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OSCILLATION PROPERTIES OF THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Oscillation properties of elements of possible bases for the solution space of a third order linear differential equation are considered.

1. **Introduction.** We will consider the differential equation

$$y''' + p(x)y' + q(x)y = 0 \tag{1}$$

and its adjoint

$$y''' + p(x)y' + (p'(x) - q(x))y = 0, \tag{2}$$

where we will assume that the coefficients are continuous on $[0, +\infty)$. In particular, we will consider equations which are of Class I or Class II as defined by Hanan [1].

We will consider a solution of (1) oscillatory if it changes sign for arbitrarily large x.

It has been shown by Utz [3], that the solution space of equation (1) can have at the same time a basis consisting of i oscillatory solutions and $3 - i$ nonoscillatory solutions, for $i = 0, 1, 2, 3$.

We will describe the types of bases possible for the solution spaces of equations (1) of Class I and Class II, with respect to the number of oscillatory solutions possible in a given basis. In doing so, we will generalize a theorem of Utz [3].

2. An equation (1) is said to be Class I if any solution for which $y(a) = y'(a) = 0$, $y''(a) > 0$ is positive on $[0, a)$. It is said to be Class II if any solution for which $y(a) = y'(a) = 0$, $y''(a) > 0$ is positive on $(a, +\infty)$. It was shown by Hanan [1] that (1) is Class I if and only if (2) is Class II.

In [1], Hanan considers a solution $y(x)$ of (1) to be oscillatory if it has an infinity of zeros in $[0, +\infty)$, but it follows from the definitions that if (1) is Class I or Class II, then this definition of oscillation implies $y(x)$ must change signs for arbitrarily large x.

We will use a method similar to that used by Lazer [2, p. 437] to prove the following lemma.

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Lemma. *If* (1) *is Class* I, *and if* (1) *has an oscillatory solution, then there exists a nontrivial nonoscillatory solution such that* $y(x) > 0$ *for* $x \in [0, +\infty)$.

Proof. Let $u(x)$, $v(x)$, $w(x)$ be a basis for the solution space of (1). Let

$$y_n(x) \equiv C_{n,1}u(x) + C_{n,2}v(x) + C_{n,3}w(x),$$

where $y_n(n) = y_n'(n) = 0$, $y_n''(n) > 0$, and where $C_{n,1}^2 + C_{n,2}^2 + C_{n,3}^2 = 1$. Suppose further, without loss of generality, that $\lim C_{n,i} = C_i$ for $i = 1, 2, 3$. Let

$$y(x) = C_1u(x) + C_2v(x) + C_3w(x).$$

Since $\{y_n(x)\}$ converges to $y(x)$ uniformly on any finite subinterval of $(0, +\infty)$, we have $y(x) \geqq 0$. Now $y(x) \not\equiv 0$ since $C_1{}^2 + C_2{}^2 + C_3{}^2 = 1$. Further, by [1] if there is an x_1 such that $y(x_1) = 0$, then y is oscillatory. Thus, $y(x) > 0$ for all x.

Using Lemma 1.1 of [2], we observe that Utz in Theorem 2 [3] is considering a special equation of Class I. Thus the following theorem will generalize the result of Utz.

Theorem 1. *If* (1) *is Class* I, *and if some solution oscillates, then the solution space of* (1) *has a basis with three oscillatory solutions, and a basis with exactly two oscillatory solutions.*

Proof. By the lemma, there is a nonoscillatory solution $w(x)$ of (1). By [1] any solution of (1) that vanishes at least once is oscillatory. Let $w(x)$, $u(x)$, $v(x)$ be solutions of (1) which form a basis, where $w(x)$ is nonoscillatory and $u(x)$ is oscillatory.

Let $a \in (0, \infty)$ such that $u(a)v(a) \neq 0$. Choose constants k_1 and k_2 such that $v(a) + k_1w(a) = 0$ and $v(a) + k_2u(a) = 0$. Then, $y_1(x) \equiv v(x) + k_1w(x)$ is oscillatory, $y_2(x) \equiv v(x) + k_2u(x)$ is oscillatory, and $u(x)$ is oscillatory. Further

$$v(x) + k_2u(x) - k_2u(x) \equiv v(x),$$

$$v(x) + k_1w(x) - v(x) \equiv k_1w(x).$$

Since $k_1 \neq 0$, $y_1(x)$, $y_2(x)$, and $u(x)$ forms a basis for the solution space of (1).

