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# OBSTRUCTIONS TO EMBEDDING AND ISOTOPY IN THE METASTABLE RANGE

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## 1. Introduction.

1.1. *Preliminary definitions and summary.* Throughout this paper, "manifold" means differentiable manifold (closed or open) without boundary, with a countable base. "Differentiable" means infinitely differentiable, and "embedding" means differentiable embedding.

Suppose $V$ and $M$ are manifolds of dimension $k$ and $n$, respectively, $V$ compact, and $f: V \to M$ is a differentiable map. An *embedding homotopy* of $f$ (abbreviated $e$-homotopy) shall be defined to be a homotopy of differentiable maps, $f_t : V \to M$, for $0 \leqq t \leqq 1$, such that $f_0 = f$ and $f_1$ is an embedding. We say that $e$-homotopies $\{f_{0,t}\}$ and $\{f_{1,t}\}$ are *isotopic* if there exists a 2-parameter homotopy of differentiable maps $f_{\tau,t} : V \to M$, for $0 \leqq \tau, t \leqq 1$, such that $f_{\tau,0} = f$ and $f_{\tau,1}$ is an embedding for all $\tau$. Let $[f_t]$ denote the isotopy class of $\{f_t\}$, and let $[V \subset M]_f$ denote the set of all isotopy classes of $e$-homotopies of $f$.

It is not difficult to show that if $f$ is an embedding, $[V \subset M]_f$ naturally has the structure of an Abelian group with identity $[f]$ (where $\{f\}$ is the constant homotopy), provided $2n > 3(k + 1)$. However, this construction is not within the scope of the present paper; we refer the reader to J. C. Becker [1] for the case when $M$ is a Euclidean space. $[V \subset R^n]_f$ becomes $E(V, n)$, the so-called embedding group.

We consider three problems in this paper. The first is existence of an $e$-homotopy of $f$, i.e., whether $[V \subset M]_f$ is nonempty; the second is enumeration of $[V \subset M]_f$; more precisely, whether two given $e$-homotopies are isotopic. The third question deals with the function $\Delta : [V \subset M]_f \to [V \subset M]$, where $[V \subset M]$ is the set of isotopy classes of embeddings of $V$ into $M$, and where, for any $e$-homotopy $\{f_t\}$ of $f$, $\Delta[f_t] = [f_1]$, the isotopy class containing $f_1$. As we see in §3.5, there is an action of $\pi_1(M^V, f)$ on $[V \subset M]_f$ whose orbits correspond to the image of $\Delta$, where $M^V$ is the space of differentiable functions $V \to M$ with the compact-open topology. In §3.8, we discuss

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that action in the special case that $k \geqq 2$, $V = S^k$, the $k$-sphere, $n = 2k + 1$, and $f$ is inessential.

We translate the problem of existence and isotopy of $e$-homotopies of $f$ into a lifting problem, using Haefliger's results [4]. In §3.1, we define a $2k$-manifold $\boldsymbol{R}^*V$ with boundary $PV$, the total space of the projective bundle associated with the tangent bundle of $V$. $\boldsymbol{R}^*V$ has the homotopy type of the reduced deleted product of $V$, $(V^2 - \Delta_V)/T$, where $\Delta_V$ is the diagonal and $T$ exchanges coordinates. In § 3.2 we define a pair of spaces $(Y', Z')$ and a map $\pi_M' : (Y', Z') \to (\boldsymbol{R}^*V, PV)$ such that $\pi_M'$ and $\pi_M' \mid Z'$ are both fibrations, and for each $e$-homotopy $\{f_t\}$ of $f$ we define a specific section of $\pi_M'$, $\Phi[f_t] : (\boldsymbol{R}^*V, PV) \to (Y', Z')$. The function $\phi : [M \subset V]_f \to \mathrm{Sec}(\pi_M')$ which sends each $[f_t]$ to $[\Phi[f_t]]$, the class containing $\Phi[f_t]$ (where $\mathrm{Sec}\,(\pi_M')$ is the set of homotopy classes of sections of $\pi_M'$, two sections being homotopic if they are connected by a homotopy of sections) is onto if $2n \geqq 3(k+1)$ and one-to-one if $2n > 3(k+1)$ (see Theorems 3.3.1 and 3.3.2). The obstruction theory for sections of fibrations of pairs, developed in §2, can then be applied.

The first obstruction to finding an $e$-homotopy of $f$ lies in $H^n(\boldsymbol{R}^*V; \pi_{n-1})$, and higher obstructions lie in $H^{n+i}(\boldsymbol{R}^*V; \pi_{n+i-1})$ for $i \geqq 1$, where $\pi_{n+i-1}$ is a sheaf of Abelian groups over $\boldsymbol{R}^*V$ which is not generally even *locally* a product sheaf, for $i \geqq 0$. (When restricted to either $PV$ or $\boldsymbol{R}^*V - PV$, however, $\pi_{n+i-1}$ is locally trivial, i.e., locally a product sheaf.) The first obstruction to isotopy of two $e$-homotopies of $f$ lies in $H^{n-1}(\boldsymbol{R}^*V; \pi_{n-1})$; higher obstructions lie in $H^{n+i-1}(\boldsymbol{R}^*V; \pi_{n+i-1})$ for $i \geqq 1$.

Thus (cf. Theorems 2.5.1 and 3.3.2), $[V \subset M]_f$ is in one-to-one correspondence with $H^{2k}(\boldsymbol{R}^*V; \pi_{2k})$ if $k \geqq 2$ and $n = 2k + 1$. This correspondence is canonical if $f$ is an embedding; $[f]$ then corresponds to 0. Identifying the two sets in that case, we then say that $[V \subset M]_f$ is an Abelian group.

1.2. *Applications.* Suppose now that $V = S^k$, for $k \geqq 2$, and $n = 2k + 1$. Let $x \in M$ be a basepoint, and let $f : S^k \to M$ be a basepoint-preserving embedding. Define $d : \pi_1(M, x) \to Z_2$ to be the orientation homomorphism, i.e., the kernel of $d$ is the image of the fundamental group of the orientation covering space of $M$.

THEOREM 1.2.1. *$[S^k \subset M]_f$ is generated by elements $\langle g \rangle$ for all $g \in \pi_1(M, x)$, subject only to the following relations:*

(i) $\langle 1 \rangle = 0$, *where* 1 *is the identity of* $\pi_1(M, x)$.

(ii) $\langle g^{-1} \rangle = (-1)^{k+1}(-1)^{d(g)}\langle g \rangle$ *for all* $g \in \pi_1(M, x)$.

The reader can easily verify that if $\pi_{k+1}(M, x) = 0$, the evaluation on the basepoint of $S^k$, $M^{S^k} \to M$, induces an isomorphism $\pi_1(M^{S^k}, f) \to \pi_1(M, x)$, provided $f$ is inessential; we identify these groups for convenience.

THEOREM 1.2.2. *Suppose $f$ is small, i.e., $f(S^k)$ is contained in a single chart of $M$, and $\pi_{k+1}(M) = 0$. The action of $\pi_1\langle M, x\rangle$ on $[S^k \subset M]_f$ is given by $(\langle g\rangle, h) \mapsto (-1)^{d(h)}\langle h^{-1}gh\rangle$ for all $g, h \in \pi_1(M, x)$.*

In the following applications, 0 will be a small embedding, $\pi_1$ will be the fundamental group of the space into which we are embedding $S^k$.

THEOREM 1.2.3 (HACON [3]). *For $k \geqq 2$, $[S^k \subset S^1 \times S^{2k}]_0$ is isomorphic to the direct sum of countably many copies of the integers, and the action of $\pi_1 \cong Z$ is trivial.*

Suppose now that $k \geqq 2$, and $P_r$ is a real projective $r$-space, for $k + 2 \leqq r \leqq 2k + 1$. Let $G = [S^k \subset P_r \times R^{2k-r+1}]_0$.

THEOREM 1.2.4. *Case* I. *If $k$ and $r$ are even, $G \cong Z_2$ and the action of $\pi_1$ is trivial. Case* II. *If $k$ and $r$ are both odd, $G \cong Z_2$ and the action of $\pi_1$ is trivial. Case* III. *If $k$ is even and $r$ is odd, $G \cong Z$ and the action of $\pi_1$ is trivial. Case* IV. *If $k$ is odd and $r$ is even, $G \cong Z$ and the action of $\pi_1 \cong Z_2$ is nontrivial; the generator of $\pi_1$ takes every element of $G$ to its inverse.*

Theorems 1.2.3 and 1.2.4 follow immediately from 1.2.1 and 1.2.2, as the reader may easily verify.

1.3. *Embeddings in a Euclidean space.* Let $V$ be a compact $k$-dimensional manifold, as before, and let $M = R^n$. Our obstruction theory then reduces to a simpler theory. The first obstruction to embedding $V$ in $R^n$ lies in $H^n(\boldsymbol{R}^*V; \mathcal{Z})$, where $\mathcal{Z} = Z$ if $n$ is even and $\mathcal{Z}$ is the twisted integer sheaf (sometimes called $Z^T$) if $n$ is odd. Higher obstructions lie in $H^{n+i}(\boldsymbol{R}^*V; \mathcal{Z} \otimes \pi_i)$ for $i \geqq 1$, where $\pi_i$ is the stable $i$-stem in the homotopy of spheres. The first obstruction to isotopy of two embeddings lies in $H^{n-1}(\boldsymbol{R}^*V; \mathcal{Z})$; higher obstructions lie in $H^{n+i-1}(\boldsymbol{R}^*V; \mathcal{Z} \otimes \pi_i)$ for $i \geqq 1$. $V$ embeds in $R^n$ if and only if all obstructions vanish, provided $2n \geqq 3(k + 1)$; two embeddings are isotopic if and only if all obstructions vanish, provided $2n > 3(k + 1)$ [7], [8].

