ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 3, Summer 1971

SOME REPRESENTATION THEOREMS

A. DE KORVIN AND R. J. EASTON

1. **Introduction.** Much has been written concerning integral representations of continuous linear transformations on spaces of functions. See, for example, [2], [3], [4], [5], [6], [7], [8], [9], [10] and [11]. In all of these articles the functions were defined on a compact Hausdorff space. In [1], the following representation theorem is given. Let H be a normal topological space and let $C_B{}^*(H, R)$ denote the dual of the space of all bounded continuous real-valued functions defined on H. Then, if $\Phi \in C_B{}^*(H, R)$, $\Phi(f) = \int_H f\,du$, where u is a finitely additive, regular, bounded, real-valued measure defined on the field generated by the closed subsets of H. The purpose of this paper is to obtain a similar representation theorem in the vector valued setting of [2], [3], [4], [7], [9], [10] and [11]. The functions are defined on a normal topological space H, with their range spaces being totally bounded subsets of a linear normed space X. The map Φ is bounded and linear from this space of functions to a linear normed space Y and the measure K has values in $B(X, Y^{**})$, the space of all bounded linear maps from X to the bidual of Y.

In the second part of the paper, results similar to those of R. J. Easton and D. H. Tucker in [2] are obtained. A Lebesgue type theory is developed and a representation theorem is obtained.

The authors point out that these techniques would yield similar results in the setting of Goodrich [5] and [6] and Swong [8].

The authors would also like to thank the referee for his comments, in particular those which allow us to state Theorem 3.2 in its present form.

2. **Notations.** Let H be a normal topological space and let X and Y be linear normed spaces; let $C_B(H, X)$ denote the space of all X-valued, continuous, and bounded functions defined on H. Let $C_{TB}(H, X)$ denote the functions of $C_B(H, X)$ which are totally bounded, i.e. their range is a totally bounded subset of X, F denotes the field generated by the closed subsets of H, and $S_F(H, X)$ denote the simple functions, over F, from H to X. The dual and bidual of Y will be denoted by Y^* and Y^{**} respectively.

Received by the editors February 22, 1970 and, in revised form, May 18, 1970.
AMS 1970 *subject classifications*. Primary 28A45, 46G10.

3. **Representation theorems.** If $E \in F$, we denote the characteristic function of E by χ_E, and for $x \in X$, the X-valued function $\chi_E \cdot x(t) = \chi_E(t) \cdot x$. Similarly, $f \cdot x$ is defined for any real-valued function f, and any x in X.

Let f^* be any element of $C_B^*(H, X)$ and let f be any element of $C_B(H, R)$. Define Φ in $C_B^*(H, R)$ by the equation

$$\Phi(f) = f^*(f \cdot x),$$

for x in X. We now make use of the representation theorem on p. 262, Theorem 2 of [**1**], to obtain a unique regular, finitely additive, bounded, real-valued measure defined on F, which we denote by u_{x, f^*}, such that

$$\Phi(f) = \int_E f \, du_{x, f^*}.$$

Define

$$\chi_{E, x}(f^*) = u_{x, f^*}(E).$$

Then

$$\sup_{\|f^*\| \leqq 1} |\chi_{E,x}(f^*)| = \sup_{\|f^*\| \leqq 1} |u_{x, f^*}(E)| \leqq \sup_{\|f^*\| \leqq 1} \|u_{x, f^*}\|$$

(variation of u_{x, f^*})

and since

$$\|u_{x, f^*}\| = \|\Phi\| = \sup_{\|f\| \leqq 1} |\Phi(f)| = \sup_{\|f\| \leqq 1} |f^*(f \cdot x)| \leqq \|x\|$$

if $\|f^*\| \leqq 1$, we have $\|\chi_{E,x}\| \leqq \|x\|$. Furthermore, $\chi_{E,x}$ is clearly linear on $C_B^*(H, X)$ and hence $\chi_{E,x} \in C^{**}(H, X)$.

