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P-FRACTIONS AND THE PADÉ TABLE[1]

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The regular continued fraction of a positive real number x_0 is obtained by writing x_0 as the sum of the greatest integer $[x_0]$ in x_0 and a remainder r_1, $0 \leqq r_1 < 1$, that is, $x_0 = [x_0] + r_1$. If $r_1 > 0$ we replace the "small" number r_1 by the "large" one $1/r_1 = x_1$ and repeat the process with x_1, that is;

$$x_0 = [x_0] + r_1 = [x_0] + \frac{1}{1/r_1}$$

$$= [x_0] + \frac{1}{x_1} = [x_0] + \frac{1}{[x_1] + r_2}.$$

Continuing in this fashion and setting $[x_i] = b_i$, $i = 0, 1, 2, \cdots$, we arrive at the finite or infinite regular continued fraction for x_0

$$x_0 = b_0 + \frac{1}{b_1 +}\,\frac{1}{b_2 +}\cdots.$$

We follow an analogous procedure for the power series

$$f = \sum_{n=-N_0}^{\infty} a_n x^n$$

$$= a_{-N_0} x^{-N_0} + \cdots + a_{-1}x^{-1} + a_0 + a_1 x + \cdots.$$

The "small" part of f is the series $\sum_1^\infty a_n x^n$ whose first non-vanishing term we denote by $a_{N_1} x^{N_1}$ and formally write

$$(a_{N_1} x^{N_1} + a_{N_1+1} x^{N_1+1} + \cdots)(a'_{-N_1} x^{-N_1} + a'_{-N_1+1} x^{-N_1+1} + \cdots) = 1,$$

where a'_{-N_1+n} is uniquely determined by $a_{N_1}, \cdots, a_{N_1+n}$, for $n = 0, 1, 2, \cdots$. We set $\sum_{-N_0}^{0} a_n x^n = b_0$ and have

$$f = \sum_{n=-N_0}^{\infty} a_n x^n = b_0 + 1 \Big/ \sum_{n=-N_1}^{\infty} a'_n x^n.$$

The process is continued to produce a finite or infinite continued fraction, called the principal part expansion of f,

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$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots,$$

where b_n is a polynomial in $1/x$ *of degree* N_n and $N_n \geqq 1$, $n = 1, 2, 3, \cdots, b_0$ may be a constant, including zero.

A *P-fraction* is defined to be any continued fraction

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots,$$

where the b_n's are polynomials in $1/x$ of degrees N_n with $N_n \geqq 1$ for $n = 1, 2, 3, \cdots$. We denote its nth approximant by A_n/B_n and state some theorems [**1, 2, 3**].

THEOREM 1. *To each P-fraction corresponds a unique power series,* $f = \sum_{n=-N_0}^{\infty} a_n x^n$, *such that the power series expansion of* A_n/B_n *agrees with* f *up to but not including the term of degree* $2N_1 + \cdots + 2N_n + N_{n+1}$.

THEOREM 2. *To different P-fractions correspond different power series.*

THEOREM 3. *A power series* $\sum_{-N_0}^{\infty} a_n x^n$ *corresponds to its own principal part expansion.*

THEOREM 4. *A P-fraction is finite if and only if the corresponding series is a rational function.*

Regular C-fractions are not P-fractions but some other C-fractions are. The P-fraction of a series $\sum_0^\infty c_n x^n$ (without principal part) is identical to its associated continued fraction when it exists, that is, when all the persymmetric determinants $\phi_m = |c_{i+j-1}|$, $i, j = 1, 2, \cdots, m$, $m = 1, 2, \cdots$ are different from zero. We set $\phi_0 = 1$.

If, on the other hand, $\phi_m \neq 0$ if and only if $m = M_n$, $n = 0, 1, 2, \cdots$ and $0 = M_0 < M_1 < M_2 < \cdots$, then the degree of b_n in the principal part expansion of f is $N_n = M_n - M_{n-1}$, $n = 1, 2, \cdots$, and $b_0 = c_0$. It is easy to show that by making arbitrary small perturbations of the coefficients, from c_n to c_n^*, we can assume that the corresponding determinants ϕ_n^* are all different from zero, so that the series $\sum_0^\infty c_n^* x^n$ has an associated continued fraction. This fact is used to establish the connection between an arbitrary P-fraction and associated continued fractions [**4**].

THEOREM 5. *Let* $f = \sum_0^\infty c_n x^n$ *be a power series with P-fraction P for which the determinants* ϕ_m *differ from zero if and only if* $m = M_n$, $n = 0, 1, 2, \cdots$, $0 = M_0 < M_1 < M_2 \cdots$. *Let* $f^* =$

$\sum_0^\infty c_n{}^* x^n$ be such that $|c_n - c_n{}^*| < \epsilon$ and $\phi_n{}^* \neq 0$, $n = 0, 1, 2, \cdots$, and let the associated continued fraction of f^* be A with approximants K_m/L_m. Then that contraction P^* of A whose approximants are those K_m/L_m where $m = M_n$, $n = 0, 1, 2, \cdots$ approaches P as $\epsilon \to 0$ in the sense that, after a possible equivalence transformation, the elements of P^* approach those of P and the coefficients of K_{M_n}/L_{M_n} approach those of A_n/B_n.

P-fractions are related to the Padé table as follows.

THEOREM 6. *Let $f = \sum_0^\infty c_n x^n$ be any power series with $c_0 \neq 0$, s any fixed integer ($s = 0, \pm 1, \pm 2, \cdots$) and*

$$b_0{}^{(s)} + \frac{1}{b_1{}^{(s)}} + \frac{1}{b_2{}^{(s)}} + \cdots$$

with approximants $A_n{}^{(s)}/B_n{}^{(s)}$ be the P-fraction of $x^s f = \sum_0^\infty c_n x^{n+s}$, then $\{A_n{}^{(s)}/x^s B_n{}^{(s)}\}$ is the sequence of consecutive distinct fractions down diagonal numbers s, $[m, m - s]$ or $[m + s, m]$, of the Padé table of f. In particular ($s = 0$), the approximants of the P-fraction of f are the distinct fractions of the main diagonal.

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