2. **Fibrations of pairs.**

2.1. *Preliminary definitions.* Throughout this section, we let $(K, L)$ be an oriented simplicial pair, $(Y, Z)$ another pair of spaces, and $\pi : (Y, Z) \to (K, L)$ a map of pairs such that $\pi$ and $\pi \mid Z$ are both fibrations. We say that $\pi$ is a *fibration of pairs* over $(K, L)$. If $c : (K, L) \to (Y, Z)$ is a map of pairs such that $\pi \circ c$ is the identity on $K$, we say that $c$ is a *section* of $\pi$. We say that two sections of $\pi$, $c_0$ and $c_1$, are *homotopic* if there is a homotopy $c_t$, for $0 \leqq t \leqq 1$, of sections of $\pi$. For each cell $\sigma \subset K$, let $E_\sigma = \pi^{-1}\sigma$ if $\sigma \not\subset L$, and let $E_\sigma = \pi^{-1}\sigma \cap Z$ if $\sigma \subset L$. Let $\pi_\sigma : E_\sigma \to \sigma$ be the restriction of $\pi$ to $E_\sigma$.

We can immediately pose two questions. First: When does $\pi$ have a section; and second: When are two sections homotopic? Suppose $A$ is a subcomplex of $K$, and $h$ is a section of $\pi$ over $A$, i.e., a map of pairs $h : (K \cap A, L \cap A) \to (Y, Z)$ such that $\pi \circ h$ is the identity. The relative versions of our questions are: When can $h$ be extended over $K$, and when are two such extensions homotopic rel $A$?

2.2. *The sheaf of homotopy groups.* We shall define a sheaf $\pi_n = \pi_n(\pi) = (\mathcal{G}, p)$ over the space $K$, which we call the sheaf of $n$th homotopy groups of $\pi$ for any $n \geqq 1$, provided that $E_\sigma$ is $n$-simple for all cells $\sigma$. As a set, $\mathcal{G}$ will be defined to be the union, over all cells $\sigma \subset K$, of $\operatorname{Int} \sigma \times \pi_n(E)$; and $p(x, a) = a$ for all $\sigma$, all $x \in \operatorname{Int} \sigma$, and all $a \in \pi_n(E_\sigma)$. The stalk of $\pi_n$ over $x$ we identify with $\pi_n(E_\sigma)$. In order to describe the topology of $\mathcal{G}$ it is only necessary to describe continuous sections over open stars of cells, where, if $\sigma$ is a cell of $K$, St $\sigma$, the open star of $\sigma$, is the union of the interiors of all cells of which $\sigma$ is a face. We then say that a section $f : \operatorname{St} \sigma \to \mathcal{G}$ is *continuous* if for any cell $\tau \supset \sigma$ and any $x \in \operatorname{Int} \tau$, $f(x) = (x, i_{\#} a_0)$, where $i : E_\sigma \to E_\sigma$ is the inclusion, $x_0$ is the barycenter of $\sigma$, and $a_0 = f(x_0)$. We can thus immediately identify the group of continuous sections of $\mathcal{G}$ over St $\sigma$ with $\pi_n(E_\sigma)$, by evaluating each section at $x_0$.

For any subcomplex $A \subset K$ (not necessarily $L$) let $C^* = (C^*(K, A; \pi_n), \delta)$ be the graded differential complex of degree 1 where $C^k(K, A; \pi_n)$ is defined to be the set of all $k$-cochains, i.e., functions $c$ whose domain is the set of $k$-cells of $K$, where $c(\sigma) \in \pi_n(E_\sigma)$ for each $k$-cell $\sigma$, and where $c(\sigma) = 0$ if $\sigma \subset A$. If $c \in C^k(K, A; \pi_n)$ is any $k$-cochain, we define $\delta c \in C^{k+1}(K, A; \pi_n)$ as follows: For any $(k+1)$-cell $\tau$, let $\delta c(\tau) = \sum [\sigma; \tau] c(\sigma)$, where the sum is over all $k$-cells $\sigma \subset \tau$, and where $[\sigma; \tau] = \pm 1$ is the incidence number. According to Theorem 5.1 of [6], we may identify $H^*(K, A; \pi_n)$ with the homology of the graded complex $C^*$. If $A$ is empty, we write $C^k(K; \pi_n)$ for $C^k(K, \emptyset; \pi_n)$, etc.

2.3. *The obstruction cochain.* Let $f : (\bar{K}^n, \bar{L}^n) \to (Y, Z)$ be a section of $\pi$ over $\bar{K}^n = K^n \cup A$, where $K^n$ is the $n$-skeleton of $K$. We consider the question of extension of $f$ to the $(n+1)$-skeleton. Let $c^{n+1} = c^{n+1}(f)$, an element of $C^{n+1}(K, A; \pi_n)$, be defined as follows: If $\sigma \subset K$ is an $(n+1)$-cell and if $\phi : S^n \to \partial\sigma$ is the standard homeomorphism (whose degree is determined by the orientation of $\sigma$), we let $c^{n+1}(\sigma) \in \pi_n(E_\sigma)$ be the homotopy class represented by the composition $f \circ \phi : S^n \to E_\sigma$. As in the usual obstruction setting, we have some theorems, which we state without proof (see Hu [5, Chapter 6]).

THEOREM 2.3.1. *The obstruction cochain $c^{n+1}(f)$ is an invariant of the homotopy class of $f$* rel A, *i.e., if $f_t : (\bar{K}^n, \bar{L}^n) \to (Y, Z)$, for $0 \leqq t \leqq 1$, is a homotopy of extensions of $h$, then $c^{n+1}(f_0) = c^{n+1}(f_1)$.*

THEOREM 2.3.2. *An extension of $f$ over $\bar{K}^{n+1}$ exists if and only if $c^{n+1}(f) = 0$.*

THEOREM 2.3.3. *$c^{n+1}(f)$ is a cocycle.*

We may thus define $\gamma^{n+1}(f) \in H^{n+1}(K, A; \pi_n)$ to be the cohomology class of $c^{n+1}(f)$.

THEOREM 2.3.4. *$\gamma^{n+1}(f) = 0$ if and only if $f \mid \bar{K}^{n-1}$ can be extended to $\bar{K}^{n+1}$.*

2.4. *The difference cochain.* Suppose $f_0$ and $f_1$ are sections of $\pi$ over $\bar{K}^n$ which are extensions of $h$, and that $g_t$, for $0 \leqq t \leqq 1$, is a homotopy of extensions of $h$ over $\bar{K}^{n-1}$, $g_i = f_i \mid \bar{K}^{n-1}$ for $i = 0, 1$. Let $d^n = d^n(f_0, f_1; g_t) \in C^n(K, A; \pi_n)$ be defined as follows.

Let $\pi \times 1 : (Y \times I, Z \times I) \to (K \times I, L \times I)$ be the obvious fibration pair. We define a section $F$ of $\pi \times 1$ over $\bar{K}^n \times \partial I \cup \bar{K}^{n-1} \times I$ as follows:

$$F(x, t) = \begin{cases} (f_t(x), t) & \text{if } t = 0 \text{ or } 1, \text{ for all } x \in \bar{K}^n, \\ (g_t(x), t) & \text{if } x \in \bar{K}^{n-1}, \text{ for all } t \in I. \end{cases}$$

Now $c^{n+1}(F) \in C^{n+1}(K \times I,\ A \times I \cup K \times \partial I;\ \pi_n(\pi \times 1))$. But that group is isomorphic in an obvious way to $C^n(K, A; \pi_n)$, since $\pi_n(\pi \times 1) = p^{-1}\pi_n$, where $p : K \times I \to K$ is the projection. Let $d^n$ be the image of $c^{n+1}(F)$ under that isomorphism. We state without proof analogues of the usual theorems on difference cochains.

THEOREM 2.4.1. *$d^n(f_0, f_1; g_t)$ is a homotopy invariant.*

THEOREM 2.4.2. *$\{g_t\}$ can be extended to a homotopy of $f_0$ with $f_1$ if and only if $d^n(f_0, f_1; g_t) = 0$.*

THEOREM 2.4.3. $\delta d^n(f_0, f_1; g_t) = c^{n+1}(f_1) - c^{n+1}(f_0)$.

Thus, if $f_0$ and $f_1$ can both be extended to $\bar{K}^{n+1}$, $d^n(f_0, f_1; g_t)$

is a cocycle; let $\delta^n(f_0, f_1; g_t) \in H^n(K, A; \pi_n)$ be its cohomology class.

THEOREM 2.4.4. *If $k_t$, $0 \leqq t \leqq 1$, is a homotopy of $f_1 \mid \bar{K}^{n-1}$ with $f_2 \mid \bar{K}^{n-1}$, where $f_2$ is another extension of $h$ over $\bar{K}^n$, then $d^n(f_0, f_2; r_t) = d^n(f_0, f_1; g_t) + d^n(f_1, f_2; k_t)$, where $r_t = g_{2t}$ if $0 \leqq t \leqq \frac{1}{2}$, $r_t = k_{2t-1}$ if $\frac{1}{2} \leqq t \leqq 1$.*

THEOREM 2.4.5. *If $f_0$ and $f_1$ can both be extended to $\bar{K}^{n+1}$, then $g_t \mid \bar{K}^{n-2}$ can be extended to a homotopy of $f_0$ with $f_1$ if and only if $\delta^n(f_0, f_1; g_t) = 0$.*

THEOREM 2.4.6. *For any $f_0$ and any homotopy $g_t$, as before, and, for any $d \in C^n(K, A; \pi_n)$, there exists an extension $f_1'$ of $g_1$ such that $d^n(f_0, f_1'; g_t) = d$.*

2.5. *A classification theorem.* Suppose that $\pi$ is $(n-1)$-connected, i.e., each $E_\sigma$ is connected, and $\pi_k = 0$ for all $k < n$, for some integer $n \geqq 1$. Suppose also that $\dim K \leqq n$. Let $[K, h; \pi]$ be the set of rel $A$ homotopy classes of extensions of $h$ over $K$. (If $A$ is empty, write $[K; \pi]$.)

