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ON COMPUTATIONAL APPLICATIONS OF THE THEORY OF MOMENT PROBLEMS

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SUMMARY. Many computational problems can be formulated as the task to evaluate a linear functional L for a given function φ when L is subject to a finite number of constraints.

In this paper we discuss tasks of this form. $L(\varphi)$ can be evaluated numerically either by approximating φ with linear combinations of a given system of functions $u_1, u_2, \cdots, u_n$ or by approximating L with a finite sum. In this way one can treat effectively such problems as the evaluation of a class of slowly convergent Fourier integrals, finding the limit value of sequences and the approximation of functions.

In our theoretical analysis we shall use the theory of the moment problem and consider generalizations of an optimization problem first studied by A. A. Markov and P. L. Čebyšev. We extend the results in various directions using the theory of semi-infinite programming.

1. **Introduction.** Let $[a, b]$ be a closed bounded interval and denote with $C[a, b]$ the space of functions f which are continuous on $[a, b]$ and normed by $\|f\| = \max_{a \leq t \leq b}|f(t)|$. Let L be a bounded linear functional defined on $C[a, b]$ and let $u_1, u_2, \cdots$ be a sequence of functions in $C[a, b]$. In this paper we shall discuss general and effective ways of solving:

TASK S: *Compute $L(\varphi)$ when $L(u_r) = \mu_r$, $r = 1, 2, \cdots$. The sequence $\mu_1, \mu_2, \cdots$ is given numerically and $\varphi(t)$ can be evaluated at any point t in $[a, b]$.* (No explicit representation of L is assumed to be known.)

THEOREM 1. *Let φ be a given function in $C[a, b]$ and $u_1, u_2, \cdots$ a sequence in $C[a, b]$ such that to every $\epsilon > 0$ one can find a finite linear combination $\sum_{r=1}^{N} c_r u_r$ meeting the condition*

$$\left\| \sum_{r=1}^{N} c_r u_r - \varphi \right\| < \epsilon. \tag{1}$$

Let L be a bounded linear functional on $C[a, b]$ and put $\mu_r = L(u_r)$, $r = 1, 2, \cdots$. Then the value of $L(\varphi)$ is uniquely defined by the conditions

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$$L(u_r) = \mu_r, \; r = 1, 2, \cdots.$$

We mention the following examples of sequences u_r which meet the condition (1).

$u_r(t) = t^{r-1}, \varphi \in C[a, b]$,

$u_r(t) = t^{\lambda_r t}$ where $\lambda_1 = 0, \lambda_r < \lambda_{r+1}, \sum_{r=2}^{\infty} 1/\lambda_r = \infty$ and $\varphi \in C[0, 1]$.

See [3] p. 197 and p. 232.

We list some different computational problems which can be regarded as special cases of Task S.

EXAMPLE 1. *Summation of certain slowly convergent series. Let* $\varphi \in C[0, 1]$. We want to evaluate

$$L(\varphi) = \sum_{j=1}^{\infty} \frac{1}{j^2} \varphi\left(\frac{1}{j}\right).$$

Put $u_r(t) = t^{r-1}$. We find

$$L(u_r) = \sum_{j=1}^{\infty} \frac{1}{j^2} \frac{1}{j^{r-1}} = \zeta(r + 1), \; r = 1, 2, \cdots$$

where ζ is Riemann's ζ-function. This is tabulated but can also be evaluated numerically by using Euler-Maclaurin's formula on the defining series. For further details see [12], [19].

EXAMPLE 2. *Numerical quadrature by inconvenient weight-function.* The special example

$$\int_0^1 e^{\sin t} \ln(1/t)\, dt$$

is used to illustrate the general idea. The function $\ln(1/t)$ has a singularity at the origin and hence Romberg's scheme or Newton-Cote's rules cannot be expected to be effective. We put

$$L(\varphi) = \int_0^1 \varphi(t) \ln(1/t)\, \mathrm{dt}$$

and take $u_r(t) = t^{r-1}$. Hence $L(u_r) = r^{-2}, r = 1, 2, \cdots$.