Also, $u(x)$, $y_2(x)$, and $w(x)$ is a basis for the solution space of (1).

We will now consider an equation of Class II.

Theorem 2. *If* (1) *is Class* II, *and if some solution oscillates, then the solution space of* (1) *has a basis consisting of exactly i oscillatory solutions, for* $i = 0, 1, 2$.

PROOF. Since (1) has an oscillatory solution, (2) has an oscillatory solution by [1]. Also, since (1) is Class II, (2) is Class I. Let $u_1(x)$, $u_2(x)$, $u_3(x)$ be a basis for the solution space of (2) such that $u_1(x)$ is nonoscillatory, and such that $u_2(x)$ and $u_3(x)$ are oscillatory. Then $u_1(x)$, $u_2(x)$, and

$$u_3(x) + \lambda u_1(x) \equiv w_3(x)$$

is a basis where λ is chosen such that

$$u_3(a) + \lambda u_1(a) = u_2(a) = 0$$

for some $a \in [0, +\infty)$. Note that $u_1(x)$ is nonoscillatory, but $u_2(x)$ and $w_3(x)$ are oscillatory. Now

$$U_1(x) \equiv u_1(x)u_2'(x) - u_2(x)u_1'(x),$$

$$U_2(x) \equiv u_1(x)w_3'(x) - w_3(x)u_1'(x),$$

$$U_3(x) \equiv u_2(x)w_3'(x) - u_2'(x)w_3(x),$$

is a basis for the solution space of (1). It is clear that $U_1(x)$, and $U_2(x)$ are oscillatory solutions. Now $U_3(a) = U_3'(a) = 0$ implies $U_3(x)$ is nonoscillatory since it is a nontrivial solution of (1) which is Class II.

Let $u_1(x)$, $u_2(x)$, $u_3(x)$ be a basis for the solution space of (2) such that each is oscillatory. Let $a \in [0, +\infty)$ be such that $u_1(a) = 0$. Not both $u_2(a)$ and $u_3(a) = 0$. Suppose $u_3(a) \neq 0$. Choose a constant λ such that $u_2(a) + \lambda u_3(a) = 0$. Let

$$v_2(x) \equiv u_2(x) + \lambda u_3(x).$$

Now $u_1(x)$, $v_2(x)$, and $u_3(x)$ is a basis for (2) where each oscillates. Since $u_1(a) = v_2(a) = 0$ and they are linearly independent, their zeros separate on $(a, +\infty)$ [1]. Suppose b is the first zero of $u_1(x)$ to the right of a, and c is the first zero of $v_2(x)$ to the right of a. Suppose further that $b < c$. Since

$$u_1(b)v_2(c) - u_1(c)v_2(b) \neq 0$$

we can solve

$$0 = c_1u_1(b) + c_2v_2(b) + c_3u_3(b),$$

$$0 = c_1u_1(c) + c_2v_2(c) + c_3u_3(c),$$

where $c_3 \neq 0$. Let

$$v_3(x) \equiv c_1u_1(x) + c_2v_2(x) + c_3u_3(x).$$

Since $c_3 \neq 0$, $u_1(x)$, $v_2(x)$, $v_3(x)$ is a basis for (2) where each is oscillatory. Now

$$W_1(x) \equiv u_1(x)v_2'(x) - v_2(x)u_1'(x),$$
$$W_2(x) \equiv u_1(x)v_3'(x) - v_3(x)u_1'(x),$$
$$W_3(x) \equiv v_2(x)v_3'(x) - v_3(x)v_2'(x),$$

is a basis for (1). Note that $W_1(a) = W_1'(a) = W_2(b) = W_2'(b) = W_3(c) = W_3'(c) = 0$, and since (1) is Class II each is nonoscillatory.

The fact that (1) also has a basis with exactly one oscillatory solution follows immediately.