THEOREM 2.5.1. *$[K, h; \pi]$ can be put into one-to-one correspondence with $H^n(K, A; \pi_n)$.*

PROOF. By successive application of Theorem 2.3.2 on the skeleta of $K$, we can choose a section $f_0$ of $\pi$ such that $f_0 \mid A = h$. Now let $f$ be any other extension of $h$ over $K$. By Theorem 2.4.2, $f_0 \mid \bar{K}^{n-1}$ and $f \mid \bar{K}^{n-1}$ are homotopic rel $A$. Pick a homotopy $\{g_t\}$. Let $[f] \in [K, h; \pi]$, the homotopy class of $f$, correspond to the difference cohomology class $\delta^n(f_0, f; g_t)$. By Theorems 2.4.1, 2.4.2, and 2.4.3, this correspondence is well defined; by Theorem 2.4.5 it is one-to-one, and by 2.4.6 it is onto.

3. **Existence of embeddings and isotopies.**

3.1. *The space $R^*M$.* Let $M$ be any $n$-dimensional manifold, for any integer $n$. Let $SM$ and $PM$ be the total spaces of the sphere bundle and the projective bundle, respectively, associated to the tangent bundle of $M$. Let $RM = M^2 - \Delta_M$, the deleted product of $M$, and let $R^*M = RM/T$, where $T$ is the map which exchanges coordinates. We call $R^*M$ the reduced deleted product of $M$. Let $\phi : RM \cup SM \rightarrow R^N \times R^N \times S^{N-1}$ be the map where

$$\phi(x, y) = (gx, gy, (gx - gy)/\|gx - gy\|)$$

for all $(x, y) \in RM$, and $\phi(v) = (g\pi v, g\pi v, g_* v \|g_* v\|^{-1})$ for any unit tangent vector $v \in SM$, where $g : M \to R^N$ is any distal embedding of $M$ in any Euclidean space, and $\pi : SM \to M$ is the projection. Let $\mathbf{R}M$ be the topological space $(RM \cup SM, \mathcal{T})$, where $\mathcal{T}$ is the unique topology which makes $\phi$ an embedding. $\mathbf{R}M$ also has the structure of a differentiable manifold with boundary $SM$; we leave verification to the reader. Let $T : SM \to SM$ also denote the antipodal map on each fiber of $\pi$; $T$ then acts continuously on $\mathbf{R}M$; we define $\mathbf{R}^*M$ to be the quotient space $\mathbf{R}M/T$, also a $2n$-manifold, with boundary $PM$. We remark that $\mathbf{R}M$ and $\mathbf{R}^*M$ have the same homotopy types as $RM$ and $R^*M$, respectively, since if we remove the boundary of any manifold, it does not change the homotopy type.

If $V$ is another manifold and if $f : V \to M$ is an embedding, maps $Rf : RV \to RM$, $R^*f : R^*V \to R^*M$, $\mathbf{R}f : \mathbf{R}V \to \mathbf{R}M$, and $\mathbf{R}^*f : \mathbf{R}^*V \to \mathbf{R}^*M$ are naturally defined. $Rf(x, y) = (fx, fy)$, etc.

Let $R^\infty =$ the union of $R^N$, for all $N \geqq 1$, with the weak topology. We then define $R(M \times R^\infty)$, $R^*(M \times R^\infty)$, $S(M \times R^\infty)$, $P(M \times R^\infty)$, $\mathbf{R}(M \times R^\infty)$, and $\mathbf{R}^*(M \times R^\infty)$, to be the unions of the corresponding constructions on $M \times R^N$, over all integers $N \geqq 1$, with the weak topology.

3.2. *The obstructions to embedding and isotopy.* Let $M$ be an $n$-dimensional manifold. We replace the inclusion of pairs $(\mathbf{R}^*M, PM) \subset (\mathbf{R}^*(M \times R^\infty), P(M \times R^\infty))$ with a fibration of pairs $\pi_M : (Y, Z) \to (\mathbf{R}^*(M \times R^\infty), P(M \times R^\infty))$ of the same homotopy type. Specifically, let $Y = \{\alpha \in \mathbf{R}^*(M \times R^\infty)^I \mid \alpha(1) \in \mathbf{R}^*M\}$, and

$$Z = \{\alpha \in Y \mid \alpha(t) \in P(M \times R^\infty), \text{all } t\},$$

where $\mathbf{R}^*(M \times R^\infty)^I$ is the space of all paths in $\mathbf{R}^*(M \times R^\infty)$ with the compact-open topology. We let $\pi_M(\alpha) = \alpha(0)$ for all $\alpha \in Y$.

Let $V$ be a compact manifold of dimension $k$, and $f : V \to M$ a differentiable map. Choose, once and for all, an embedding $\boldsymbol{i} : V \to R^\infty$. Let $(Y', Z')$ be the pullback, as in the diagram:

$$(3.2\text{-}1) \qquad \begin{array}{ccc} (Y', Z') & \xrightarrow{\;p_2\;} & (Y, Z) \\ \Big\downarrow{\scriptstyle \pi_M'} & & \Big\downarrow{\scriptstyle \pi_M} \\ (\mathbf{R}^*V, PV) & \xrightarrow{\;\mathbf{R}^*(f, i)\;} & (\mathbf{R}^*(M \times R^\infty), P(M \times R^\infty)) \end{array}$$

Specifically, we let $Y' = \{(r, \alpha) \in \mathbf{R}^*V \times Y \mid \mathbf{R}^*(f, \boldsymbol{i})(r) = \alpha(0)\}$, and $Z' = Y' \cap PV \times Z$; $\pi_M'(r, \alpha) = r$ and $p_2(r, \alpha) = \alpha$ for all $(r, \alpha) \in Y'$.

Now if $f$ is homotopic to an embedding, $\pi_M'$ has a section; specifically, if $\{f_t\}$ is an $e$-homotopy of $f$, let $\Phi[f_t](r) = (r, \alpha) \in Y'$

for all $r \in \boldsymbol{R}^*V$, where, for any $0 \leqq t \leqq 1$, $\alpha(t) = \boldsymbol{R}^*(f_{2t}, \boldsymbol{i})(r)$ if $0 \leqq t \leqq \frac{1}{2}$, $\alpha(t) = \boldsymbol{R}^*(f_1, (2-2t))\boldsymbol{i}(r)$ if $\frac{1}{2} \leqq t \leqq 1$. If $\{f_t\}$ and $\{g_t\}$ are $e$-homotopies of $f$ which are isotopic, $\Phi[f_t]$ and $\Phi[g_t]$ are homotopic as sections of $\pi_M{}'$. The converses of these two statements are true in a suitable metastable range, as we shall see in the next paragraph; the obstructions to finding a section of $\pi_M{}'$, and to finding a homotopy of two sections, as defined in §2, we call the obstructions to embedding and isotopy, respectively. We let $\pi_i$ denote the sheaf of homotopy groups $\pi_i(\pi_M{}')$ for each integer $i \geqq 1$; the first obstruction to finding an embedding of $V$ in $M$ homotopic to $f$ lies in $H^n(\boldsymbol{R}^*V; \pi_{n-1})$; higher obstructions lie in $H^{n+i}(R^*V; \pi_{n+i-1})$ for $i \geqq 1$. The first obstruction to finding an isotopy of $\{f_t\}$ and $\{g_t\}$ (which can also be thought of as the first obstruction to finding an isotopy of $f_1$ with $g_1$ which is homotopic to $\{r_t\}$, where $r_t = f_{1-2t}$ if $0 \leqq t \leqq \frac{1}{2}$, $r_t = g_{2t-1}$ if $\frac{1}{2} \leqq t \leqq 1$) lies in $H^{n-1}(\mathbf{R}^*V; \pi_{n-1})$; higher obstructions lie in $H^{n+i-1}(\mathbf{R}^*V; \pi_{n+i-1})$ for $i \geqq 1$.

3.3. *The restatement of Haefliger's results.*

THEOREM 3.3.1. *Suppose* $2n \geqq 3(k+1)$. *Then* $f$ *is homotopic to an embedding if and only if* $\pi_M{}'$ *has a section. Furthermore, if* $\Phi$ *is a section of* $\pi_M{}'$, $f$ *has an e-homotopy* $\{f_t\}$ *such that* $\Phi[f_t]$ *is homotopic to* $\Phi$.

PROOF. If $\{f_t\}$ is an $e$-homotopy of $f$, $\Phi[f_t]$ is the desired section. Suppose $\Phi : (\mathbf{R}^*V, PV) \to (Y', Z')$ is a section of $\pi_M{}'$. Consider the diagram

$$\begin{array}{ccccc}
R^*V \times I & \xleftarrow{q \times 1} & RV \times I & \xrightarrow{Q \times I} & V^2 \times I \\
\downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle G[\Phi]} \\
R^*(M \times R^\infty) & \xleftarrow{q'} & R(M \times R^\infty) & \xrightarrow{Q'} & M^2
\end{array}$$

where $q$ and $q'$ are the quotient maps, $Q = 1 \cup \pi : \boldsymbol{R}V \to V^2$, and $Q'$ is the composition $(p_1)^2 \circ (1 \cup \pi)$, where $p_1 : M \times R^\infty \to M$ is projection to the first factor. The map $\phi^*$ is defined by $\phi^*(r, t) = \alpha(t)$ for all $(r, t) \in \boldsymbol{R}^*V \times I$, where $\Phi(r) = (r, \alpha) \in Y'$. $G[\Phi]$ and $\phi$ are the unique maps which make the diagram commute and which satisfy the equation $\phi(r, 0) = \boldsymbol{R}(f, \boldsymbol{i})(r)$ for all $r \in \boldsymbol{R}V$. Now let $g_t : V_2 \to M_2$, for all $0 \leqq t \leqq 1$, be the homotopy where $g_t(x, y) = G[\Phi](x, y, t)$ for all $(x, y) \in V^2$. Then $\{g_t\}$ is an equivariant homotopy; that is, $T \circ g_t = g_t \circ T$ for all $t$, and $g_1$ is isovariant, i.e., $g_1{}^{-1}\Delta_M = \Delta_V$. According to Theorem 1(a) of Haefliger [4], $f$ is homotopic to an embedding of $V$ into $M$.