EXAMPLE 3. *Analytic continuation of certain functions.* Let $\mu_1, \mu_2, \cdots$ be a sequence admitting the representation

$$\mu_r = \int_0^1 t^{r-1}\, d\alpha(t), \; r = 1, 2, \cdots, \text{where} \int_0^1 |d\alpha(t)| < \infty.$$

We want to evaluate the function F defined by

$$F(z) = \sum_{r=1}^{\infty} z^{r-1}\mu_r. \tag{2}$$

We get

$$F(z) = \int_0^1 \frac{1}{1-tz}\, d\alpha(t) \text{ when } \int_0^1 t^{r-1}\, d\alpha(t) = \mu_r,\ r = 1, 2, \cdots.$$

The integral defines F for all z such that $\varphi(t) = (1 - tz)^{-1}$ is continuous on $[0, 1]$ and gives hence the analytic continuation of F outside the region of convergence of the series (2). See [7], [18].

EXAMPLE 4. *Evaluation of a general class of slowly convergent Fourier integrals.* Consider the task to compute

$$F(\omega) = \int_0^{\infty} e^{i\omega t} f(t)\, dt$$

when f admits the representation

$$f(t) = \int_0^{\infty} e^{-xt}\, d\alpha(x),\ t \geqq 0$$

where α is of bounded variation over $[0, \infty]$ and the numerical values of f are known in the set $0 = t_1 < t_2 < \cdots$. Put $h = t_2 - t_1$. After transformations we get

$$F(\omega) = -h \int_0^1 (\ln \lambda + ih\omega)^{-1}\, d\beta(\lambda),$$

$$f(t_r) = \int_0^1 \lambda^{t_r/h}\, d\beta(\lambda),\ r = 1, 2, \cdots.$$

This is a special case of task S above and the conditions of theorem 1 are met if $\sum_{r=2}^{\infty} t_r^{-1}$ is divergent. See [18], [22] and [24]. In a computational solution of task S one can only use a finite number of elements of the sequence $L(u_1), L(u_2), \cdots$, say $\mu_1, \mu_2, \cdots, \mu_n$. We discuss two approaches namely: (A) approximate φ with a finite linear combination of $u_1, u_2, \cdots$ and (B) replace L with a functional L_0 which is of the form

$$L_0(\varphi) = \sum_{i=1}^{q} m_i \varphi(t_i),\ t_i \in [a, b],\ q < \infty. \tag{3}$$

APPROACH A. Let

$$Q_n = \sum_{r=1}^{n} y_r u_r \tag{4}$$

be given and put $\epsilon_n = \|\varphi - Q_n\|$. Then

$$L(Q_n) = \sum_{r=1}^{n} y_r \mu_r \text{ and } |L(\varphi) - L(Q_n)| \leqq \epsilon_n \|L\|.$$

It is possible to take Q_n as a linear combination of the form (4) which minimizes $\|\varphi - Q_n\|$ but this is a major computational task. But Powell [27] has shown that in the case $u_r(t) = t^{r-1}$ (compare examples 1–3 above) a very good realization of this goal is achieved if we select Q_n as the polynomial of degree less than n which interpolates φ in $t_1, t_2, \cdots, t_n$ where

$$t_i = \frac{a+b}{2} + \frac{b-a}{2} \cos \theta_i, \ \theta_i = \frac{i-0.5}{n} \pi, \ i = 1, 2, \cdots, n.$$

Hence Q_n can be constructed by means of n^2 arithmetic operations if we use the algorithms in [1], [15] and [16].

APPROACH B. Determine abscissae t_i and masses m_i such that $L_0(u_r) = L(u_r)$, $r = 1, 2, \cdots, n$ where L_0 is defined by (3). We note that if Q_n in (4) interpolates φ in t_i then

$$\sum_{r=1}^{n} y_r \mu_r = \sum_{i=1}^{q} m_i \varphi(t_i)$$

and hence the two approaches A and B give the same estimates for $L(\varphi)$.

2. **Moment problems.** In this section we shall describe how to derive error bounds associated with the approaches A and B to Task S by means of the theory of moment problems. We consider first the case when L is a positive bounded linear functional over $C[a, b]$, i.e., $f(t) \geqq 0$, $t \in [a, b]$ implies $Lf \geqq 0$. Then there is an α which is nondecreasing and bounded over $[a, b]$ and such that L admits the representation

$$Lf = \int_a^b f(t)\, d\alpha(t), \ \|L\| = \int_a^b d\alpha(t).$$

We extend later our results to general continuous linear functionals over $C[a, b]$ when an upper bound for $\|L\|$ is known. We define first two general optimization problems.

