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CONTINUED FRACTIONS IN BANACH SPACES

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We assume that the proper generalization of continued fractions to complex Banach spaces is the limit of holomorphic maps generated by composition of linear fractional maps. The deepest results are those of Mac Nerney [10], [11], in which a major part of the work of H. S. Wall on positive definite continued fractions is generalized to $B(H)$, the space of bounded operators on a Hilbert Space. In case the map is from the plane into $B(H)$, Mac Nerney also obtains connections with moment problems. Russian work on linear fraction maps and applications may be found in [9].

In $B(H)$ we are forced to use symmetry in our definition due to the lack of commutativity. Mac Nerney observed this in his work and we note that the Möbius transformations in $B(H)$ have the symmetric form $L \cdot (I - BB^*)^{-1/2}(Z + B)(I + B^*Z)^{-1}(I - B^*B)^{1/2}$ where $*$ denotes the adjoint, L is a linear isometry, and Z and B are in the open unit ball [7]. This lack of commutativity makes the recursion relations for continued fractions extremely complicated [10, p. 675], we shall try to establish convergence by other means.

We consider the continued fraction $A_1/I + A_2/I + \cdots$ to be generated by transformations $t_n(w) = A_n(I + w)^{-1}$ and define the continued fraction as $\lim_{n\to\infty} T_n(0) = \lim_{n\to\infty} t_1 \cdot t_2 \cdots t_n(0)$. We prefer the limit in the uniform operator topology on $B(H)$, but in some cases only strong limits are obtained.

Suppose first that each A_i is a positive operator, $(A_i x, x) \geqq 0$, for each x in H. Since the product of positive operators may not be positive unless they commute, and since we wish the limit to be a positive operator we modify the t_n to $t_n(w) = A_n(I + w)^{-1} A_n^*$ or $(\sqrt{A_n})(I + w)^{-1}(\sqrt{A_n})$. Then if $w \geqq 0$, $t_n(w) \geqq 0$. The usual order for operators $A \leqq B$ means $(Ax, x) \leqq (Bx, x)$.

Proposition. *If $A_i \geqq 0$, then the odd approximants decrease and exceed the even approximants which increase. Hence the even and odd approximants converge strongly. If the A_i are uniformly bounded and commute the continued fraction converges in the uniform operator topology to a positive operator.*

If the A_i commute, then via the Gelfand map the algebra of operators generated by A_i is isomorphic to continuous real valued functions and the positive operators correspond to positive functions. Since the

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even and odd approximants for real valued uniformly bounded continuous functions converge monotonically to a continuous function, the convergence is uniform; this via the Gelfand map corresponds to convergence in the uniform operator topology. We conjecture that commutativity can be removed in the last statement, but have only been able to prove convergence under uncomfortable constraints.

APPLICATION. *Suppose* $A > \lambda^2 I$ *for* $\lambda > 0$. *Then the continued fraction* $(A - \lambda^2 I)/2\lambda I + (A - \lambda^2 I)/2\lambda I + \cdots$ *converges uniformly to a positive operator* X *so that* $(X + \lambda I)^2 = A$.

The techniques of the proof are similar to those used by McFarland [12]. One needs to use the fact that $A_n \to A$, $B_n \to B \Rightarrow A_n B_n \to AB$ in the uniform operator topology. The above algorithm appears preferable to the usual algorithm in which only an increasing sequence of approximants to the square root of an operator are constructed and only a strong limit is obtained. An interesting continued fraction algorithm for fractional powers of accretive operators was obtained by Fair [3].

We now state a convergence criterion which is a simple variation of the contraction mapping principle, that seems appropriate for applications to continued fractions. This idea is essentially due to E. P. Merkes (unpublished).

LEMMA. *Suppose* X *is a Banach space (or complete metric space). For* $n = 1, 2, 3, \cdots$, *let* $Z_{n-1} \subset X$, $a_n \in Z_n$, *and* t_n *be a map so that* $t_n(Z_n) \subset Z_{n-1}$. *Let* $f_n = t_1 \cdot t_2 \cdots t_n(a_n)$, *then* $\{f_n\}$ *converges provided* $\|t_n(U_1) - t_n(U_2)\| \leqq K_n\|U_1 - U_2\|$ *on* Z_n, *some infinite subsequence of* $\{Z_n\}$ *is uniformly bounded, and* $\Pi K_n = 0$.