Examining the details of Haefliger's proof, however, we observe that it is possible to construct an $e$-homotopy $\{f_t\}$ of $f$ and a 2-parameter homotopy $h_{\tau,t} : V^2 \to M^2$, $0 \leqq \tau, t \leqq 1$, such that $h_{0,t} = g_t$ and $h_{1,t} = f_t^2$ for all $t$; $h_{\tau,0} = g_0$ and $h_{\tau,1}$ is isovariant for all $\tau$; and $h_{\tau,t}$ is equivariant for all $\tau, t$. Using $\{h_{\tau,t}\}$, we may show that $\Phi$ is homotopic to $\Phi[f_t]$; we leave the details to the reader.

**Theorem** 3.3.2. *Suppose $2n > 3(k + 1)$. Then two e-homotopies of $f$, $\{f_t\}$ and $\{g_t\}$, are isotopic if and only if $\Phi[f_t]$ is homotopic to $\Phi[g_t]$.*

**Proof.** If $\{f_{\tau,t}\}$ is an isotopy of $\{f_t\}$ with $\{g_t\}$, then $\{\Phi[f_{\tau,t}]\}_{0 \leqq \tau \leqq 1}$ is a homotopy of $\Phi[f_t]$ with $\Phi[g_t]$. Conversely, suppose $\Phi_\tau$, for $0 \leqq \tau \leqq 1$, is a homotopy of sections of $\pi_M{}'$ such that $\Phi_0 = \Phi[f_t]$ and $\Phi_1 = \Phi[g_t]$. For each $\tau$, let $G[\Phi_\tau] : V^2 \times I \to M^2$ be the map as constructed in the proof of 3.3.1. Let $h_{\tau,t} : V^2 \to M^2$, for $0 \leqq \tau \leqq 1$, be the 2-parameter homotopy where $h_{\tau,t}(x, y) = G[\Phi_\tau](x, y, t)$ for all $(x, y) \in V^2$. Note that $h_{\tau,0} = f^2$ and $h_{\tau,1}$ is isovariant for all $\tau$; $h_{0,t} = f_{2t}{}^2$ and $h_{1,t} = g_{2t}$ for all $0 \leqq t \leqq \frac{1}{2}$, and $h_{0,t} = f_1{}^2$ and $h_{1,t} = g_1{}^2$ for all $\frac{1}{2} \leqq t \leqq 1$; and $h_{\tau,t}$ is equivariant for all $\tau, t$. Thus $h_{\tau,1}$, for $0 \leqq \tau \leqq 1$, is an isovariant homotopy of $f_1{}^2$ with $g_1{}^2$ which is equivariantly homotopic, rel $f_1{}^2$ and $g_1{}^2$, to the homotopy $r_\tau{}^2 : V^2 \to M^2$, $0 \leqq \tau \leqq 1$, where $r_\tau = f_{1-2\tau}$ if $0 \leqq \tau \leqq \frac{1}{2}$, $g_{2\tau-1}$ if $\frac{1}{2} \leqq \tau \leqq 1$. Haefliger's construction [4, Theorem 1 (b)] then gives us an isotopy of $f_1$ with $g_1$ which is homotopic to $\{r_\tau\}$. The construction of the isotopy of $\{f_t\}$ with $\{g_t\}$ is routine, and left to the reader.

3.4. *The structure of the sheaf $\pi_{n-1}(\pi_M)$.* In this paragraph, we insist that $n \geqq 2$.

**Lemma** 3.4.1. *The inclusion $R(M \times R^\infty) \to (M \times R^\infty)^2$ is a homotopy equivalence.*

**Proof.** Let $h_t : R^\infty \to R^\infty$, for $0 \leqq t \leqq 1$, be the isotopy where $h_0$ is the identity and where, for any integer $m \geqq 1$ and any $(m + 1)^{-1} \leqq t \leqq m^{-1}$, $h_t(x_1, x_2, \cdots) = (y_1, y_2, \cdots)$, where $y_i = x_i$ for all $1 \leqq i < m$, $y_i = x_{i-1}$ for all $i > m + 1$, and $y_m = x_m \cos\theta$ and $y_{m+1} = x_m \sin\theta$, where $\theta = \frac{1}{2}\pi(t(m^2 + m) - m)$. Note that $h_1$ is a homeomorphism of $R^\infty$ to the hyperplane $H_0$ of all points in $R^\infty$ with first coordinate 0. Let $g_t : R^\infty \to R^\infty$, for $0 \leqq t \leqq 1$, be the isotopy where $g_t(x_1, x_2, \cdots) = (x_1 + t, x_2, \cdots)$, i.e., translation along the $x_1$-axis. We define a homotopy $r_t : (M \times R^\infty)^2 \to (M \times R^\infty)^2$, $0 \leqq t \leqq 1$, as follows:

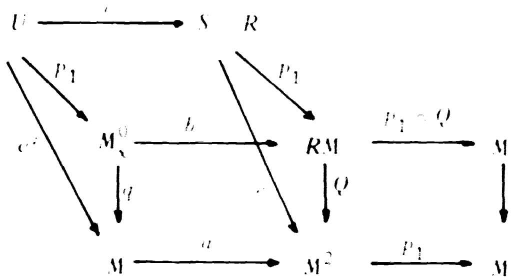
$$r_t(x, v, y, w) = \begin{cases} (x, h_{2t}v, y, h_{2t}w) & \text{if } 0 \leqq t \leqq \frac{1}{2}, \\ (x, h_1 v, y, g_{2t-1}w) & \text{if } \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

for all $x$, $y \in M$ and $v$, $w \in R^\infty$. Note that $r_0$ is the identity, $r_1(M \times R^\infty)^2 \subset R(M \times R^\infty)$, and $r_t(R(M \times R^\infty)) \subset R(M \times R^\infty)$ for all $t$; thus $r_1$ is a homotopy inverse of the inclusion, and we are done.

Let $Q = 1 \cup \pi : RM \to M^2$ be the quotient map, where $\pi : SM \to M = \Delta_M$ is the projection. Let $e : R \to M^2$ be a fibration replacing $Q$, i.e., $R = \{(r, \alpha) \in RM \times (M^2)^I \mid Q \circ \alpha(1) = r\}$, and let $e(r, \alpha) = \alpha(0)$. Let $S = \{(r, \alpha) \in R \mid p_1 \circ \alpha \text{ is constant}\}$, where $p_1 : M^2 \to M$ is projection to the first factor. We pick a basepoint $x \in M$ and a local orientation of $M$ at $x$, which we represent by a homeomorphism $\omega : S^{n-1} \to SM_x$, $SM_x$ being the set of unit tangents of $M$ at $x$. For each loop $\sigma$ of $M$, we define a map $\chi[\sigma] : S^{n-1} \to R_x$, where $R_x = e^{-1}(x, x)$, as follows: $\chi[\sigma](v) = (\omega(v), \alpha)$, where $\alpha(t) = (x, \alpha(t))$, for all $v \in S^{n-1}$. The homotopy class of $\chi[\sigma]$ clearly depends only on the homotopy class of $\sigma$, hence if $[\sigma] = g \in \pi_1(M, x)$, we define $\chi(g)$ to be the homotopy class of $\chi[\sigma]$.

Lemma 3.4.2. *As an Abelian group, $\pi_{n-1}(R_x)$ is freely generated by the set of all $\chi(g)$, for $g \in \pi_1(M, x)$.*

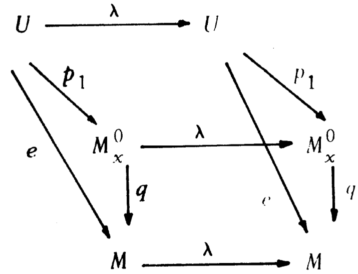
Proof. Consider the commutative diagram



where $ay = (x, y)$ for all $y \in M$, $M_x^0 = (M - \{x\}) \cup SM_x$, with the topology which makes $b = a \cup \pi$ an embedding, where $U = \{(r, \alpha) \in M_x^0 \times M^I \mid q \circ \alpha(1) = r\}$, and $p_1(r, \alpha) = r$, $e(r, \alpha) = \alpha(0)$, and $c(r, \alpha) = (br, \beta)$ for all $(r, \alpha) \in U$, where $\beta(t) = (x, \alpha(t))$ for all $t$. Now since $p_1 : M^2 \to M$ and $p_1 \circ Q$ are both fibrations with fibers $M$ and $M_x^0$, respectively, and since $p_1 : U \to M_x^0$ and $p_1 : R \to RM$ are homotopy equivalences (as the reader can easily check), the inclusion $S \subset R$ is a homotopy equivalence. Also (where $U_x = (e')^{-1}x$), $c$ maps $U_x$ homeomorphically to $S_x = R_x \cap S$, which is of the homotopy type of $R_x$.

Let $\lambda : M \to M$ be a universal covering of $M$, and pick $x \in \lambda^{-1}x$.

Let $M_x{}^0$ and $U$ be the pullbacks, as in the diagram




$M$ is simply connected. By a Serre spectral sequence argument, we can show that the fiber of $e$ (hence also that of $e$) is of the homotopy type, through dimension $n - 1$, of the loop space of the cofiber of $q$, which is a wedge of $n$-spheres, one for each element of $\pi_1(M, x)$. We leave the remaining details to the reader.

LEMMA 3.4.3. *If $\pi_r(M) = 0$ for all $1 < r \leqq m$ for some integer $2 \leqq m \leqq n$, then $\pi_{n+m-2}(R_x)$ is isomorphic to $\pi_{n-1}(R_x) \otimes \pi_{m-1}$, where $\pi_{m-1}$ is the stable $(m - 1)$-stem in the homotopy of spheres.*

PROOF. Let $\pi_{n-1}(R_x) \otimes \pi_{n-1} \to \pi_{n+m-2}(R_x)$ be the homomorphism which sends each $g \otimes h$ to $g \circ h$. We refer the reader to the proof of Theorem 3.4.2 above. Since $M$ is $m$-connected, $R_x$ has the homotopy type of the loop space of a wedge of $n$-spheres up through dimension $n + m - 2$; we omit the details.