Let $[a, b]$ be a closed bounded finite interval, $u_1, u_2, \cdots, u_n$ and φ given continuous functions over $[a, b]$ and $\mu_1, \mu_2, \cdots, \mu_n$ given real numbers. We consider the two problems (P) and (D) defined as follows:

(P)	(D)
$\sup_{\alpha} \int_a^b \varphi(t)\, d\alpha(t)$	$\inf_{y} \sum_{r=1}^{n} y_r \mu_r$
subject to $\int_a^b u_r(t)\, d\alpha(t) = \mu_r$	$\sum_{r=1}^{n} y_r u_r(t) \geqq \varphi(t),\ t \in [a, b]$
$r = 1, 2, \cdots, n, \alpha \nearrow$.	

(P stands here for primal, D for dual).

The tasks (P) and (D) are called a dual pair of semi-infinite linear programs. The solution of P could be used to find upper bounds for $L(\varphi)$ if L is positive. Sometimes this can be achieved easier. We prove

Lemma 1. *Let α_0 be a feasible solution of (P), y a feasible solution of* (D). *Then*

$$\int_a^b \varphi(t)\, d\alpha_0(t) \leqq \sum_{r=1}^{n} y_r \mu_r.$$

Proof. Since α_0 and y are feasible solutions $\sum_{r=1}^{n} y_r u_r(t) \geqq \varphi(t)$ and $\mu_r = \int_a^b u_r(t)\, d\alpha_0(t)$, $r = 1, 2, \cdots, n$. Hence

$$\sum_{r=1}^{n} y_r \mu_r = \int_a^b \sum_{r=1}^{n} y_r u_r(t)\, d\alpha_0(t) \geqq \int_a^b \varphi(t)\, d\alpha_0(t).$$

In order to describe the main results about (P) and (D) we need some concepts:

The subset M_n of R^n defined by

$$M_n = \{v \mid v_r = \int_a^b u_r(t)\, d\alpha(t),\ r = 1, 2, \cdots, n \text{ for some increasing } \alpha \text{ of bounded variation}\}$$

is called the *moment cone* associated with (P). Compare [26].

Remark. M_n is the smallest cone containing the curve

$$U = \{u(t) = u_1(t), u_2(t), \cdots, u_n(t) \mid t \in [a, b]\}.$$

Hence (P) is consistent if and only if $\mu = \mu_1, \mu_2, \cdots, \mu_n$ belongs to M_n.

The system $u_1, u_2, \cdots, u_n$ is said to meet *Krein's condition* on $[a, b]$ if there are constants $c_1, c_2, \cdots, c_n$ such that

$$\sum_{r=1}^{n} c_r u_r(t) > 0, \; t \in [a, b].$$

Using Helly's selection principle we can prove:

LEMMA 2. *The moment cone is closed if $u_1, u_2, \cdots, u_n$ meet Krein's condition.*

Compare [14] and [20]. We are now ready to state:

THEOREM 2. *Let $u_1, u_2, \cdots, u_n$ meet Krein's condition. Then one can prove the assertions:*

1. (D) *is always consistent.*
2. (D) *has unbounded solutions if and only if* (P) *is inconsistent.*
3. *If* (P) *is consistent* (P) *and (D) have equal optimal values.*
4. *The optimal value of* (P) *is assumed for a point-mass distribution with q mass-points, $q \leqq n$.*

The proof of these results can be based on the separation theorem for convex sets in R^n and Carathéodory's theorem on the representation of the convex hull. See [20].

THEOREM 3. *Let y be an optimal solution of* (D) *and let the optimal solution of* (P) *be α^*, a point-mass distribution with increase m_i at t_i, $i = 1, 2, \cdots, q$. Then the following relations hold:*

$$\sum_{i=1}^{q} m_i u_r(t_i) = \mu_r, \quad r = 1, 2, \cdots, n, \tag{5}$$

$$\sum_{r=1}^{n} y_r u_r(t_i) = \varphi(t_i), \quad i = 1, 2, \cdots, q. \tag{6}$$

If $u_1, u_2, \cdots, u_n$ and φ have a continuous derivative of the first order then

$$\sum_{r=1}^{n} y_r u_r{}'(t_i) = \varphi'(t_i), \qquad \text{if } a < t_i < b. \tag{7}$$

PROOF. (5) expresses the fact that α^* is a feasible solution of (P). Theorem 2 statement 3 gives $\int_a^b \varphi(t)\, d\alpha^*(t) = \sum_{r=1}^{n} y_r \mu_r$ or

$$\int_a^b \left(\sum_{r=1}^n y_r u_r(t) - \varphi(t) \right) d\alpha^*(t) = 0.$$

Thus if α^* has a positive increase m_i at t_i then

$$\sum_{r=1}^n y_r u_r(t_i) = \varphi(t_i)\ , \ i = 1, 2, \cdots, q.$$

This proves (6).

