sphere Σ in E^n fails to be locally flat and Σ has exterior (interior) tangent balls over W, then W has codimension two in E^n.*

Proof. Suppose W contains an $(n - 1)$-cell D. For each positive integer i define $X_i = \{p \in D \mid \Sigma$ has an exterior tangent ball at p of radius $1/i\}$. The hypothesis insures that $D = \bigcup_{i=1}^{\infty} X_i$ and each X_i is closed. Therefore, a Baire Category argument yields an integer i and an $(n - 1)$-cell D' in

D such that $D' \subset X_i$. The points of a round $(n-1)$-sphere T in E^n can be considered as the set of directions for vectors in E^n. Let $N_1, N_2, \ldots, N_m$ be closed neighborhoods in T whose union covers T such that, for fixed j, the angle between any two vectors of N_j is less than $\pi/2$. Define $Y_j = \{p \in D' \mid \Sigma$ has a normal $o(p)$ at p corresponding to an exterior tangent ball of radius $1/i$ at p such that the direction of $o(p)$ lies in $N_j\}, j = 1, 2, \ldots, m$. Since each N_j is closed and the balls over D' may be taken to all be of radius $1/i$, it follows that each Y_j is closed and that $D' = \bigcup_{j=1}^{m} Y_j$. Consequently some Y_j contains an open subset of D' and this Y_j then contains an $(n-1)$-cell D'' in D. By the definition of N_j it is clear that a collection B of exterior tangent balls of radius $1/i$ exists which is locally stable over D''. A contradiction to the fact that $D'' \subset W$ comes from Theorem 2.1.

COROLLARY 2.5. *If W is the set of points where an $(n-1)$-sphere Σ in E^n fails to be locally flat and Σ has double tangent balls over W, then W has codimension two in E^n.*

PROOF. Corollary 2.5 follows immediately from Corollary 2.4.

There are examples in E^3 showing that no larger codimension is possible in the conclusion of Corollary 2.4. As a first step observe that the Fox-Artin 2-sphere **[14]**, assumed to be wild from the interior, can be embedded in E^3 so it has interior tangent balls at each point. Now to construct the desired example take a round 3-ball R and a 2-sphere S such that $S \cap R$ is an equatorial arc A in the boundary of R. Let $\{a_i\}$ be a countable set in (Bd R)-A whose closure is $A \cup (\bigcup\{a_i | i = 1, 2, 3, \ldots\}) = W$, and attach to $S - A$ a null sequence of disjoint Fox-Artin feelers, one toward each a_i from some point of S, to obtain a 2-sphere Σ whose wild set is precisely W. If the construction is carefully done, R is tangent to Σ at every point of W. The other interior tangent balls are not the same size as R but they clearly exist if the construction is not carried out in a deliberately mischievous manner.

Similar examples in higher dimensions can be constructed by rigidly spinning this 3-dimensional one.

The example in §1 gives an $(n-1)$-sphere Σ in E^n that is locally flat modulo a subset W such that Σ has uniform double tangent balls over W. The next result gives some limitations on attempts to extend these balls to a uniform exterior (or interior) collection over all of Σ.

COROLLARY 2.6. *If W is a closed subset of an $(n-1)$-sphere Σ in E^n such that Σ is locally flat modulo W, Σ has uniform double tangent balls over W, and Σ has a uniform exterior (interior) set of tangent balls over an open subset U of Σ containing W, then Σ is flat.*

PROOF. The hypothesis provides a positive number δ, a collection B_e

of exterior tangent balls of radius δ over U, and a collection B_i of interior tangent balls of radius δ over W. The uniqueness, up to size, of the double tangent balls over W insures that, for each $p \in W$, an $(n-1)$-cell D may be found in U with $p \in \operatorname{Int} D$ such that B_e is locally stable over D. By Theorem 2.1 Σ is locally flat at each point of W and the result follows.

The next corollary is obvious when B is a uniform exterior collection of tangent balls and perhaps not difficult to prove as stated, but the motiviation for defining a locally stable collection originally came from deciding how to generalize this corollary.