We henceforth express elements of $\pi_{n-1}(R_x)$ as formal sums of the $\chi(g)$ for values of $g \in \pi_1(M, x)$.

Let $\mu : \pi_{n-1}(R_x) \times \pi_1(M^2, (x, x)) \to \pi_{n-1}(R_x)$ be the usual (right) action of the fundamental group of a base on the homotopy of a fiber. We shall identify $\pi_1(M^2, (x, x))$ with $\pi_1(M, x) \oplus \pi_1(M, x)$ in the usual way.

LEMMA 3.4.4. *If $g, h \in \pi_1(M, x)$, then $\mu(\chi(g), (h, 1)) = \chi(h^{-1}g)$, where $1 \in \pi_1(M, x)$ is the identity.*

PROOF. Let $\sigma$ be a loop in $M$ which represents $g$, and $\tau$ a loop which represents $h$. Consider a map $\nu : S^{n-1} \times I \to S$ defined as follows: $\nu(v, t) = (\omega(v), \alpha)$ for all $v \in S^{n-1}$ and all $t \in I$, where

$$\alpha(u) = \begin{cases} (x, \tau(-(t + 1)u + t)) & \text{if } 0 \leqq u \leqq t/(t + 1), \\ (x, \sigma((t + 1)u - t)) & \text{if } t/(t + 1) \leqq u \leqq 1. \end{cases}$$

Note that $[\nu( \ ,0)] = \chi(g)$, $[\nu( \ ,1)] = \chi(h^{-1}g)$, and $[\nu( \ ,t)] \in \pi_{n-1}(e^{-1}(\tau(t), x))$ for all $t \in I$, and we are done.

LEMMA 3.4.5. *If* $g, h \in \pi_1(M, x)$, $\mu(\chi(g),(1,h)) = (-1)^{d(h)}\chi(gh)$, *where d is the orientation homomorphism (cf. 1.2) and* $\mu$ *is the usual action of the fundamental group of the base on the homotopy of the fiber.*

PROOF. Let $\sigma$ and $\tau$ be loops in $M$ representing $g$ and $h$, respectively. Let $\omega_t : S^{n-1} \to SM_{\tau(t)}$, for $0 \leqq t \leqq 1$, be a homotopy such that $\omega_0 = \omega$. Note then that $\omega_1 = \omega \circ \epsilon$, where $\epsilon : S^{n-1} \to S^{n-1}$ is a map of degree $(-1)^{d(h)}$. We define $\xi : S^{n-1} \times I \to S$ as follows: $\xi(v, t) = (\omega_t(v), \alpha)$ for all $v \in S^{n-1}$ and $t \in I$, where

$$\alpha(u) = \begin{cases} (x, \sigma((1+t)u)) & \text{if } 0 \leqq u \leqq (1+t)^{-1}. \\ (x, \tau((1+t)u - 1)) & \text{if } (1+t)^{-1} \leqq u \leqq 1. \end{cases}$$

Our proof is complete, since $[\xi(\ ,0)] = \chi(g)$.

$$[\xi(\ ,1)] = \chi(gh)[\epsilon] = (-1)^{d(h)}\chi(gh),$$

and

$$[\xi(\ ,t)] \in \pi_{n-1}(e^{-1}(x, \tau(t))) \quad \text{for all } t \in I.$$

As before, let $T : M^2 \to M^2$ and $T : \boldsymbol{R}M \to \boldsymbol{R}M$ be as defined in §3.1. Let $T$ operate on the path space $(M^2)^I$ by composition. Now $R$ is an invariant subspace of $\boldsymbol{R}M \times (M^2)^I$ under $T$, but $S$ is not.

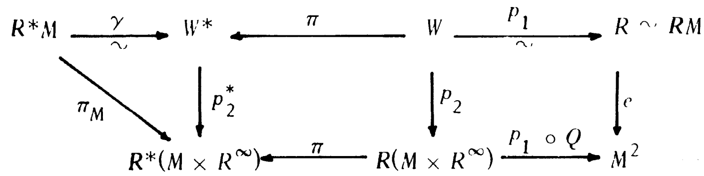
LEMMA 3.4.6. *If* $g \in \pi_1(M, x)$, $T_*\chi(g) = (-1)^n(-1)^{d(g)}\chi(g^{-1})$.

PROOF. Let $\tau$ be a loop in $M$ which represents $g$, and let $\omega_t : S^{n-1} \to SM_{\tau(t)}$ be the homotopy as defined in the proof of Lemma 3.4.5 above. Let $\zeta : S^{n-1} \times I \to R$ be the homotopy where, for all $v \in S^{n-1}$ and $t \in I$, $\zeta(v, t) = (\omega_t, \alpha)$, where $\alpha(u) = (\tau(ut), \tau(1 - u + ut))$ for all $0 \leqq u \leqq 1$. Now $\zeta(v, t) \in R_x$ for all $(v, t)$; thus $[\zeta(\ ,1)] = [\zeta(\ ,0)] = \chi(g)$, while

$$\begin{aligned}[T \circ \zeta(\ ,1)] &= \chi(g^{-1})[\epsilon] \\ &= (-1)^n(-1)^{d(g)}\chi(g^{-1}),\end{aligned}$$

where $\epsilon = T \circ \omega_1 \circ \omega_0^{-1}$, $T$ being the antipodal map on $S^{n-1}$. We are done.

Let $Q' = 1 \cup \pi : \boldsymbol{R}(M \times R^\infty) \to (M \times R^\infty)^2$ be the quotient map. Let $W = \{(r, r) \in R \times \boldsymbol{R}(M \times R^\infty) \mid e(r) = (p_1{}^2 \circ Q')r\}$. Since $T$ acts on $R$, $M^2$, and $\boldsymbol{R}(M \times R^\infty)$, and $T \circ e = e \circ T$ and $T \circ p_1{}^2 \circ Q' = p_1{}^2 \circ Q' \circ T$, $T$ also acts on $W$. Let $W^* = W/T$. Consider diagram (3.4-1) below, in which $W$ is the pullback:




where, for any $\alpha \in Y$, i.e., $\alpha : I \to R^*(M \times R^\infty)$ and $\alpha(1) \in R^*M$, we let $\boldsymbol{\alpha} : I \to R(M \times R^\infty)$ be one of the two paths where $\pi \circ \boldsymbol{\alpha} = \alpha$; we then define $\gamma(\alpha)$ to be the unordered pair

$$((\boldsymbol{\alpha}(1), p_1{}^2 \circ Q' \circ \boldsymbol{\alpha}), \boldsymbol{\alpha}(0)), ((T \circ \boldsymbol{\alpha}(1), T \circ p_1{}^2 \circ Q' \circ \boldsymbol{\alpha}), \ T \circ \boldsymbol{\alpha}(0)) \in W^*.$$

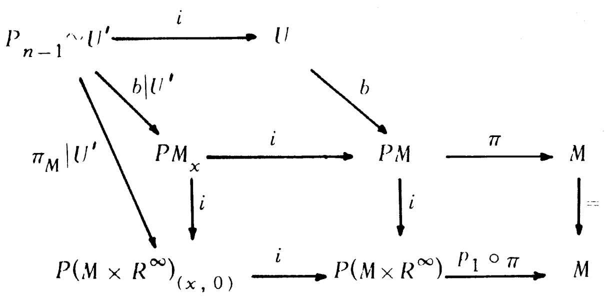
Since by Lemma 3.4.1, $p_1{}^2 \circ Q'$ is a homotopy equivalence, $\gamma$ is a homotopy equivalence.

Pick $v \in S(M \times R^\infty)$ to be a unit vector at $(x, 0) \in M \times R^\infty$. Let $v^* = \{v, -v\} \in P(M \times R^\infty)$, and let $Y_v = \pi_M{}^{-1}v^* \subset Y$. Let $W_v = (p_2)^{-1}v \subset W$ and $W_v{}^* = (p_2{}^*)^{-1}v^* \subset W^*$. Now $(p_1 \circ \pi^{-1} \circ \gamma) : Y_v \to R_x$ is a homotopy equivalence; we define $Y(g) = (p_1 \circ \pi^{-1} \circ \gamma)_\#{}^{-1}\chi(g)$ for all $g \in \pi_1(M, x)$; $\pi_{n-1}(Y_v)$ is freely generated by the $Y(g)$.

Let $U_v = U \cap Y_v$. We let $\theta : S^{n-1} \times I \to S(M \times R^\infty)_{(x,0)}$ be any map such that $\theta(w, 0) = v$ and $\theta(w, 1) = \omega(w)$ for all $w \in S^{n-1}$, and let $\eta : S^{n-1} \to U_v$ be the map where, for all $w \in S^{n-1}$, $\eta(w) = \alpha$ with $\alpha(t) = \pi \circ \theta(w, t)$ for all $0 \leqq t \leqq 1$, where $\pi : S(M \times R^\infty) \to P(M \times R^\infty)$ is the covering map. Since $S(M \times R^\infty)_{(x,0)} \cong S^\infty$, $\theta$ exists and is unique up to homotopy rel $S^{n-1} \times \partial I$, hence $\eta$ exists and is unique up to homotopy. Let $\psi \in \pi_{n-1}(U_v)$ be the class containing $\eta$.

Lemma 3.4.7. *$\pi_{n-1}(U_z) \cong Z$ and is generated by $\psi$.*

Proof. Let $U' = \{\alpha \in P(M \times R^\infty)^I_{(x,0)} \mid \alpha(1) \in PM_x\}$, and consider the commutative diagram




where each map labeled "$i$" is an inclusion and $b(\alpha) = \alpha(1)$ for all

$\alpha \in U$. Now $b$ and $b \mid U'$ are both homotopy equivalences, $PM_x$ and $P(M \times R^\infty)_{(x,0)}$ (which are, respectively, homeomorphic to real projective spaces $P_{n-1}$ and $P_\infty$) are the fibers of $\pi$ and $p_1 \circ \pi$, respectively; thus the inclusion $U' \subset U_v$ is a homotopy equivalence. $U'$ is of the homotopy type of $S^{n-1}$, the fiber of the inclusion $P_{n-1} \subset P_\infty$; and $\eta : S^{n-1} \to U'$ is a homotopy equivalence. The result follows.

