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SOME UNSOLVED PROBLEMS CONCERNING COUNTABLY COMPACT SPACES

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Introduction. As is well known, the concept of countable compactness played an important role in much early work in general topology and was only ousted from its central position after A. Tychonoff showed that the more restrictive property of compactness was productive. Nevertheless present estimates of the gap between countable compactness and compactness cannot be very precise, for in each of the following questions if "countably compact" is replaced by "compact", then it is known that the resulting question has an affirmative answer. Yet the questions, as stated, appear to us to be quite challenging.

(1) Is every countably compact Hausdorff space that satisfies the first axiom of countability completely regular? (normal?)

(2) Is every countably compact quasi-topological group a topological group? (Raised by A. D. Wallace [34]).

(3) Does the existence of a countably compact Hausdorff space that is not compact imply the axiom of choice for sets of some given infinite cardinal?

(4) Is every countably compact Hausdorff space with a G_δ-diagonal metrizable? (Raised by B. A. Anderson in [1] and in an alternate form by R. W. Heath; Question 1 of [20].)

(5) Is every countably compact Hausdorff quasi-developable space a compact metric space? (Raised by H. R. Bennett in [5]).

(6) Is every countably compact perfectly normal space compact? (Raised by M. P. Berri, J. R. Porter and R. M. Stephenson; Problem 13 of [6]).

In this paper we are interested in the bearing that quasi-uniform spaces have upon the study of countably compact spaces that are not compact. We make no claim that the study of quasi-uniform spaces sheds light on all the above problems; our results are confined to partial solutions of problems 2, 4, 5 and 6. Nevertheless, it is known that a topological space is (countably compact, compact) if and only if its (upper semi-continuous, fine transitive) quasi-uniformity is precompact, and it is quite possible that the similarity of these charac-

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terizations of countable compactness and compactness may yet be of value in solving some of the problems listed above.

In section two we review both those facts about quasi-uniform spaces that are needed in the present paper and those results that appear to us to bear upon the study of non-compact countably compact spaces.

Section three concerns problem 2. We show that every quasi-topological group admits a natural compatible quasi-uniformity and that this quasi-uniformity is weakly locally symmetric if and only if the given quasi-topological group is, in fact, a topological group. It follows that every quasi-metrizable (= first countable) countably compact quasi-topological group is a topological group. We show that every quasi-uniformity compatible with a compact Hausdorff space is weakly locally symmetric; hence we obtain an alternate proof of the known result that every compact Hausdorff quasi-topological group is a topological group. It is noteworthy that if every quasi-uniformity compatible with a countably compact Hausdorff space is weakly locally symmetric, then an affirmative answer to problem 2 for Hausdorff spaces would follow from our results. Unfortunately, the quasi-uniform space problem is, itself, unsolved and appears to be an interesting problem in its own right.

In section 4 we study subquasi-metrizable spaces. We show that a topological space is a $\sigma^{\#}$-space if and only if it admits a coarser non-archimedean quasi-metric. We do not know if being subquasi-metrizable is equivalent to being a $\sigma^{\#}$-space, although this equivalence holds in any space whose fine (= universal) quasi-uniformity has a transitive base. We also modify a proof given by R. E. Hodel of D. Burke's result that every $\sigma^{\#}$ regular $w\Delta$ space is developable. From our modification it follows that every regular countable compact subquasi-metrizable space is a compact metric space.

Section 5 concerns countably compact spaces that have a (weak) G_δ-diagonal. We show that every topological space that has a G_δ-diagonal and a compatible quasi-uniformity with the Lebesgue property is subquasi-metrizable. We also show that every orthocompact (hereditarily orthocompact) space with a (weak) G_δ-diagonal is a $\sigma^{\#}$-space. In particular it follows that every orthocompact regular countably compact space with a G_δ-diagonal is a compact metric space and that every hereditarily orthocompact regular countable compact quasi-developable space is a compact metric space.

2. **Quasi-uniform spaces.** The concept of a quasi-uniformity on a set X was introduced by L. Nachbin [27]. If X is a nonempty set and $\mathcal{U}$ is a filter on $X \times X$, then $\mathcal{U}$ is a *quasi-uniformity* on X if and only if

(i) for each $U \in \mathcal{U}$, $\Delta = \{(x, x) \mid x \in X\} \subset U$.

(ii) for each $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V \circ V \subset U$.