Let now t_i satisfy $a < t_i < b$, and be a mass-carrying point of $[a, b]$. Put $\psi(t) = \sum_{r=1}^n y_r u_r(t) - \varphi(t)$. Since y is feasible $\Psi(t) \geqq 0$, $t \in [a, b]$. We have just proved that $\Psi(t_i) = 0$. Therefore $\Psi'(t_i) = 0$ which is (7).

Hence we can construct the solutions of (P) and (D) if we can solve the nonlinear systems obtained by combining (5), (6), and (7). Compare [14] and [20].

We note that we have no assurance that the inf is assumed in (D).

Theorems 2 and 3 can easily be extended in various directions. Thus we directly establish that analogous statements hold for the problems ($\underline{\text{P}}$) and ($\underline{\text{D}}$) defined thus:

$$(\underline{\text{P}}) \qquad \begin{gathered} \min_{\alpha} \int_a^b \varphi(t)\, d\alpha(t)\ , \\ \int_a^b u_r(t)\, d\alpha(t) = \mu_r\, ,\ r = 1, 2, \cdots, n, \\ \alpha \nearrow. \end{gathered}$$

$$(\underline{\text{D}}) \qquad \begin{gathered} \sup_y \sum_{r=1}^n y_r \mu_r\ , \\ \sum_{r=1}^n y_r u_r(t) \leqq \varphi(t)\ ,\ t \in [a, b]. \end{gathered}$$

A difficulty in practical computation is that q in general cannot be determined in advance. We know that $q \leqq n$ but in many cases of practical interest q is close to $n/2$. We will treat some cases when a much better bound than $q \leqq n$ is available. For this we need a definition. The integer Z defined by

$$Z = \sum_{i=1}^q (\text{sign}(t_i - a) + \text{sign}(b - t_i))$$

is called the *index* of the point set $t_1, t_2, \cdots, t_q$ in $[a, b]$.

We note that if $t_1, t_2, \cdots, t_q$ are the mass-carrying points of an optimal solution of (P) then Z is equal to the number of equations in (6) and (7).

THEOREM 4. *Let $u_1, u_2, \cdots, u_n$ and φ form a Čebyšev system over* [a, b]. *Then μ can be represented with a point-mass distribution of index less than n if and only if μ is a boundary point of M_n. If μ belongs to the interior of M_n then there are exactly two representations of index n one of which gives the optimal solution of* (P), *the other of* ($\underline{\mathrm{P}}$).

See [26]. A particular instance of theorem 4 is when $u_r(t) = t^{r-1}$ and $\varphi^{(n)}(t) > 0$ on $[a, b]$. Then (P) and ($\underline{\mathrm{P}}$) are solved by means of standard methods, e.g., the algorithms in [11] An application to the summation of certain power series is given in [7]. We mention also that the task to estimate the ℓ^2-norm of the error in the computed solution of a linear system can be formulated as a moment problem provided certain general assumptions are made on the matrix of coefficients. An algorithmic solution is given in [10].

THEOREM 5. *Let $u_r(t) = t^{r-1}$, $r = 1, 2, \cdots, n$ and let φ be a polynomial of degree $\ell \geqq n$. Then there is an optimal solution of* (P) *whose index satisfies $Z \leqq \ell$.*

PROOF. Let y be an optimal solution of (D) and put

$$\Psi(t) = \sum_{r=1}^{n} y_r t^{r-1} - \varphi(t).$$

Since y is optimal it is also feasible and hence $\Psi(t) \geqq 0$. Ψ is a polynomial of degree ℓ and can therefore have only ℓ zeros counted with multiplicity. Since all optimal solutions of (P) have their mass-points in the zeros of Ψ we must have $Z \leqq \ell$. Q.E.D.