We now identify the simple function $\chi_E \cdot x$ with the element $\chi_{E, x}$ of $C_B^{**}(H, X)$ since this identification is an isometric isomorphism. See [**11**], for more detail. From this point on we will not distinguish between the simple function $\chi_E \cdot x$ and its corresponding element $\chi_{E, x}$ in C_B^{**}.

LEMMA 3.1. *If $\{e_1, e_2, \cdots, e_n\}$ is any partition of H, with $e_i \in F$ and $x_i \in X$, $i = 1, 2, \cdots, n$, then*

$$\left\| \sum_{i=1}^{n} \chi_{e_i} \cdot x_i \right\|_{C^{**}} \leqq \max_i \|x_i\|_X.$$

PROOF. Consider $f^* \in C_B^*(H, X)$, with $\|f^*\| \leqq 1$. Then

$$\left\| \left(\sum_{i=1}^{n} \chi_{e_i} \cdot x_i \right) (f^*) \right\| = \left| \sum_{i=1}^{n} u_i(e_i) \right|$$

where $u_i = U_{x_i, f^*}$ Each u_i is a regular, finitely additive, measure on F with finite variation. Hence for each i there exists a closed set $c_i \subset e_i$ such that

$$\|u_i\|(e_i - c_i) < \epsilon/3n.$$

Since H is normal there exists disjoint open sets o_i such that $c_i \subset o_i$ and

$$\|u_i\|(o_i - c_i) < \epsilon/3n,$$

also there exist closed G_δ sets $c_i{}'$ such that

$$c_i \subset c_i{}' \subset o_i.$$

Therefore

$$\left| \sum_{i=1}^{n} u_i(e_i) - \sum_{i=1}^{n} u_i(c_i{}') \right|$$

$$\leqq \left| \sum_{i=1}^{n} u_i(e_i) - \sum_{i=1}^{n} u_i(c_i) \right| + \left| \sum_{i=1}^{n} u_i(c_i) - \sum_{i=1}^{n} u_i(o_i) \right|$$

$$+ \left| \sum_{i=1}^{n} u_i(o_i) + \sum_{i=1}^{n} u_i(c_i{}') \right| \leqq \epsilon.$$

Hence

$$\sum_{i=1}^{n} u_i(e_i) \leqq \epsilon + \sum_{i=1}^{n} u_i(c_i{}').$$

Since H is normal and each c_i is a closed G_δ, pick a sequence $\{f_{k,i}\}$ of continuous real-valued functions such that $0 \leqq f_{k,i}(t) \leqq 1$ for all t and $f_{k,i}(t) = 1$ on $c_i{}'$, the support of f_{k_i}, supp $f_{k_i} \subset o_i$, and $f_{k,i} \searrow^k \chi_{c_i'}$, for each i. Then

$$\left| \sum_{i=1}^{n} u_i(c_i{}') \right| = \left| \lim_k \sum_{i=1}^{n} \int_H f_{k,i} du_i \right|,$$

since

$$\int_H |f_{k,i} - \chi_{c_i'}| = \int_{u_{k_i} - c_i} |f_{k,i} - \chi_{c_i'}| du_i$$
$$\leqq 2\|u_i\|(u_{k_i} - c_i{}')$$

where $u_{k_i} \supset$ supp f_{k_i} and the u_{k_i} may be chosen such that $\|u_i\|(u_{k_i} - c_i{}') < 1/k$. We have

$$\int_H f_{k_i} du_i = f^*(f_{k_i} \cdot x_i),$$

and so,

$$\left| \lim_k \sum_{i=1}^n \int_H f_{k_i} \cdot du_i \right| = \lim_k \left| f^*\left(\sum_{i=1}^n f_{k_i} \cdot x_i \right) \right|$$

$$\leqq \overline{\lim_k} \left\| \sum_{i=1}^n f_{k_i} \cdot x_i \right\| \leqq \overline{\lim_k} \sup_{t \in H} \left| \sum_{i=1}^n f_{k_i}(t) \cdot x_i \right|$$

$$\leqq \overline{\lim_k} \sup_{t \in H} \sum_{i=1}^n |f_{k_i}(t)| \, \|x_i\|$$

$$\leqq \overline{\lim_k} \sup_{t \in H} \sum_{i=1}^n |f_{k_i}(t)| \max_i \|x_i\|.$$