A (sub)base $\mathcal{B}$ for $\mathcal{U}$ is *transitive* provided that for each $B \in \mathcal{B}$, $B \circ B = B$. A quasi-uniformity with a transitive subbase is called a *transitive quasi-uniformity*. If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\tau_{\mathcal{U}} = \{A \subset X : \text{if } x \in A, \text{ there is } V \in \mathcal{U} \text{ with } V(x) \subset A\}$ is a topology for X. If X is a set, $\mathcal{U}$ is a quasi-uniformity on X and τ is a topology on X, then $\mathcal{U}$ is *compatible* with (X, τ) provided that $\tau = \tau_{\mathcal{U}}$. Every topological space admits a compatible transitive quasi-uniformity [28, Theorem 1] and among the compatible transitive quasi-uniformities there is always a finest one, which is denoted by $\mathcal{FT}$. The following result is an evident modification of a result of V. S. Krishnan [23].

Proposition 2.1 [23]. *Let (X, τ) be a topological space and let U be the collection of all upper semi-continuous functions on (X, τ). For each $\epsilon > 0$ and each $f \in U$, let*

$$U_{(f,\epsilon)} = \{(x, y) \in X \times X : f(y) - f(x) < \epsilon\}.$$

The quasi-uniformity $\mathcal{U}$ generated by $\{U_{(f,\epsilon)} : f \in U, \epsilon > 0\}$ is compatible with τ.

The quasi-uniformity of Proposition 2.1 is called the upper semi-continuous quasi-uniformity and is denoted by $\mathcal{USC}$. It is shown in [15] that $\mathcal{USC}$ is a transitive quasi-uniformity.

A quasi-uniform space $(X, \mathcal{U})$ has the *Lebesgue property* provided that for each $\tau_{\mathcal{U}}$-open cover $\mathcal{C}$ of X there exists $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ is a refinement of $\mathcal{C}$. A quasi-uniformity $\mathcal{U}$ is *precompact* provided that if $U \in \mathcal{U}$, then there is a finite subset F of X such that $X = \bigcup \{U(x) : x \in F\}$.

A result of some importance is that a T_1 space is quasi-metrizable if and only if it admits a compatible quasi-uniformity with a countable base. This result is embedded in J. L. Kelley's presentation of the metrization lemma [22, page 185]; however, a direct and simple proof follows from an elegant lemma of A. H. Frink. Frink's lemma was originally given for symmetric distance functions. It is easily verified that Frink's result also holds for non-symmetric distance functions.

Lemma. [17]. *Let X be a set and let* d *be a function that assigns a non-negative real number* $\mathrm{d}(a, b)$ *to every ordered pair* $(a, b) \in X \times X$. *Suppose that* d *satisfies the following conditions,*

(i) $\mathrm{d}(a, b) = 0$ *if and only if* $a = b$,

(ii) *For each* $\epsilon > 0$, *if* $\mathrm{d}(a, b) < \epsilon$ *and* $\mathrm{d}(b, c) < \epsilon$, *then* $\mathrm{d}(a, c) < 2\epsilon$.

Then (X, τ_{d}) *is quasi-metrizable.*

The function d given above need not be a quasi-metric.

PROPOSITION 2.2. *A T_1 space (X, τ) is quasi-metrizable if and only if it admits a compatible quasi-uniformity with a countable base.*

PROOF. Let (X, τ) be a T_1 space and let $\mathcal{U}$ be a compatible quasi-uniformity with a countable base $\{U_i\}_{i=1}^{\infty}$. Without loss of generality we may assume that $U_{i+1}^2 \subset U_i$ for $i \geqq 1$. Define a real-valued function d with domain $X \times X$ as follows.

(i) $\mathrm{d}(a, a) = 0$,

(ii) for $a \neq b$, $\mathrm{d}(a, b) = 1/2^i$ where i is the least positive integer n such that $b \notin U_n(a)$. It is easily verified that d satisfies the conditions of Frink's lemma. The remaining implication may be established in the obvious manner.

The following two propositions appear to bear upon the problems discussed in this paper, however they are not used subsequently in this paper.

PROPOSITION 2.3 [3]. *A topological space is countably compact if and only if $\mathcal{U}\mathcal{S}\mathcal{C}$ is precompact.*

PROPOSITION 2.4 [15]. *A topological space is compact if and only if $\mathcal{F}\mathcal{T}$ is precompact.*

3. Quasi-topological Groups.

DEFINITION. A *quasi-topological group* is a T_1 topological semigroup $(X, \circ, \tau)$ such that $(X, \circ)$ is a group.