Using the same arguments we prove the more general result:

THEOREM 6. *Use the same notations as in the preceding theorem but assume now that φ is of the form*

$$\varphi(t) = Q_0(t) + Q_1(t)/Q_2(t)$$

where Q_i is a polynomial of degree ℓ_i, $i = 0, 1, 2$, and $Q_2(t) \neq 0$ $t \in [a, b]$. Then

$$Z \leqq \max(n - 1 + \ell_2, \ell_0 + \ell_2, \ell_1).$$

We want to consider problems when α is permitted to have nonnegative variation and prove:

THEOREM 7. *The following tasks are a dual pair of semi-infinite programs*

$$\sup_{\alpha} \int_a^b \varphi(t)\, d\alpha(t)$$

$$\int_a^b u_r(t)\, d\alpha(t) = \mu_r, \quad r = 1, 2, \cdots, n,$$

$$\int_a^b |d\alpha(t)| \leqq L,$$

$$\inf_{y} \sum_{r=1}^{n} y_r \mu_r + y_0 L$$

$$\left| \sum_{r=1}^{n} y_r u_r(t) - \varphi(t) \right| \leqq y_0, t \in [a, b].$$

PROOF. Write α as $\alpha^+ - \alpha^-$ where α^+ and α^- are increasing. Our first problem takes the form

$$\sup_{\alpha+,\alpha-} \int_a^b \varphi(t)\, d\alpha^+(t) - \int_a^b \varphi(t)\, d\alpha^-(t),$$

$$(8) \quad \begin{aligned} &\int_a^b u_r(t)\, d\alpha^+(t) - \int_a^b u_r(t)\, d\alpha^-(t) = \mu_r, r = 1, 2, \cdots, n, \\ &\int_a^b d\alpha^+(t) + \int_a^b d\alpha^-(t) + \epsilon = L, \\ &\quad \alpha^+ \nearrow \qquad \alpha^- \nearrow \qquad \epsilon \geqq 0. \end{aligned}$$

The dual of this problem reads

$$\inf_{y} \sum_{r=1}^{n} y_r \mu_r + L y_0,$$

$$\sum_{r=1}^{n} y_r u_r(t) + y_0 \geqq \varphi(t), \quad t \in [a, b],$$

$$- \sum_{r=1}^{n} y_r u_r(t) + y_0 \geqq -\varphi(t), \quad t \in [a, b],$$

$$y_0 \geqq 0.$$

This can be rewritten as

$$\inf_{y} \sum_{r=1}^{n} y_r \mu_r + L y_0,$$

$$\left| \sum_{r=1}^{n} y_r u_r(t) - \varphi(t) \right| \leqq y_0, \quad y_0 \geqq 0.$$

For all feasible solutions of this last mentioned problem $y_0 > 0$ and hence we must have $\epsilon = 0$ in (8) if φ is not a linear combination of $u_1, u_2, \cdots, u_n$. Compare [25].

We mention two applications of the last theorem:

Error bounds for quadrature rules. We seek $L(\varphi) = \int_0^1 \varphi(t)\, d\alpha(t)$ where φ is continuous on $[0, 1]$ and α of bounded variation over the same interval. We make the approximation

$$L(\varphi) \approx L_0(\varphi) = \sum_{i=1}^{q} m_i \varphi(t_i).$$

We want a bound for the error $E(\varphi) = L(\varphi) - I_0(\varphi)$. The abscissae t_i and weights m_i are determined numerically and we know that

$$\int_0^1 t^{r-1}\, d\alpha(t) - \sum_{i=1}^{q} m_i t_i^{r-1} = \epsilon_r, \quad r = 1, 2, \cdots, n.$$