But, since the supp $f_{k,i}$ are disjoint and $0 \leqq f_{k,i}(t) \leqq 1$,

$$\sum_{i=1}^n f_{k_i}(t) \leqq 1 \quad \text{for all } t \in H.$$

Hence

$$\left| \sum_{i=1}^n u_i(c_i{}') \right| \leqq \max_i \|x_i\|.$$

Thus

$$\left(\sum_{i=1}^n \chi_{e_i} \cdot x_i \right) (f^*) \leqq \epsilon + \max_i \|x_i\|$$

for all $\epsilon > 0$ and for all f^*, $\|f^*\| \leqq 1$, so

$$\left\| \sum_{i=1}^n \chi_{e_i} \cdot x_i \right\| \leqq \max_i \|x_i\|.$$

As before X and Y are linear normed spaces and $B(X, Y^{**})$ will denote the space of all bounded linear transformations from X to Y^{**}. Let K be any finitely additive set function defined on F with values in $B(X, Y^{**})$.

DEFINITION 3.1. The set function K is said to be weakly regular if for each x in X and y^* in Y^*, the real-valued set function $y^*K(\cdot)x$ is regular.

DEFINITION 3.2. The set function K is said to satisfy the Gowurin property if there exists a constant P such that for any partition $e_1, e_2, \cdots, e_n$ of H, with e_i in F and for any choice of x_i in X, the following holds:

$$\left\| \sum_{i=1}^{n} K(e_i)\cdot x_i \right\|_{Y^{**}} \leqq P\cdot \max_i \|x_i\|_X.$$

The greatest lower bound of the constants P is called the X-Gowurin constant for K.

DEFINITION 3.3. A function f from H to X is said to be integrable with respect to K if for each $\epsilon > 0$ there exists an ϵ-partition of H with respect to f, and there exists a point y^{**} in Y^{**} such that for each $d > 0$, there exists a partition P of H into elements of F, such that if $e_1, e_2, \cdots, e_n$ is any refinement of P, with e_i in F, then for any choice of t_i in e_i,

$$\left\| y^{**} - \sum_{i=1}^{n} K(e_i)f(t_i) \right\|_{Y^{**}} < d.$$

We denote the point y^{**} by $\int_H dK\cdot f$.

Note. It is clear that any function of the form $\sum_{i=1}^{n} \chi_{E_i}\cdot x_i$, E_i in F, and x_i in X, is integrable and

$$\int_H dK\left(\sum_{i=1}^{n} \chi_{E_i}\cdot x_i\right) = \sum_{i=1}^{n} K(E_i)\cdot x_i.$$

Now let T denote a continuous linear transformation from $C_B(H, X)$ to Y.

LEMMA 3.2. *For each such T, there exists a finitely additive, weakly regular, Gowurin set function K, defined on F with values in $B(X, Y^{**})$, given by*

$$K(e)\cdot x = T^{**}(\chi_e\cdot x),$$

for each e in F and x in X.

PROOF. Consider a partition $\{e_1, e_2, \cdots, e_n\}$ of H, with e_i in H and $x_1, x_2, \cdots, x_n$ in X. We have

$$\left\| \sum_{i=1}^{n} K(e_i)\cdot x_i \right\|_{Y^{**}} = \left\| T^{**}\left(\sum_{i=1}^{n} \chi_{e_i}\cdot x_i\right)\right\|_{Y^{**}}$$

$$\leqq \|T\| \left\| \sum_{i=1}^{n} \chi_{e_i}\cdot x_i \right\|_{C_B^{**}} \leqq \|T\| \max_i \|x_i\|,$$

from Lemma 3.2. Now for y^* in Y^* and x in X, let $\lambda(e) = y^*K(e)\cdot x$, then

$$\begin{aligned}\lambda(e) &= y^*K(e)\cdot x = y^*(T^{**}(\chi_e\cdot x)) = (\chi_e\cdot x)(T^*(y^*))\\ &= u_{x,\,T^*(y^*)}(e)\end{aligned}$$

where $u_{x,\,T^*(y^*)}$ is regular.