It is known that a regular Hausdorff quasi-topological group is a topological group provided that it is either a completely regular maximal Lindelöf space or a locally compact space ([29] and [11]. In [18] it was asserted that every countably compact Hausdorff quasi-topological group is also a topological group; however, in [34] A. D. Wallace pointed out an error in the arguments of Lemma 2 of [18] and raised Question 2.

We omit the proof of the following proposition, since the argument needed to establish this proposition is standard.

PROPOSITION 3.1. *Let $(X, \circ, \tau)$ be a quasi-topological group and for each $U \in \eta_e$ define $L(U) = \{(x, y) \in X \times X : x^{-1}y \in U\}$. Let β be a base for η_e, and let $L = \{L(U) : U \in \beta\}$.* Then L *is a base for a compatible quasi-uniformity for* (X, τ).

The quasi-uniformity generated by L is denoted by $\mathcal{L}$ and is called the *left quasi-uniformity* for $(X, \circ, \tau)$ *with respect to* β. The *right* and *central* quasi-uniformities $\mathcal{R}$ and $\mathcal{C}$ are defined in the following manner. For each $U \in \eta_e$ let $R(U) = \{(x, y) \in X \times X : yx^{-1} \in U\}$ and let $C(U) = R(U) \cap L(U)$. Let β be a base for η_e. Then $\mathcal{R}$ and $\mathcal{C}$

are the quasi-uniformities generated by $R = \{R(U) : U \in \beta\}$ and $C = \{C(U) : U \in \beta\}$ respectively. We note that $\mathcal{L}$ and $\mathcal{R}$ are not in general, conjugates (i.e., $\mathcal{L} \neq \mathcal{R}^{-1}$); consequently $\mathcal{C}$ is not, in general, a uniformity.

PROPOSITION 3.2. *A quasi-topological group* $(X, \circ, \tau)$ *is quasi-metrizable if and only if* (X, τ) *satisfies the first axiom of countability.*

PROOF. Suppose that (X, τ) satisfies the first axiom of countability. Then η_e has a countable base $\mathcal{B}$ and the corresponding quasi-uniformity $\mathcal{L}$ is a compatible quasi-uniformity for (X, τ) with a countable base. The result now follows from Proposition 2.2.

DEFINITION [26]. A quasi-uniform space $(X, \mathcal{U})$ is *weakly locally symmetric* provided that if $x \in X$ and $U \in \mathcal{U}$, then there is a symmetric entourage $V \in \mathcal{U}$ such that $V(x) \subset U(x)$.

DEFINITION [10]. A topological space (X, τ) is R_0 (also called *essentially* T_1) provided that if $x, y \in X$, then $\overline{\{x\}} = \overline{\{y\}}$ or $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$.

It is known that a topological space admits a compatible weakly locally symmetric quasi-uniformity if and only if it is an R_0 space [26, Theorem 3.6].

PROPOSITION 3.3. *Let* $(X, \circ, \tau)$ *be a quasi-topological group. Then* $(X, \circ, \tau)$ *is a topological group if and only if one of* $\mathcal{L}$, $\mathcal{R}$ *or* $\mathcal{C}$ *is weakly locally symmetric.*

PROOF. If $(X, \circ, \tau)$ is a topological group, then it is well known that $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}$ are uniformities. We prove the converse for $\mathcal{L}$. Let $\rho : X \to X$ be defined by $\rho(x) = x^{-1}$. In order to show that ρ is continuous it suffices to show that ρ is continuous at e. Let $e \in A \in \tau$. Then there exists $W \in \eta_e$ and a symmetric entourage $V \in \mathcal{L}$ with $V(e) \subset A$ and $L(W) \subset V$. Then $W = \{y : (y^{-1}, e) \in L(W)\} \subset \rho^{-1}(V(e)) \subset \rho^{-1}(A)$ and ρ is continuous at e.

LEMMA. *A quasi-uniform space* $(X, \mathcal{U})$ *is weakly locally symmetric if and only if* $\tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$.

PROOF. Suppose first that $\tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$. Let $x \in X$ and let $U \in \mathcal{U}$. Then since $x \in \operatorname{int} U(x)$, there exists $V \in \mathcal{U}$ such that $V^{-1}(x) \subset \operatorname{int} U(x) \subset U(x)$ and there exists $W \in \mathcal{U}$ such that $W \subset U \cap V$. Then $W \cup W^{-1}$ is a symmetric entourage and $W \cup W^{-1}(x) = W(x) \cup W^{-1}(x) \subset U(x) \cup V^{-1}(x) \subset U(x)$. Thus $(X, \mathcal{U})$ is weakly locally symmetric.