Let $u_1(x)$, $u_2(x)$, and $u_3(x)$ be a basis for (2) such that $u_1(a) = u_2(a) = 0$ for some $a \in [0, +\infty)$ and such that $u_3(x) > 0$ for all x. Then

$$U_1(x) \equiv u_1(x)u_3'(x) - u_3(x)u_1'(x),$$
$$U_2(x) \equiv u_2(x)u_3'(x) - u_3(x)u_2'(x),$$
$$U_3(x) \equiv u_1(x)u_2'(x) - u_2(x)u_1'(x).$$

As before, $U_1(x)$, $U_2(x)$, and $U_3(x)$ form a basis for (1), $U_1(x)$ and $U_2(x)$ are oscillatory, but $U_3(x)$ is nonoscillatory.

THEOREM 3. *If* (1) *is Class* II, *and if the solution space of* (1) *has a basis consisting of three oscillatory solutions, then the solution space of* (2) *has a basis with one oscillatory and two nonoscillatory solutions.*

PROOF. Let $U_1(x)$, $U_2(x)$, and $U_3(x)$ be as in the last paragraph. Since $U_1(x)U_2'(x) - U_2(x)U_1'(x) = ku_3(x) \neq 0$, since $k \neq 0$ and $u_3(x) > 0$, the zeros of $U_1(x)$ and $U_2(x)$ separate. If (1) has a basis with three oscillatory solutions, then some oscillatory solution $z(x)$ must be of the form

$$z(x) \equiv U_3(x) + c_1U_1(x) + c_2U_2(x).$$

Let $x_1 < x_2 < \cdots$ be the consecutive zeros of $z(x)$. Define

$$y_n(x) \equiv k_{1,n}U_1(x) + k_{2,n}U_2(x),$$

where $k_{1,n}^2 + k_{2,n}^2 = 1$ and $y_n(x_n) = 0$. The zeros of $y_n(x)$ and $z(x)$ separate to the left of x_n by [1]. Suppose, without loss of generality, that $\lim k_{n,i} = k_i$ for $i = 1, 2$. Let

$$y(x) \equiv k_1U_1(x) + k_2U_2(x).$$

Since $\{y_n(x)\}$ converges to $y(x)$ uniformly on $[x_j, x_{j+1}]$, and since each $y_n(x)$ for $n > j + 2$ changes signs on $[x_j, x_{j+1}]$, $y(x)$ must have a zero on $[x_j, x_{j+1}]$. Since $k_1^2 + k_2^2 = 1$, $y(x)$ and $z(x)$ are clearly linearly independent. Thus by [1] $y(x)$ and $z(x)$ cannot have two zeros in common. Hence for $j \geqq N$ for some $N > 0$, $y(x)$ has a zero in (x_j, x_{j+1}).

Since $y(x)$ is a solution to (1) which is of Class II, it must change signs in (x_j, x_{j+1}).

Suppose

$$y(x_0)z'(x_0) - z(x_0)y'(x_0) = 0.$$

Then the equations

$$l_1y(x_0) + l_2z(x_0) = 0, \qquad l_1y'(x_0) + l_2z'(x_0) = 0,$$

can be solved for l_1 and l_2 not both zero.

Let

$$w(x) \equiv l_1y(x) + l_2z(x).$$

Since $w(x_0) = w'(x_0) = 0$, $w(x)$ is of constant sign for $x > x_0$. But this is not possible since when $j \geqq N$, $x_j > x_0$, and $l_2z(x) \geqq 0$ on $[x_j, x_{j+1}]$ there is an $a \in (x_j, x_{j+1})$ such that $l_1y(a) \geqq 0$ and $b \in (x_{j+1}, x_{j+2})$ such that $l_1y(b) \leqq 0$. Thus

$$l_1y(a) + l_2z(a) \geqq 0 \quad \text{and} \quad l_1y(b) + l_2z(b) \leqq 0.$$

Hence

$$y(x)z'(x) - z(x)y'(x)$$

is a nonoscillatory solution of (2).