LEMMA 3.3. *For f in $C_B(H, R)$ and x in X, $f\cdot x$ is integrable.*

PROOF. Consider $\epsilon > 0$. Since f is bounded and continuous there exists an ϵ-partition $P = \{e_1, e_2, \cdots, e_n\}$ of H with respect to f, with e_i in F for each i. Hence let $y^{**} = T^{**}(f\cdot x)$ and let $\{E_1, E_2, \cdots, E_m\}$ be any refinement of P, with $E_j \in F$. Then if $x_j \in f\cdot x(E_j)$, $x_j = r_j\cdot x$ where $r_j \in f(E_j)$. Therefore,

$$\begin{aligned}&\left\| y^{**} - \sum_{j=1}^{m} K(E_j)\cdot x_j \right\|_{Y^{**}}\\ &\quad= \left\| T^{**}(f\cdot x) - T^{**}\left(\sum_{j=1}^{m} \chi_{E_j}\cdot r_j\cdot x\right)\right\|_{Y^{**}}\\ &\quad\leqq \|T\|\cdot\left\| f\cdot x - \sum_{j=1}^{m} \chi_{E_j}\cdot r_j\cdot x\right\|_{C_B^{**}}.\end{aligned}$$

Consider c^* in $C_B{}^*$ such that $\|c^*\| \leqq 1$, then since

$$\int_H \chi_{E_j}\cdot du_{c^*,\,r_j x} = \int_H r_j\cdot\chi_{E_j}\cdot du_{c^*,\,x}$$

we have

$$\begin{aligned}&\left|\left(f\cdot x - \sum_{j=1}^{m} \chi_{E_j}\cdot r_j\cdot x\right)(c^*)\right|\\ &\quad= \left|\int_H f\cdot du_{c^*,\,x} - \int_H \sum_{j=1}^{m} \chi_{E_j}\cdot r_j du_{c^*,\,x}\right| \leqq \epsilon\cdot\|x\|_X.\end{aligned}$$

The previous lemma along with Lemma 3.1 gives us the following Riesz representation theorem.

THEOREM 3.1. *Let T be a continuous linear transformation from $C_B(H, X)$ to Y, then there exists a unique weakly regular, finitely additive, $B(X, Y^{**})$ valued, Gowurin set function defined on F, such that*

$$[T(f\cdot x)]^{**} = \int_H dK(f\cdot x)$$

for all f in $C_B(H, R)$ and all x in X.

PROOF. The existence of K follows from Lemma 3.1, and from Lemma 3.3, we have that for all f in $C_B(H, R)$ and x in X,

$$T^{**}(f\cdot x) = [T(f\cdot x)]^{**} = \int_H dK(f\cdot x).$$

The uniqueness will follow from the same technique as we will use in Theorem 3.2.

THEOREM 3.2. *Let T be a continuous linear transformation from $C_B(H, X)$ to Y, then there exists a unique, finitely additive, weakly regular, Gowurin set function defined on F, with values in $B(X, Y^{**})$ such that every f in $C_{TB}(H, X)$ is integrable with respect to K. Moreover,*

$$T^{**}(f) = \int_H dK\cdot f,$$

for all f in $C_{TB}(H, X)$.

PROOF. From Lemma 3.2, let K be the finitely additive, weakly regular, Gowurin set function, defined on F, which is given by the equation

$$K(e)\cdot x = T^{**}(\chi_e\cdot x).$$

From Theorem 3.1 it will follow that each f in $C_{TB}(H, X)$ is integrable with respect to K, and that

$$T^{**}(f) = \int_H dK\cdot f,$$

once it is shown that the collection of functions of the form $\{f\cdot x\}$, f in $C_B(H, R)$ and x in X, are dense in $C_{TB}(H, X)$ in the uniform norm. This is shown as follows: given f in $C_{TB}(H, X)$, and $\epsilon > 0$, there exists a finite cover $N(f(t_i), \epsilon/2)$ of the range of f. Let

$$V_i = \{t \mid \|f(t) - f(t_i)\| < \epsilon\},$$

and

$$W_i = \{t \mid \|f(t) - f(t_i)\| < \epsilon/2\},$$

then

$$\bigcup_{i=1}^{n} W_i \supset H \quad \text{and} \quad \overline{W}_i \subset V_i.$$