THEOREM. *Suppose for* $n = 1, 2, 3, \cdots, 0 < g_{n-1} < 1$; $\|A_n\| \leqq K_n(1 - g_{n-1})g_n$, $K_n \leqq 1$, *and* $\Pi K_n((1 - g_n)/g_{n+1}) = 0$. Then $A_1/I + A_2/I + \cdots$ *converges.*

PROOF. Let $Z_n = \{x : \|x - I\| \leqq 1 - g_{n+1}\}$, $t_n(Z) = I + A_n Z^{-1}$. If $Z \in Z_n$, then Z is invertible and $\|Z^{-1}\| \leqq 1/g_{n+1}$. Hence $t_n(Z)$ is in Z_{n-1}. Now $\|t_n(U_1) - t_n(U_2)\| \leqq \|A_n\| \; \|U_1{}^{-1}(U_2 - U_1)U_2{}^{-1}\| \leqq K_n(1 - g_n)/g_{n+1} \; \|U_2 - U_1\|$. The lemma may now be applied with $a_n = I$.

See [4] for a different approach.

APPLICATION. *Suppose,* $\lambda > 0$, $\|A/2 - I\| < 1$, *then the continued fraction* $(A - \lambda^2 I)/2\lambda I + (A - \lambda^2 I)/2\lambda I + \cdots$ *converges to an operator* X *so that* $\|X\| < \lambda$ *and* $(X + \lambda I)^2 = A$.

The elements in the above expansion are less than 1/4 in norm so that the theorem applies to give convergence.

We apply the lemma to a situation in which the norms of the operators may be large. We state the theorem in a simple way which is easily extended to the general case proved in the plane in [8].

THEOREM. *Suppose for* $n = 1, 2, 3, \cdots, \|A_{2n-1}\| \leqq k/4$ *for* $k < 1$, *and* $\|A_{2n}^{-1}\| \leqq 4/9$, *then the continued fraction* $A_1/I + A_2/I + \cdots$ *converges.*

PROOF. Let $Z_{2n} = \{Z : \|Z - I\| \leqq 1/2\}$, $Z_{2n-1} = \{Z : \|Z\| \leqq 2\}$. Let $t_{2n+1}(Z) = I + A_{2n+1}Z$, $t_{2n}(Z) = I + A_{2n}Z^{-1}$. If $Z \in Z_{2n+1}$, then $\|t_{2n+1}(Z) - I\| \leqq \|A_{2n+1}Z\| \leqq 1/2$ so $t_{2n+1}(Z_{2n+1}) \subset Z_{2n}$. Let $Z \in Z_{2n}$ then $\|Z\| \leqq 3/2$ and $\|(t_{2n}(Z) - I)^{-1}\| = \|ZA_{2n}^{-1}\| \leqq 3/2 \cdot 4/9 = 2/3$. Let $w = (t_{2n}(Z) - I)^{-1}$. For Z in Z_{2n}, $\|(I + w)^{-1}\| \leqq 3$. One can show that $t_{2n}(Z)^{-1} = (I + w)^{-1}w$ and hence $\|t_{2n}(Z)^{-1}\| \leqq 2$ or $t_{2n}^{-1}(Z_{2n}) \subset Z_{2n-1}$. Now $\|t_{2n+1}(U_1) - t_{2n+1}(U_2)\| \leqq \|A_{2n+1}(U_1 - U_2)\| \leqq k/4 \|U_1 - U_2\|$ for U_1, U_2 in Z_{2n+1}. For $U_i (i = 1, 2)$ in Z_{2n}, let $w_i = (t_{2n}(U_i) - I)^{-1}$, then $\|t_{2n}^{-1}(U_1) - t_{2n}^{-1}(U_2)\| = \|w_1(I + w_1)^{-1} - (I + w_2)^{-1}w_2\| = \|(I + w_2)^{-1}(w_1 - w_2)(I + w_1)^{-1}\| \leqq 9\|w_1 - w_2\| \leqq 9 \cdot 4/9 \|U_1 - U_2\|$. An application of the lemma supplies convergence of $T_n(I)$. Note that symmetry can be obtained by defining A_i as the product of B_iC_i in our theorems. We leave the details to the reader. Finally we suggest that the following theorem of Earle and Hamilton [1] in conjunction with our contracting lemma may be useful for future work.

THEOREM. Let *f be a holomorphic map of an open bounded subset B of a Banach space strictly inside B. Then there is a Finsler metric under which f is a contraction.*

We also suggest as an interesting problem the solution of the quadratic equation $Z^2 - AZ - B = 0$ by continued fractions in Banach spaces, since this equation is of interest in stability theory of differential equations [2]. Some continued fraction solutions of this equation have been obtained by Fair [5], [6].

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