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DIOPHANTINE APPROXIMATION IN A VECTOR SPACE

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1. Throughout this paper, we suppose that S is a real inner product space of dimension at least two and that e is a point of S with unit norm. We denote by S' that subspace of S which has the property that if z belongs to it, then $((z, e)) = 0$, and let u denote a point of S' which has unit norm. For each point z of S, we denote the point $2((z, u))u - z$ by $\bar{z}$ and the point $\bar{z}/\|z\|^2$ by $1/z$. (We assume that there is adjoined to S a "point at infinity" with the usual conventions.) It should be noted that if S is one of E^2, E^3, E^5, and E^9, e is the unit vector with last coordinate 1, and u is the unit vector with first coordinate 1, then $1/z$ restricted to S' reduces to the ordinary reciprocal for real numbers, complex numbers, quaternions, and Cayley numbers, respectively.

Suppose that U is a subset of S' having the following properties:

(i) each element of U is a point of S' with unit norm,

(ii) u belongs to U,

(iii) if x belongs to U, then so do $-x$ and $\bar{x}$,

(iv) if x and y belong to U, then $2((x, y))$ is integral, and

(v) if z is a point of S', there exists a finite sequence $x_1, x_2, \cdots, x_k$, with each term in U, and a finite sequence $n_1, n_2, \cdots, n_k$, with each term an integer, such that $\|z - (n_1x_1 + n_2x_2 + \cdots + n_kx_k)\| < 1$. It is not difficult to see that such a set U exists even when S is infinite dimensional. Notice that when S is one of E^2, E^3, E^5, and E^9 with e and u as above, we may take U to be the set of all units of an appropriate ring of integers.

2. We will now give some definitions which facilitate the statement of the diophantine approximation result below.

A point z of S' is said to be *integral with respect to U* (or *U-integral*) if and only if there exists a finite sequence $x_1, x_2, \cdots, x_k$, with each term in U, and a finite sequence $n_1, n_2, \cdots, n_k$, with each term an integer, such that $z = n_1x_1 + n_2x_2 + \cdots + n_kx_k$. A point z of S' is said to be *rational with respect to U* (or *U-rational*) if and only if there exists a finite sequence $b_0, b_1, b_2, \cdots, b_k$, with each term U-integral, such that z is the value of the continued fraction

$$(2.1) \qquad b_0 + \frac{1}{b_1 +}\,\frac{1}{b_2 +}\,\cdots\,\frac{1}{+ b_k}.$$

A point of S' which is not U-rational is said to be *irrational with respect to U* (or *U-irrational*). If each one of $b_0, b_1, b_2, \cdots, b_k$ is U-integral, we denote by $D(b_0, b_1, b_2, \cdots, b_k)$ the number

$$\|b_k\|\,\|b_{k-1}+1/b_k\|\cdots\cdots\left\|b_1+\frac{1}{b_2+}\quad\cdots\quad\frac{1}{+b_k}\right\|$$

and we say that $b_0, b_1, b_2, \cdots, b_k$ is *primary* whenever it is true that if each one of $a_0, a_1, a_2, \cdots, a_n$ is U-integral and

$$b_0+\frac{1}{b_1}\,\frac{1}{+b_2+}\quad\cdots\quad\frac{1}{+b_k}=a_0+\frac{1}{a_1}\,\frac{1}{+a_2+}\quad\cdots\quad\frac{1}{+a_n},$$

then $D(b_0, b_1, b_2, \cdots, b_k) \leqq D(a_0, a_1, a_2, \cdots, a_n)$. When $b_0, b_1, b_2, \cdots, b_k$ is primary and z is the value of (2.1), then $D(b_0, b_1, b_2, \cdots, b_k)$ is denoted by $Q(z)$.

It should be noted that in the examples mentioned above (E^2, E^3, E^5, and E^9), that with a suitable choice of U, the definitions of U-integral, U-rational, and U-irrational are equivalent to the ordinary definitions of integral, rational, and irrational. The number $Q(z)$ corresponds to the modulus of the denominator of z "expressed in lowest terms". (It may be shown that for every point z which is U-rational, $Q(z)$ does exist.)

3. Let F denote the set of all points z of S such that $\|z\| < 1$ and, for every point x of U, $\|z\| \leqq \|z - x\|$ and let m denote the greatest number t such that, for every point z of F, $((z, e)) \geqq t$.

THEOREM. *If $c \geqq 1/(2m)$, then for every point w of S' which is irrational with respect to U, there exist infinitely many points z of S' which are rational with respect to U such that*

$$\|w - z\| < c/Q^2(z), \tag{3.1}$$

while if $c < 1/5^{1/2}$, there is a point w of S' which is irrational with respect to U such that there are at most a finite number of points z of S' which are rational with respect to U such that (3.1) *holds true.*

A proof of this theorem and some related results will appear elsewhere. The techniques used in the proof of this theorem resemble those used by Ford in [1]. We let M denote the set of all transformations T from S onto S having the property that there exists a finite sequence $b_0, b_1, b_2, \cdots, b_{2k}$, with each term U-integral, such that for every point z of S, $T(z)$ is

$$b_0+\frac{1}{b_1}\,\frac{1}{+b_2+}\quad\cdots\quad\frac{1}{+b_{2k}+z}.$$

The set M forms a group under composition and corresponds to the group of Picard used by Ford. In fact, if S is E^3 with $e = (0, 0, 1)$, $u = (1, 0, 0)$, and U is the set consisting of u, $-u$, $(0, 1, 0)$, and

$(0, -1, 0)$, then it is the group of Picard extended to E^3. The set F corresponds to the fundamental region used by Ford and the collection of all of its images under elements of M to the subdivision of the upper half-space.

Reference

1. L. R. Ford, *On the closeness of approach of complex rational fractions to a complex irrational number*, Trans. Amer. Math. Soc., v. 27 (1925), pp. 146–154.

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