Let H_i be the union of all the $\overline{W}_i$ contained in V_i. By Urysohn's lemma there exists $\{g_i\}$ continuous with $0 \leqq g_i \leqq 1$, $g_i = 1$ on H_i, and $g = 0$ off V_i. Let

$$h_1 = g_1, \qquad h_2 = (1 - g_1) \cdot g_2, \cdots,$$
$$h_n = (1 - g_1)(1 - g_2) \cdots (1 - g_{n-1})g_n,$$

then

$$h_1 + h_2 + \cdots + h_n = 1, \quad \text{and } h_i = 0 \text{ off } V_i,$$

and

$$\left\| \sum_{i=1}^{n} h_i x_i - f \right\| < \epsilon$$

where $x_i = f(t_i)$.

Now suppose K' is any other weakly regular, finitely additive, Gowurin, $B(X, Y^{**})$ valued set function defined on F such that $\int_H dK \cdot f$ exists for all f in $C_{TB}(H, X)$. Then for f in $C_B(H, R)$ and x in X, we have from Theorem 3.1 for y^* in Y^*,

$$\langle T(f \cdot x), y^* \rangle = \left\langle \int_H dK'(f \cdot x), y^* \right\rangle.$$

Therefore

$$\langle T(f \cdot x), y^* \rangle = \langle f \cdot x, T^*(y^*) \rangle = \int_H f\, du_{x,\, T^*(y^*)}.$$

Since

$$U_{x,\, T^*(y^*)}(e) = \langle \chi_e \cdot x, T^*(y^*) \rangle = \langle T^{**}(\chi_e \cdot x), y^* \rangle$$
$$= \langle K(e) \cdot x, y^* \rangle,$$

let

$$\lambda(e) = \langle K'(e) \cdot x, y^* \rangle.$$

Then by showing that

$$\int_H f \cdot d\lambda = \int_H f\, du_{x,\, T^*(y^*)}$$

for all f in $C_B(H, R)$, since λ and $u_{x,\, T^*(y^*)}$ are regular, we conclude from the uniqueness of the measure in [**1**, p. 262], that $\lambda = u_{x,\, T^*(y^*)}$.

Consider χ_e, e in F, then

$$\int_H \chi_e d\lambda = \lambda(e) = \langle K'(e) \cdot x, y^* \rangle = \left\langle \int_H dK'(\chi_e \cdot x), y^* \right\rangle.$$

Hence,

$$\int_H f d\lambda = \left\langle \int_H dK'(f \cdot x), y^* \right\rangle,$$

and since

$$\left\langle \int_H dK'(f \cdot x), y^* \right\rangle = \int_H f \, du_{x,\, T^*(y^*)},$$

we have $\lambda = u_{x,\, T^*(y^*)}$. Thus

$$\langle K(e) \cdot x, y^* \rangle = \langle K'(e) \cdot x, y^* \rangle$$

for all x in X and y^* in Y^*, and therefore $K = K'$.

4. Definition 4.1. A function f from H to X is said to be integrable with respect to K, over a set E in F, if for each $\epsilon > 0$, there exists an ϵ-partition of E with respect to f, and there exists a point y^{**} in Y^{**} such that for each $d > 0$, there exists a partition P of E into elements of F, such that if $e_1, e_2, \cdots, e_n$ is any refinement of P, with e_i in F, then for any choice of t_i in e_i,

$$\left\| y^{**} - \sum_{i=1}^{n} K(e_i) \cdot f(t_i) \right\|_{Y^{**}} < d.$$

We denote y^{**} by $\int_E dK \cdot f$.