Hence we can write

$$E(\varphi) = \int_0^1 \varphi(t)\, d\beta(t),$$

$$\int_0^1 t^{r-1}\, d\beta(t) = \epsilon_r\,, \quad r = 1, 2, \cdots, n,$$

$$\int_0^1 |d\beta(t)| = \sum_{i=1}^{q} |m_i| + \int_0^1 |d\alpha(t)|.$$

The task to maximize $E(\varphi)$ is a semi-infinite program of the type considered in theorem 7. Its dual reads

$$\inf \sum_{r=1}^{n} y_r \epsilon_r + y_0 \left(\sum_{i=1}^{q} |m_i| + \int_0^1 |d\alpha(t)| \right),$$

$$\left| \sum_{r=1}^{n} y_r t^{r-1} - \varphi(t) \right| \leqq y_0, \quad t \in [0, 1],$$

and hence an upper bound for $E(\varphi)$ is obtained if we evaluate the preference function of the dual for any approximating polynomial $Q(t) = \sum_{r=1}^{n} y_r t^{r-1}$. In the same manner we show that

$$E(\varphi) \geqq \sum_{r=1}^{n} y_r \epsilon_r - y_0 \left(\sum_{i=1}^{q} |m_i| + \int_0^1 |d\alpha(t)| \right)$$

for any polynomial $Q(t) = \sum_{r=1}^{m} y_r t^{r-1}$ such that $|Q(t) - \varphi(t)| \leqq y_0$, $t \in [0, 1]$.

Two-sided approximation of functions. Let again $u_1, u_2, \cdots, u_n$ and φ be given functions continuous on $[0, 1]$. The problem to approximate φ as well as possible in the uniform norm over $[0, 1]$ by means of a linear combination of $u_1, u_2, \cdots, u_n$ can be written as the following semi-infinite program.

Minimize y_0 when

$$\left| \sum_{r=1}^{n} y_r u_r(t) - \varphi(t) \right| \leqq y_0 , \quad t \in [0, 1].$$

Invoking theorem 7 we immediately find that the corresponding primal problem reads

$$\sup_{\alpha} \int_0^1 \varphi(t)\, d\alpha(t),$$

$$\int_0^1 u_r(t)\, d\alpha(t) = 0 , \quad r = 1, 2, \cdots, n,$$

$$\int_0^1 |d\alpha(t) = 1.$$

Compare also [**25**].

3. **Numerical solution of the moment problem (P).**

3.1. General algorithms. If we combine (5), (6) and (7) we get a nonlinear system whose solutions can be used for the construction of the optimal solutions of (P) and (D). In the general case one must verify the inequality $\sum_{r=1}^{n} y_r u_r(t) \geqq \varphi(t)$, $t \in [a, b]$, which only for special classes of problems can be done by means of a finite number of arithmetic operations.

Algorithms for the general problem are discussed in [**14**], [**20**] and [**23**].

3.2. Special cases. As apparent from the earlier arguments many particular cases of practical interest can be treated more easily with specialized methods. The problem of theorem 4 can hence be solved by means of the methods given in [18] but if $u_r(t) = t^{r-1}$ the codes in [11] are much more efficient. We note here the problem of Stieltjes, namely $\varphi(t) = (x - t)^{-1}$, x real, $u_r(t) = t^{r-1}$, when, as is well-known, the optimal value of P can be determined without prior computation of the optimal point-mass distribution. Compare [3] and [7].

When $u_1, u_2, \cdots, u_n$ form a Čebyšev system, the algorithms by Remez are used to solve the problem of two-sided approximation of section 2. See [3].

3.3. Approximations by means of simpler moment problems. Often one can solve (P) and (D) numerically by replacing them with problems which are simpler to handle.

In many cases the supremum and infimum values of P lie very close together. Hence one may be content to determine a feasible solution of (P). Such is obtained if we use an appropriate quadrature rule. Compare the preceding discussion on mechanical quadrature. For applications to Stieltjes' integrals see [12] and [19].

If we use a mechanical quadrature such that the corresponding point-mass distribution is a feasible solution of (P) this is equivalent to solving (P) with φ replaced by an interpolating polynomial.

Another approach is to approximate (P) with the moment problem which is obtained if we replace u_r, $r = 1, 2, \cdots, n$ and φ with piecewise linear functions. Then (P) can be solved with the simplex algorithm of linear programming [4] and [8], as shown in [20]. This idea can be directly generalized to moment problems in several dimensions.

Further refinement will result if we use higher order interpolation or splines to represent u_r, $r = 1, 2, \cdots, n$ and φ.

Using theorems 5 and 6 we may approximate φ over the whole interval $[a, b]$ with a polynomial of degree higher than $n - 1$ or a rational function.

General results on the convergence of sequences of semi-infinite programs are given in [20].

3.4. Extensions. As indicated in [20] and [23] the theories and methods discussed in this paper can be generalized. Hence applications to technical problems such as air and water pollution abatement are within reach.

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