Given a collection B of exterior (interior) tangent balls over an $(n-1)$-sphere Σ in E^n, the *direction relation* R on Σ is defined by letting $R(p)$ be the set of all directions toward which the normals $o(p)$, relative to elements of B, point. Thus R relates Σ into an $(n-1)$-sphere of directions, and R depends upon B. A collection B is said to have *continuous directions* when R is a continuous function.

COROLLARY 2.7. *If an $(n-1)$-sphere Σ in E^n has a collection B of exterior (interior) tangent balls at each of its points such that B has continuous directions, then Σ is flat.*

3. **Double tangent balls over nice subsets.**

THEOREM 3.1. *Suppose Σ is an $(n-1)$-sphere in E^n that has uniform double tangent balls over a closed subset X. Then each $x \in X$ has a neighborhood N_x in X which is contained in some flatly embedded $(n-1)$-sphere in E^n. Hence, $E^n - \Sigma$ is* 1-ULC *in $E^n - X$.*

PROOF. This argument is similar to a subset of the one given for Theorem 2.1. By hypothesis there exists $\delta > 0$ such that each $x \in X$ has double tangent balls of radius δ. As a result, each $x \in X$ admits a unique exterior normal vector $o(x)$ to Σ at x with length δ.

Fix $x \in X$. Clearly it has a neighborhood N in X such that for any y in N the angle between $o(x)$ and $o(y)$ is less than $\pi/4$. The direction of $o(x)$ will be regarded as the upward vertical.

For $y \in N$ elementary trigonometry reveals that $B_y \cap L(y)$ (here $L(y)$ denotes the vertical line through y and B_y the exterior tangent ball at y with radius δ) is a line segment $A(y)$ having y as its lower endpoint and having length at least $\sqrt{2}\,\delta/2$. Let N_x denote a closed neighborhood of x in X such that $N_x \subset N$ and the diameter of N_x is less than $\sqrt{2}\,\delta/2$. In this case $A(y) \cap X = \{y\}$ for each $y \in X$. Furthermore, because of the size restriction on N_x, vertical projection p of E^n to E^{n-1} satisfies $p|N_x$ is one-to-one. Consequently, N_x is contained in an ambient translate of the flat hyperplane $E^{n-1} \times 0$.

Since N_x lies in a flat sphere, it follows from work of Cannon [9, Therem 4.3] that $E^n - \Sigma$ is 1-LC in $E^n - X$ at x. The uniform version follows automatically.

The example presented in §1 reveals that, in Corollary 3.2 below, hypothesis (3), which may seem foreign to the spirit of this paper, stands independent from the other hypotheses.

COROLLARY 3.2. *Suppose that Σ is an $(n - 1)$-sphere in E^n $(n > 3)$ satisfying*

(1) *Σ is locally flat modulo an $(n - 3)$-dimensional closed subset X of S,*
(2) *Σ has double tangent balls over X, and*
(3) *$\Sigma - X$ is* 1-ULC.

Then $E^n - \Sigma$ is 1-ULC *and, for $n \geqq 5$, Σ is flat.*

PROOF. For each positive integer i define $X_i = \{x \in X \mid \Sigma$ has double tangent balls at x of radius $1/i\}$. As before, each X_i is closed and $X = \bigcup X_i$. The hypothesis that X have codimension at least 2 relative to Σ, coupled with hypothesis (3), implies that $\Sigma - X_i$ is 1-ULC for each i (cf. [**18**, Proposition 6]). Thus, for either component U of $E^n - \Sigma$, Cl $U - X_i$ is 1-ULC (cf. [**13**, Theorem 3C.12]) and Cl $U - X =$ Cl $U - \bigcup X_i$ is also 1-ULC [**9**, Theorem 2C.4] [**11**, Theorem 3.2]; since, by the local flatness of $\Sigma - X$, ClU is 1-LC at each point of $\Sigma - X$, U itself is 1-ULC. In other words, $E^n - \Sigma$ is 1-ULC. Of course, for $n \geqq 5$, this implies the flatness of Σ[**10**; **12**].

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