THEOREM 3.4.8. (i) $(p_1 \circ \pi^{-1} \circ \gamma)_\# : \pi_{n-1}(U_v) \to \pi_{n-1}(R_x)$ *maps* $\psi$ *to* $X(1)$, *where* $1 \in \pi_1(M, x)$ *is the identity.*
(ii) $\gamma_\# (\psi) = \Upsilon(1)$.

PROOF. (i) We routinely verify that $p_1 \circ \pi^{-1} \circ \gamma \circ \eta = X[0]$, where 0 is the trivial loop at $x \in M$. Part (ii) follows immediately from (i).

Define functions $\rho : \pi_1(\boldsymbol{R}^*(M \times R^\infty), v^*) \to \pi_1(M, x) \otimes \pi_1(M, x)$ and $\delta : \pi_1(\boldsymbol{R}^*(M \times R^\infty), v^*) \to Z_2$, as follows: If

$$g \in \pi_1(\boldsymbol{R}^*(M \times R^\infty), v^*),$$

pick a loop $\sigma$ representing $g$ and let $\tau$ be the path in $\boldsymbol{R}(M \times R^\infty)$ such that $\tau(0) = v$ and $\pi \circ \tau = \sigma$. Let $\delta(g) = 0$ if $\tau(1) = v$, 1 if $\tau(1) = -v$. Now $Q'v = Q'(-v) = (x, 0)$, so $p_1{}^2 \circ Q' \circ \tau$ is a loop in $M^2$; let $\rho(g)$ be the homotopy element represented by that loop.

We remark that $\delta$ is a homomorphism but $\rho$ is not; in fact, if $g, h \in \pi_1(\boldsymbol{R}^*(M \times R^\infty), v^*)$, $\rho(gh) = \rho(g)(T^{\delta(g)}\rho(h))$, where $T$ exchanges coordinates.

Let $G[M]$ be the local system of Abelian groups (i.e., locally trivial sheaf) over $\boldsymbol{R}^*(M \times R^\infty)$ such that for each $r \in \boldsymbol{R}^*(M \times R^\infty)$, $G[M]_r = \pi_{n-1}(\pi_M{}^{-1}r)$. Let $G = G[M]_{v^*}$, and let $\mu : G \times \pi_1(\boldsymbol{R}^*(M \times R^\infty), v^*) \to G$ be the usual (right) action of the fundamental group of a space on the stalk of a local system at the basepoint. We summarize the results of §3.4 in the following theorem.

THEOREM 3.4.9. (i) $G$ *is freely generated by* $\{\Upsilon(g) \mid g \in \pi_1(M, x)\}$.
(ii) *If* $g \in \pi_1(M, x)$ *and* $h \in \pi_1(\boldsymbol{R}^*(M \times R^\infty), v^*)$, *let* $\rho(h) = (h_1, h_2)$. *Then*

$$\mu(\Upsilon(g), h) = \begin{cases} (-1)^{d(h_2)}\Upsilon(h_1{}^{-1}gh_2) & \text{if } \delta(h) = 0, \\ (-1)^{d(gh_2)}(-1)^n\Upsilon(h_2{}^{-1}g^{-1}h_1) & \text{if } \delta(h) = 1. \end{cases}$$

(iii) $\pi_{n-1}(\pi_M)$ *is the unique subsheaf of* $G[M]$ *such that* $\pi_{n-1}(\pi_M)_r = G[M]_r$ *if* $r \notin P(M \times R^\infty)$, $\pi_{n-1}(\pi_M)_{v^*}$ *is the subgroup of* $G$ *generated by* $\Upsilon(1)$, *and* $\pi_{n-1}(\pi_M) \mid P(M \times R^\infty)$ *is locally trivial, i.e., locally a product sheaf (isomorphic to* $Z$*), provided* $M$ *is connected.*
(iv) *If, for some integer* $2 \leq r \leq n-2$, $\pi_i(M, x) = 0$ *for all* $2 \leq i \leq r$,

$\pi_{n+r-2}(\pi_M) \cong \pi_{n-1}(\pi_M) \otimes \pi_{r-1}$, *where* $\pi_{r-1}$ *is the stable* $(r-1)$*-stem in the homotopy of spheres.*

3.5. *Action of* $\pi_1(M^V, f)$. Let us reconsider diagram (3.2-1). Suppose that $\{f_t\}$ is a differentiable self-homotopy of $f$, i.e., $f_0 = f_1 = f$, and each $f_t$ is differentiable. We define a map of pairs $\Gamma[f_t] : (Y', Z') \to (Y', Z')$ such that $\pi_M' \circ \Gamma[f_t] = \pi_M'$ as follows. If $(r, \alpha) \in Y'$, where $r \in \boldsymbol{R}^*V$ and $\alpha : I \to \boldsymbol{R}^*(M \times R^\infty)$ is a path such that $\alpha(0) = \boldsymbol{R}^*(f, i)(r)$ and $\alpha(1) \in \boldsymbol{R}^*M$ (cf. §3.2), let $\Gamma[f_t]\,(r, \alpha) = (r, \beta)$, where $\beta(t) = \boldsymbol{R}^*(f_{1-2t}, i)(r)$ if $0 \leqq t \leqq \frac{1}{2}$ and $\beta(t) = \alpha(2t-1)$ if $\frac{1}{2} \leqq t \leqq 1$.

We say that two maps $\Gamma_0, \Gamma_1 : (Y', Z') \to (Y', Z')$ such that $\pi_M' \circ \Gamma_i = \pi_M'$ for $i = 0, 1$ are homotopic if we can find a homotopy $\Gamma_t : (Y', Z') \to (Y', Z')$, for $0 \leqq t \leqq 1$, such that $\pi_M' \circ \Gamma_t = \pi_M'$ for all $t$. The proofs of the following remarks are routine homotopy arguments, which we omit.

REMARK 3.5.1. If $\{f_t'\}$ is another differentiable self-homotopy of $f$ which is homotopic to $\{f_t\}$ rel $f$, $\Gamma[f_t']$ is homotopic to $\Gamma[f_t]$.

REMARK 3.5.2. If $\{g_t\}$ is another differentiable self-homotopy of $f$ and if $\{h_t\}$ is the self-homotopy such that $h_t = f_{2t}$ if $0 \leqq t \leqq \frac{1}{2}$ and $h_t = g_{2t-1}$ if $\frac{1}{2} \leqq t \leqq 1$, then $\Gamma[h_t]$ is homotopic to $\Gamma[g_t] \circ \Gamma[f_t]$.

REMARK 3.5.3. If $f_t = f$ for all $t$, then $\Gamma[f_t]$ is homotopic to the identity.

We can thus define a right action $\gamma : \mathrm{Sec}(\pi_M') \times \pi_1(M^V, f) \to \mathrm{Sec}(\pi_M')$ as follows: $\gamma([c], [f_t]) = [\Gamma[f_t] \circ c]$ for any section $c$ of $\pi_M'$ and any differentiable self-homotopy $\{f_t\}$ of $f$, where $[f_t]$ is the corresponding element of the fundamental group of $M^V$. Let $\pi_i = \pi_i(\pi_M')$ for any integer $i \geqq 1$. We have a right action of $\pi_1(M^V, f)$ on the sheaf $\pi_i$, namely $\gamma_* : \pi_i \times \pi_1(M^V, f) \to \pi_i$ where, for any $r \in \boldsymbol{R}^*V$ and $g = [f_t] \in \pi_1(M^V, f)$, $\gamma_*(\ , g)$ is the automorphism $\Gamma[f_t]_\#$ on the stalk $\pi_i(\pi_M')_r$. We also let $\gamma_* : H^*(\boldsymbol{R}^*V;\ \pi_i) \times \pi_1(M^V, f) \to H^*(\boldsymbol{R}^*V; \pi_i)$ be the action obtained by applying $\gamma_*$ to the coefficient sheaf.

The following remark follows immediately from a simple naturality argument:

REMARK 3.5.4. If $g \in \pi_1(M^V, f)$ and if $c_0, c_1$ are sections of $\pi_M'$ over $(\boldsymbol{R}^*V)^m$, the $m$-skeleton of $\boldsymbol{R}^*V$, for some $m \geqq 0$, and if $h_\tau$ for $0 \leqq \tau \leqq 1$ is a homotopy of sections of $\pi_M'$ over $(\boldsymbol{R}^*V)^{m-1}$ such that $h_i = c_i \mid (\boldsymbol{R}^*V)^{m-1}$ for $i = 0$ and 1, then

$$\gamma^*(d^m(c_0, c_1; h_\tau), g) = d^m(\Gamma[f_t] \circ c_0, \Gamma[f_t] \circ c_1; \Gamma[f_t] \circ h_\tau)$$

where $\{f_t\}$ is any self-homotopy of $f$ which represents $g$.

Now let $\gamma_{\#} : [V \subset M]_f \times \pi_1(M^V, f) \to [V \subset M]_f$ be the right action defined as follows: If $\{g_t\}$ is any differentiable self-homotopy of $f$ and if $\{f_t\}$ is any $e$-homotopy of $f$, let $\gamma_{\#}([f_t], [g_t]) = [h_t]$, where $\{h_t\}$ is the $e$-homotopy: $h_t = g_{1-2t}$ if $0 \leqq t \leqq \frac{1}{2}$, $h_t = f_{2t-1}$ if $\frac{1}{2} \leqq t \leqq 1$. The actions $\gamma_*$ and $\gamma_{\#}$ are consistent, i.e., if $\phi : [V \subset M]_f \to \mathrm{Sec}(\pi_M')$ is the function defined in §1.1, $\gamma_{\#}(\phi e, g) = \phi(\gamma_*(e, g))$ for all $e \in [V \subset M]_f$, $g \in \pi_1(M^V, f)$.