Now

$$\begin{aligned} y(x)z'(x) &- z(x)y'(x) \\ &\equiv (k_1U_1(x) + k_2U_2(x))(U_3'(x) + c_1U_1'(x) + c_2U_2'(x)) \\ &\quad - (U_3(x) + c_1U_1(x) + c_2U_2(x))(k_1U_1'(x) + k_2U_2'(x)) \\ &\equiv k(k_1u_1(x) + k_1c_2u_3(x) + k_2u_2(x) + k_2c_1u_3(x)) \end{aligned}$$

where $k \neq 0$. Since k_1 and k_2 are not both zero, $y(x)z'(x) - z(x)y'(x)$ and $u_3(x)$ are linearly independent solutions of (2).

THEOREM 4. *If* (1) *is Class* I, *if some solution oscillates, and if it has a basis with two or three nonoscillatory elements, then* (2) *has a basis with three oscillatory elements.*

PROOF. If (1) has a basis with all nonoscillatory solutions, then it clearly has one with exactly one oscillatory solution. Suppose $u_1(x)$, $u_2(x)$, and $u_3(x)$ is a basis for the solution space of (1) where $u_1(x)$ is oscillatory and $u_2(x)$ and $u_3(x)$ are nonoscillatory. Let us suppose $u_2(a) = u_3(a) > 0$ for some $a \in (0, +\infty)$. Now

$$W_1(x) \equiv u_1(x)u_2'(x) - u_2(x)u_1'(x),$$

$$W_2(x) \equiv u_1(x)u_3{}'(x) - u_3(x)u_1{}'(x),$$
$$W_3(x) \equiv u_2(x)u_3{}'(x) - u_3(x)u_2{}'(x),$$

is a basis for (2). Clearly $W_1(x)$ and $W_2(x)$ are oscillatory. Since $u_2(a) = u_3(a)$, we have

$$y(x) \equiv u_2(x) - u_3(x)$$

is oscillatory. Let $a < a_1 < a_2 \cdots$ be consecutive zeros of $y(x)$. Then $y'(a_i)y'(a_{i+1}) < 0$. Thus, $W_3(a_i) = u_2(a_i)(u_3{}'(a_i) - u_2{}'(a_i))$ must have opposite signs at consecutive zeros of $y(x)$.

An example of a differential equation satisfying the hypothesis of Theorem 4 will now be given.

Example. Consider the differential equation

$$y''' + y' + [2/(\exp(x) + 2)](y + y'') = 0 \tag{3}$$

whose general solution is given by

$$y(x) = C_1 \sin x + C_2 \cos x + C_3 (1 + \exp(-x)).$$

Clearly $\sin x$, $2(1 + \exp(-x)) + \cos x$, $1 + \exp(-x)$ is a basis for the solution space of (3) with one oscillatory and two nonoscillatory elements.

By letting $y = w \exp(-\frac{1}{3} \int_0^t P(s)\, ds)$, where $P(s) \equiv 2/(\exp(x) + 2)$, (3) can be transformed into an equation of the form (1), which will have the same oscillatory properties as (3) and will be of Class I if (3) is of Class I.

The equation (3) is of Class I, for if it were not there would have to exist a nontrivial solution of (3) satisfying $y(a - \delta) = y(a) = y'(a) = 0$ for some a and positive δ. But that is not possible since

$$\begin{vmatrix} \sin a & \cos a & 1 + \exp(-a) \\ \cos a & -\sin a & -\exp(-a) \\ \sin(a - \delta) & \cos(a - \delta) & 1 + \exp(-a + \delta) \end{vmatrix}$$
$$= \exp(-a)[\cos\delta + \sin\delta - \exp\delta] + \cos\delta - 1$$
$$< \exp(-a)[\cos\delta + \sin\delta - 1 - \delta] \leqq 0.$$

Applying Theorem 4, it is clear that the adjoint of (3) satisfies the hypotheses of Theorem 3. The next theorem shows that this is not always the case.

Theorem 5. *If in* (1) $q(x) > 0$, $p(x) \leqq 0$ (*consequently it is of Class* I), $2p(x)/q(x) + d^2(q(x)^{-1})/dx^2 \leqq 0$, *and if some solution oscillates,* *then every basis for* (1) *is of one of the types of Theorem* 1.

The proof of the theorem follows directly from a result due to Lazer [2, p. 444].

References

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