Lemma 4.1. *Consider E in F. If K is any finitely additive set function defined on F with values in $B(X, Y^{**})$, which satisfies the Gowurin property over H, then for any two partitions $\{P_1, P_2, \cdots, P_n\}$ and $\{Q_1, Q_2, \cdots, Q_m\}$ of E, with P_i and Q_j in F, and for any choice of $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_m$ in X, then*

$$\left\| \sum_{i=1}^{n} K(P_i) \cdot x_i - \sum_{i=1}^{m} K(Q_j) \cdot y_j \right\|_{Y^{**}} \leqq W \max_{i,j} \|x_i - y_j\|_X,$$

where W denotes a Gowurin constant for K.

Proof. This follows directly from the fact that if $\{e_1, e_2, \cdots, e_n\}$ is a partition of E, $\{e_1, e_2, \cdots, e_n, H - E\}$ is a partition of H and the fact that K is Gowurin over H.

Theorem 4.1. *If f is any K-integrable function over H and E is any element of F, then f is K-integrable over E. Moreover, $\int_E dK \cdot f = \int_H dK(\chi_E \cdot f)$.*

Proof. Consider a sequence $\{\epsilon_n\}$ of positive numbers such that $\epsilon_n \searrow 0$ as $n \to \infty$. For each n, there exists an ϵ_n-partition of E with respect to f. If we denote this partition by $P_{\epsilon_n} = \{P_1{}^n, P_2{}^n, \cdots, P^n_{r(n)}\}$ and we let $y_n = \sum_{i=1}^{r(n)} K(P_i{}^n) f(t_i{}^n)$ for a choice of $t_i{}^n$ in

P_i^n, then from Lemma 4.1, $\{y_n\}$ forms a Cauchy sequence in Y^{**} and hence converges to a y^{**} in Y^{**} since Y^{**} is complete. Also from Lemma 4.1 it follows that the convergence to y^{**} does not depend on the choice of t_i^n in P_i^n.

For the last part of the theorem we note that if we let $P_{r(n)+1}^n = H - E$, then

$$y_n = \sum_{i=1}^{r(n)+1} K(P_i^n)\chi_E \cdot f(t_i^n)$$

which converges to $\int_H dK(\chi_E \cdot f)$.

From Theorem 3.2, we have:

COROLLARY 4.1. *Every f in $C_{TB}(H, X) \cup S_F(H, X)$ is K-integrable over every E in F.*

DEFINITION 4.2. If G is a finitely additive set function defined on F with values in a linear normed space S, the semivariation of G over a set E in F is defined to be

$$v(G, E) = \sup \left\| \sum_{i=1}^{n} r_i G(e_i) \right\|_S$$

where the supremum is taken over all finite partitions $\{e_1, e_2, \cdots, e_n\}$ of E, with e_i in F, and over all finite collections $\{r_1, r_2, \cdots, r_n\}$ of real numbers with $|r_i| \leqq 1$ for all i.

We now denote $P(H, X) = \text{span}\,(C_{TB}(H, X) \cup S_F(H, X))$ and for f in $P(H, X)$ we let $\lambda_f(E) = \int_E dK \cdot f$. Then λ_f is a finitely additive set function from F to Y^{**}. For f any bounded function from H to X, we will use the notation $\|f\|_C = \sup_{t \in H}\|f(t)\|_X$. Hence for f in $P(H, X)$ since K is Gowurin,

$$\|\lambda_f(E)\| \leqq W_K\|f\|_C$$

where W_K denotes the Gowurin constant for K over H. From Lemma 4, p. 320 of [1], we conclude that

$$V(\lambda_F, E) \leqq 2W_K\|f\|_C$$

for all E in F.

LEMMA 4.2. *For f in $P(H, X)$, $v(\lambda_f, H) = 0$ if and only if $\int_E dK \cdot f = \theta_{Y^{**}}$ for all E in F.*

PROOF. The proof follows easily from the inequality

$$\left\| \int_E dK \cdot f \right\|_{Y^{**}} \leqq v(\lambda_f, H).$$

DEFINITION 4.3. If $f_1, f_2 \in P(H, X)$ we say that f_1 is equivalent to f_2, $f_1 \sim f_2$, if and only if $\int_E dK \cdot f_1 = \int_E dK \cdot f_2$ for all E in F.

It follows from Lemma 4.2, that $f_1 \sim f_2$ if and only if $v(\lambda_{f_1 - f_2}, H) = 0$. Moreover, this relation is an equivalence relation on $P(H, X)$, and we denote the equivalence class determined by f, by $[f]$.