DEFINITION 3.5.1. Let $G$ be a group and $A$ an Abelian group. We say a function $\alpha : A \times G \to A$ is a *right affine action* of $G$ on $A$ if

(i) for all $a \in A$ and $g, h \in G$, $\alpha(a, gh) = \alpha(\alpha(a, g), h)$;

(ii) for all $a \in A$, $\alpha(a, 1) = a$, where $1 \in G$ is the identity;

(iii) for all $a, b \in A$ and $g \in G$, $\alpha(a + b, g) = \alpha(a, g) + \alpha(b, g) - \alpha(0, g)$.

Suppose now that $k \geqq 2$ and $n = 2k + 1$, and $f$ is an embedding. By Theorem 2.5.1, we may identify $[V \subset M]_f$ with $H^{2k}(\boldsymbol{R}^*V; \pi_{2k})$, where $[f]$ corresponds to 0. The following theorem follows immediately from 2.5.1 and 3.5.4:

THEOREM 3.5.5. *If $f$ is an embedding and $n = 2k + 1$, then $\gamma_{\#} : [M \subset V]_f \times \pi_1(M^V, f) \to [M \subset V]_f$ is a right affine action.*

In general (without any dimensional restriction on $V$ and $M$) let $\Delta : [V \subset M]_f \to [V \subset M]$ be the function which takes $[f_t]$ to $[f_1]$ for each $e$-homotopy $[f_t]$ of $f$, as defined in §1.1.

THEOREM 3.5.6. *If $h : V \to M$ is an embedding homotopic to $f$, $\Delta^{-1}[h]$ is precisely an orbit of $[V \subset M]_f$ under the action $\gamma_{\#}$.*

PROOF. Choose an $e$-homotopy $\{f_t\}$ of $f$ such that $f_1 = h$. Suppose that $\{g_t\}$ is a differentiable self-homotopy of $f$. Then $\gamma_{\#}([f_t], [g_t]) = [k_t]$, where $k_t = g_{1-2t}$ if $0 \leqq t \leqq \frac{1}{2}$ and $k_t = f_{2t-1}$ if $\frac{1}{2} \leqq t \leqq 1$. $\Delta[k_t] = [k_1] = [h]$. Conversely, suppose that $\{r_t\}$ is another $e$-homotopy of $f$ such that $r_1 = h$. Let $\{s_t\}$ be the self-homotopy of $f$ where $s_t = r_{2t}$ if $0 \leqq t \leqq \frac{1}{2}$ and $s_t = f_{2-2t}$ if $\frac{1}{2} \leqq t \leqq 1$. Then $\gamma_{\#}([f_t], [s_t]) = [r_t]$.

3.6. *Embeddings of $S^k$ in $M^{2k+1}$.* Suppose now that $S^k$ is the $k$-sphere, for $k \geqq 2$, and that $M$ is a connected manifold of dimension $n = 2k + 1$. The space $\boldsymbol{R}S^k$ is of the homotopy type of $S^k$, while $\boldsymbol{R}^*S^k$ has the homotopy type of real projective $k$-space, $P_k$.

DEFINITION 3.6.1. If $\mathcal{G}$ is any sheaf over $\boldsymbol{R}^*S^k$, let $\mathcal{G}^0 \subset \mathcal{G}$ be the subsheaf where $\mathcal{G}_r{}^0 = 0$ if $r \notin PS^k$, and $\mathcal{G}_r{}^0 = \mathcal{G}_r$ if $r \notin PS^k$. We remark that $H^*(\boldsymbol{R}^*S^k; \mathcal{G}^0) = H^*(\boldsymbol{R}^*S^k, PS^k; \mathcal{G})$ [2].

DEFINITION 3.6.2. If $A$ is an Abelian group and $\phi : A \to A$ is an automorphism such that $\phi^2 = 1$, the identity, let $[A, \phi]$ be the sheaf over $\boldsymbol{R}^*S^k$ obtained from the product sheaf $\boldsymbol{R}S^k \times A$ by identifying $(r, a)$ with $(Tr, \phi a)$ for all $r \in \boldsymbol{R}S^k$ and $a \in A$.

Let $E : Z \oplus Z \to Z \oplus Z$ be the "exchange" automorphism, i.e., $E(x, y) = (y, x)$ for all $x, y \in Z$.

Consider the sheaf $\pi_{n-1} = \pi_{n-1}(\pi_M{}') = (\boldsymbol{R}^*(f, \boldsymbol{i}))^{-1}\pi_{n-1}(\pi_M{}')$ over $\boldsymbol{R}^*S^k$, where $f : S^k \to M$ is any differentiable map, and $\boldsymbol{i} : S^k \to M$ is any embedding. We shall assume that $S^k$ and $M$ have base-points $s_0$ and $m_0$, respectively, and $f(s_0) = m_0$.

$\boldsymbol{R}S^k$ is simply connected, so $\pi_{n-1}$ breaks up as a direct sum (cf. Theorem 3.4.9); in fact $\pi_{n-1} \cong Z \oplus \sum_{g \neq 1} Z^0$, where $Z$ is the trivial integer sheaf, and the sum is over all $g \in \pi_1 = \pi_1(M, m_0)$ not equal to the identity. We define sets $A \subset \pi_1$ and $B \subset \pi_1$ as follows: $A$ consists of all $g \in \pi_1$ such that $g \neq 1$, $g^2 = 1$, and $d(g) = 0$, and $B$ consists of all $g \in \pi_1$ such that $g^2 = 1$ and $d(g) = 1$, where $d : \pi_1 \to Z_2$ is the orientation homomorphism. Let $\Theta$ and $\Lambda$ be the sets of unordered pairs in $\pi_1$ as follows: $\Theta$ consists of all unordered pairs $\{g, g^{-1}\}$ such that $g^2 \neq 1$ and $d(g) = 0$, and $\Lambda$ consists of all $\{g, g^{-1}\}$ such that $g^2 \neq 1$ and $d(g) = 1$. Using the action of $\pi_1(\boldsymbol{R}^*S^k) \cong Z_2$ on the stalk of $\pi_{n-1}$, we obtain directly, from Theorem 3.4.9,

**Lemma** 3.6.1. $\pi_{n-1} \cong [Z, -1] \oplus \sum_A [Z, -1]^0 \oplus \sum_B Z^0 \oplus \sum_\Theta [Z \oplus Z, -E]^0 \oplus \sum_\Lambda [Z \oplus Z, E]^0$.

It is sufficient to compute the cohomology of $\boldsymbol{R}^*S^k$ with coefficients in each of the direct summands.

**Lemma** 3.6.2. $H^{2k}(\boldsymbol{R}^*S^k; [Z, -1]) = 0$.

**Proof.** $\boldsymbol{R}^*S^k$ is of the homotopy type of a complex of dimension $k < 2k$, and $[Z, -1]$ is a local system.

**Lemma** 3.6.3. *$H^{2k}(\boldsymbol{R}^*S^k; [Z, -1]^0)$ is isomorphic to $Z$ if $k$ is odd, $Z_2$ if $k$ is even.*

**Proof.** $H^{2k}(\boldsymbol{R}^*S^k; [Z, -1]^0) = H^{2k}(\boldsymbol{R}^*S^k, PS^k; [Z, -1])$. Now $\boldsymbol{R}^*S^k$ is a $2k$-manifold with boundary $PS^k$, which is oriented if $k$ is even and unoriented if $k$ is odd. In the even case, the generator of $H^{2k}(\boldsymbol{R}^*S^k; [Z, -1]^0)$ may be taken to be the top class.

**Lemma** 3.6.4. *$H^{2k}(\boldsymbol{R}^*S^k; Z^0)$ is isomorphic to $Z$ if $k$ is even, $Z_2$ if $k$ is odd.*

**Proof.** The proof is similar to that of Lemma 3.6.3, above. We leave the details to the reader.

**Lemma** 3.6.5. $H^{2k}(\boldsymbol{R}^*S^k; [Z \oplus Z, E]^0) \cong Z$.

PROOF. We consider two cases; $k$ even and $k$ odd. We have exact sequences of sheaves

$$e_1 : 0 \to Z = [Z, 1] \xrightarrow{\alpha} [Z \oplus Z, E] \xrightarrow{\beta} [Z, -1] \to 0,$$
$$e_2 : 0 \to [Z, -1] \xrightarrow{\gamma} [Z \oplus Z, E] \xrightarrow{\epsilon} Z = [Z, 1] \to 0,$$

where the maps $\alpha$, $\beta$, $\gamma$, and $\epsilon$ can be defined on the underlying groups as follows: $\alpha x = (x, x)$ and $\gamma x = (x, -x)$ for all $x \in Z$, and $\beta(x, y) = x - y$ and $\epsilon(x, y) = x + y$ for all $x, y \in Z$. (Note that $\alpha$, $\beta$, $\gamma$, and $\epsilon$ all respect the appropriate actions; i.e., $E \circ \gamma = \gamma \circ (-1)$, etc.) Note that $\epsilon \circ \alpha$ is multiplication by 2. Corresponding to $e_1$ and $e_2$, we have exact sequences in cohomology, where $\delta_1$ and $\delta_2$ are the Bokstein homomorphisms

$$(e_1)_* : \xrightarrow{\delta_1} Z \xrightarrow{\alpha_*} H^{2k}(\boldsymbol{R}^*S^k, PS^k; [Z \oplus Z, E]) \xrightarrow{\beta_*} Z_2 \to 0,$$
$$(e_2)_* : \xrightarrow{\delta_2} Z_2 \xrightarrow{\gamma_*} H^{2k}(\boldsymbol{R}^*S^k, PS^k; [Z \oplus Z, E]) \xrightarrow{\epsilon_*} Z \to 0,$$

where $\epsilon_* \circ \alpha_*$ is multiplication by 2. General algebraic considerations show that $\epsilon_*$ must be an isomorphism, and we are done. If $k$ is odd, the proof is the same with the roles of the sequences $e_1$ and $e_2$ reversed.

LEMMA 3.6.6. $H^{2k}(\boldsymbol{R}^*S^k; [Z \oplus Z, -E]^0) \cong Z$.

PROOF. Analogous to $e_1$ and $e_2$ in the proof of Lemma 3.6.5, above, $[Z \oplus Z, -E]$ may be expressed both as an extension of Z by $[Z, -1]$ and as an extension of $[Z, -1]$ by Z. We proceed as above.