DEFINITION 4.4. For f in $P(H, X)$, we define

$$\|[f]\|_1 = \|[f]\|_K^1 = v(\lambda_f, H).$$

The fact that $\|\cdot\|_K^1$ is a norm on the collection $\tilde{P}(H, X)$ of equivalence classes $[f]$, f in $P(H, X)$ follows from Lemma 4.2 and the following easily established lemma.

LEMMA 4.3. *For f_1 and f_2 in $P(H, X)$, if we denote $\lambda_1(E) = \int_E dK \cdot f_1$, $\lambda_2(E) = \int_E dK \cdot f_2$, and $(k\lambda_1)(E) = \int_E dK \cdot (kf_1)$, then*

(a) $v(\lambda_1 + \lambda_2, H) \leqq v(\lambda_1, H) + v(\lambda_2, H)$, *and*

(b) $v(k\lambda_1, H) = |k| v(\lambda_1, H)$.

DEFINITION 4.5. We define the space $L_K{}^1(H, X)$ to be the completion of $\tilde{P}(H, X)$, the completion being in the norm $\|\cdot\|_K^1$.

COROLLARY. *The collection $\tilde{S}_F(H, X)$ is dense in the space $L_K{}^1(H, X)$ in the norm $\|\cdot\|_K^1$.*

PROOF. This follows directly from the inequality

$$v(\lambda_f, H) \leqq 2W_K \|f\|_C$$

for all f in $P(H, X)$ and for all E in F.

REMARK. It is pointed out by the authors that the results of §3 in [2] could all be proved in this setting, the reader is referred to that paper for statements and proofs.

5. **A representation theorem.** Consider K as above and suppose that G is any finitely additive set function defined on F with values in $B(X, Z^{**})$, Z being a linear normed space.

DEFINITION 5.1. The set function G is said to be strongly Lipschitz with respect to K if and only if there exists a constant P such that for any E in F, any partition $\{e_1, e_2, \cdots, e_n\}$ of E, with e_i in F, and any collection $\{x_1, x_2, \cdots, x_n\}$ of elements of X, then there exists a partition $\{E_1, E_2, \cdots, E_m\}$ of H and a collection $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ of real numbers with E_j in F and $|\alpha_j| \leqq 1$, such that

$$\left\| \sum_{i=1}^{n} G(e_i) \cdot x_i \right\|_{Z^{**}} \leqq P \left\| \sum_{i,j}^{n,m} K(e_i \cap E_j)\alpha_j x_i \right\|_{Y^{**}}$$

The greatest lower bound of the numbers P is called the strong Lipschitz constant for G with respect to K.

LEMMA 5.1. *If K satisfies the Gowurin property and G is strongly Lipschitz with respect to K, then G is Gowurin over H.*

PROOF. The proof follows directly from the definition, the Gowurin constant for G being less than or equal to the product of the Gowurin constant for K and the strong Lipschitz constant for G.

The representation theorem and its converse are now stated, the proofs being very similar to those in [2].

THEOREM 5.1. *Let A be any continuous linear transformation from $L_K{}^1(H, X)$ to a linear normed space Z, then there exists a finitely additive set function G, defined on F with values in $B(X, Y^{**})$, such that G satisfies the strong Lipschitz condition with respect to K, and such that $[A(f)]^{**} = \int_H dG \cdot f$, for all f in $L_K{}^1(H, X)$, the integral being defined as before.*

PROOF. We simply point out that A is continuous on $L_K{}^1(H, X)$ in the norm $\| \cdot \|_C$, and refer the reader to [2] for details.

THEOREM 5.2. *Let G be any additive set function defined on F with values in $B(X, Z^{**})$, where G is strongly Lipschitz with respect to K. Then $\int_H dG \cdot f$ exists for all f in $L_K{}^1(H, X)$ and defines a* continuous linear transformation from $L_K{}^1(H, X)$ *to* Z^{**}.

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INDIANA STATE UNIVERSITY, TERRE HAUTE, INDIANA 47809