From Lemmas 3.6.1 through 3.6.6, we immediately obtain

THEOREM 3.6.7. *$[S^k \subset M]_f$ is isomorphic to $\sum_A Z \oplus \sum_B Z_2 \oplus \sum_{\Theta \cup \Lambda} Z$ if $k$ is odd, and to $\sum_A Z_2 \oplus \sum_B Z \oplus \sum_{\Theta \cup \Lambda} Z$ if $k$ is even.*

3.7. *Explicit geometric construction of $[S^k \subset M]_f$.* We retain the notation of §3.5, and assume that $f : S^k \to M$ is an embedding, where $f(s_0) = x$; $s_0$ is the basepoint of $S^k$. Recall that we let $v \in S(M \times R^\infty)$ such that $\pi v = (x, 0)$. We can insist that $\boldsymbol{i} : S^k \to R$ be an embedding where $\boldsymbol{i}(s_0) = 0$.

Let $\sigma$ be a $2k$-cell of $\boldsymbol{R}^*S^k$ such that, for some $w^* \in PS^k$, $w^* \in \partial\sigma$ and $\boldsymbol{R}^*(f, \boldsymbol{i})(w^*) = v^* = \{v, -v\} \in P(M \times R^\infty)$. Pick a cell $\tau \subset \boldsymbol{R}S^k$ such that $\pi\tau = \sigma$ and $w \in \partial\tau$ such that $\pi w = w^*$. Choose any ordered pair $(s_1, s_2) \in \operatorname{Int} \tau$, and let $N_1$ and $N_2$ be closed ball-shaped neighborhoods of $s_1$ and $s_2$, respectively, such that $N_1 \times N_1 \subset \operatorname{Int} \tau$. Let $\alpha : I \to \tau$ be a path such that $\alpha(0) = (s_1, s_2)$, $\alpha(1) = w$, and $\alpha(t) \in \operatorname{Int} \tau$ for all $t < 1$. Then, for all $0 \leqq t \leqq 1$, $\alpha(t) = (\alpha_1(t),$

$\alpha_2(t)$), where $\alpha_i : I \to S^k$ is any path from $s_i$ to $s_0$, for $i = 1$ and 2. Pick any $g \in \pi_1 = \pi_1(M, x)$. Let $\beta : I \to M$ be a simple smooth path such that $\beta(0) = f(s_2)$, $\beta(1) = f(s_1)$, and the loop $(f \circ s_2^{-1}) \cdot \beta(f \circ s_1^{-1})$ represents $g$. Let $B$ be a neighborhood of $\beta(I)$ homeomorphic to a $(2k + 1)$-ball such that $B \cap f(S^k) = f(N_1) \cup f(N_2)$. Let $f_t : S^k \to M$, for $0 \leqq t \leqq 1$, be any homotopy of differentiable maps such that $f_0 = f$, $f_t|(S^k - N_2) = f|(S^k - N_2)$ for all $t$, and $f_t(N_2) \subset B$ for all $t$, and where the map $F : S^k \times I \to M \times I$, where $F(s, t) = f_t(s)$ for all $s \in S^k$ and $0 \leqq t \leqq 1$, has just one double point; namely $F(s_1, \frac{1}{2}) = F(s_2, \frac{1}{2})$, and $F(S^k \times I)$ meets itself transversely at $(f(s_1), \frac{1}{2})$.

The liftings $\Phi[f_t]$ and $\Phi[f]$ are certainly homotopic on the $(2k - 1)$-skeleton of $\boldsymbol{R}^*S^k$; in fact we may define $g_u : ((\boldsymbol{R}^*S^k)^{2k-1}, PS^k) \to (Y', Z')$ for $0 \leqq u \leqq 1$, explicitly, using the homotopy $\{f_t\}$ (we omit the details; $\{g_u\}$ is essentially the $\Phi$-construction (cf. 3.2) restricted to the $(2k - 1)$-skeleton). Now consider the difference class:

$$d^{2k} = d^{2k}(\Phi[f], \Phi[f_t]; g_u) \in C^{2k}(\boldsymbol{R}^*S^k; \pi_{2k}).$$

We can identify the stalk of $\pi_{2k}$ over $w^*$ with that of $\pi_{2k}(\pi_M)$ over $v^*$, and we have

LEMMA 3.7.1. *$d^{2k}(\sigma) = \pm\Upsilon(g)$ and $d^{2k}(\sigma') = 0$ for any $2k$-cell $\sigma' \neq \sigma$. Furthermore, we may insist $d^{2k}(\sigma) = \Upsilon(g)$, by redefining $\{f_t\}$ if necessary.*

PROOF. Using the $\Phi$-construction, we may extend the homotopy $\{g_u\}$ over $\sigma'$ for any $\sigma' \neq \sigma$, hence $d^{2k}(\sigma') = 0$. Now (cf. 2.4) $d^{2k}(\sigma)$ is represented by a map $h : \partial(\sigma \times I) \to Y$ such that, for all $(a, t) \in \partial(\sigma \times I)$ and all $0 \leqq u \leqq 1$,

$$h(a, t) = \begin{cases} \boldsymbol{R}^*(f_{2tu}(a), \boldsymbol{i}) & \text{if } 0 \leqq u \leqq \frac{1}{2}, \\ \boldsymbol{R}^*(f_t(a), (2 - 2u)\boldsymbol{i}) & \text{if } \frac{1}{2} \leqq u \leqq 1, \end{cases}$$

whose composition with $p_1 \circ \pi^{-1} \circ \gamma$ (as in diagram 3.4-1) is homotopic to $\pm\Upsilon(g)$. The sign is ambiguous, because there are essentially two ways an $r$-manifold can intersect itself transversely in a $2r$-manifold. Both ways are possible in this case, hence we are done.

We now define $\langle g \rangle \in [S^k \subset M]_f$ to be $[f_t]$, where $\{f_t\}$ is described above. Theorem 1.2.1 then follows immediately from Theorem 3.6.7.

3.8. *Free isotopy classes.* In this paragraph, we assume that $f: S^k \to M$ is a small embedding, i.e., $f(S^k)$ lies in a single chart of $M$. Again, we assume that $k \geqq 2$ and $n = 2k + 1$. We now investigate the affine action of $\pi_1(M^{S^k}, f)$ on $[S^k \subset M]_f$. Let $s_0 \in S^k$ and $m_0 \in M$ be basepoints, and assume that $f$ is basepoint-preserving.

DEFINITION 3.8.1. Let $\{f_t\}$ be a differentiable self-homotopy of $f$. We say that $\{f_t\}$ is *small* if $f_t(S^k)$ lies in a single chart of $M$ for each $t$, and we say that $\{f_t\}$ is *large* if $f_t(s_0) = m_0$ for all $t$.

We remark that the subsets $L_f$ and $S_f$ of $\pi_1(M^{S^k}, f)$ represented by large and small self-homotopies of $f$, respectively, are subgroups, and that $L_f \cong \pi_{k+1}(M, m_0)$ and $S_f \cong \pi_1(M, m_0)$. $L_f$ is normal, and $\pi_1(M^{S^k}, f)$ is a semidirect product of $L_f$ with $S_f$; we leave this fact as an exercise.

THEOREM 3.8.1. *If $x \in \pi_1(M^{S^k}, f)$ is represented by a small self-homotopy $\{f_t\}$, then $\gamma(\langle g\rangle, x) = (-1)^{d(h)}\langle h^{-1}gh\rangle$ for all $g \in \pi_1(M, m_0)$, where $h$ is the element of $\pi_1(M, m_0)$ represented by the loop $\{f_t(s_0)\}$.*

PROOF. $\langle g\rangle$ is represented by a homotopy which extends a pseudopod out from $f(S^k)$, around a loop $\sigma$ representing $g$, then linking $f(S^k)$ with linking number 1. The action of $x$ drags the entire image $f(S^k)$ around the loop $\alpha$, where $\alpha(t) = f_t(s_0)$ for all $t$; the pseudopod is now forced to follow the loop $\alpha^{-1}\sigma\alpha$ and link with linking number $(-1)^{d(h)}$.

THEOREM 3.8.2. *If $x \in L_f$, then $\gamma(\langle g\rangle, x) = \langle g\rangle + \gamma(0, x)$ for all $g \in \pi_1(M, m_0)$.*

PROOF. Since $x$ is represented by a large self-homotopy $\{f_t\}$, we may assume that $\{f_t\}$ leaves a neighborhood of $s_0$, $N$, fixed; we can insist that $N = B \cap \bigcup_t f_t(S^k)$, where $B$ is the $(2k+1)$-ball used to construct $\langle g\rangle$ in §3.7. Our theorem follows, because the difference cochain may be evaluated separately on $N$ and $S^k - N$, and the results added.

Theorem 1.2.2 follows directly from Theorem 3.8.1; we may extend this result slightly, using 3.8.2, as follows:

THEOREM 3.8.3. *If $f: S^k \to M$ is a basepoint-preserving small embedding, then the subset of $[S^k \subset M]$ consisting of those isotopy classes homotopic to $f$ can be put into one-to-one correspondence with the set of orbits of the cokernel of a homomorphism $\Xi: \pi_{k+1}(M, m_0) \to [S^k \subset M]_f$ by a right action of $\pi_1(M, m_0)$; provided $k \geqq 2$ and* dim $M = 2k + 1$.

PROOF. Let $\iota: \pi_{k+1}(M, m_0) \to \pi_1(M^{S^k}, f)$ be the monomorphism onto $L_f$ induced by the map $S^{k+1} \to \Omega S^k$, and let $\Xi$ be defined by:

$\Xi(x) = \gamma(0,_\iota(x))$. By Theorem 3.8.2, $\Xi$ is a homomorphism. We can easily check that the action of $S_f \cong \pi_1(M, m_0)$ on $[S^k \subset M]_f$ is consistent with the usual right action of the fundamental group of a space on a higher homotopy group, via $\Xi$. We leave the details to the reader.

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