Now suppose that $\mathcal{U}$ is weakly locally symmetric. Let $x \in A \in \tau_{\mathcal{U}}$. Then there exists $U \in \mathcal{U}$ such that $x \in U(x) \subset A$, and there is a symmetric $V \in \mathcal{U}$ such that $V(x) \subset U(x)$. Thus $A \in \tau_{\mathcal{U}^{-1}}$ and $\tau_{\mathcal{U}} \subset \tau_{\mathcal{U}^{-1}}$.

PROPOSITION 3.4. *If (X, τ) is a (countably) compact R_0 space, then every compatible quasi-uniformity (with a countable base) is weakly locally symmetric.*

PROOF. We prove only the parenthetical result, since the proposition concerning compact R_0 spaces follows similarly. Let (X, τ) be a countably compact R_0 space and suppose that $\mathcal{U}$ is a compatible quasi-uniformity with a countable base $\mathcal{B}$ that is not weakly locally symmetric. By the preceding lemma, there exist $x \in X$ and $U \in \mathcal{U}$ such that the $\tau_{\mathcal{U}} -$ int $U(x) \notin \tau_{\mathcal{U}^{-1}}$. Thus if $K = X - (\tau_{\mathcal{U}} - \text{int}\, U(x))$, then there exists $q \in \tau_{\mathcal{U}} -$ int $U(x)$ such that for each $V \in \mathcal{U}$, $V^{-1}(q) \cap K \neq \emptyset$. Let $\mathcal{F}$ be the filter generated by $\{V^{-1}(q) \cap K : V \in \mathcal{U}\}$. Then $\mathcal{F}$ has a cluster point p. Note that $p \in K$ and that for each $V \in \mathcal{U}$, $p \in \overline{V^{-1}(q)}$. Let $W \in \mathcal{U}$ with $V \circ V \subset W$. Now $p \in \overline{V^{-1}(q)} \subset V^{-1} \circ V^{-1}(q)$, whence $(p, q) \in W$. Since (X, τ) is R_0, $q \in \bigcap \{W(p) : W \in \mathcal{U}\}$ so that $q \in \overline{\{p\}}$ and $q \in \overline{\{q\}} = \overline{\{p\}} \subset \overline{K} = K - a$, contradiction.

COROLLARY. [11, Theorem 1]. *Every compact quasi-topological group is a topological group.*

COROLLARY. *Every countably compact quasi-topological group that satisfies the first axiom of countability is a compact metrizable topological group.*

PROOF. Let $(X, \circ, \tau)$ be a countably compact quasi-topological group that satisfies the first axiom of countability. Let β be a countable base for η_e and let $\mathcal{L}$ be the left quasi-uniformity for $(X, \circ, \tau)$ with respect to β. Then $\mathcal{L}$ has a countable base so that by the preceding proposition $\mathcal{L}$ is weakly locally symmetric. By Proposition 3.3 $(X, \circ, \tau)$ is a topological group; and the result follows since every topological group that satisfies the first axiom of countability is metrizable.

4. Topologies comparable to quasi-metric topologies.

DEFINITION. Let (X, τ) be a topological space and let $\mathcal{C}$ be a collection of open sets such that if $x \in X$, then $\bigcap \{C \in \mathcal{C} : x \in C\} \in \tau$. Then $\mathcal{C}$ is a Q-collection. If $\mathcal{C}$ is an open cover of X, then $\mathcal{C}$ is a *Q-cover of* X. If $\mathcal{C}$ is a Q-collection, the $A_x^{\mathcal{C}} = \bigcap \{C \in \mathcal{C} \mid x \in C\}$.

DEFINITION. Let (X, τ) be a topological space and let $\mathcal{B}$ be a (sub) base for τ. If there exists a sequence $\{\mathcal{B}_i\}_{i=1}^{\infty}$ of Q-collections such that $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ then $\mathcal{B}$ is a *σ-Q(sub) base for* τ.

A space (X, τ) has a σ-Q-base if and only if it has a σ-Q-subbase.

THEOREM 4.1 [14, Theorem 3.2]. *Let (X, τ) be a T_1 topological space. Then the following are equivalent,*

(i) *There exists a σ-Q-base for τ,*

(ii) *(X, τ) is generated by a non-Archimedean quasi-metric,*

(iii) *(X, τ) has a compatible transitive quasi-uniformity with a countable base.*

DEFINITION [25]. A cover $\mathcal{V}$ of a set X is *separating* if given $x, y \in X$ with $x \neq y$, there is $V \in \mathcal{V}$ such that $x \in V$, $y \notin V$. A topological space with a σ-closure preserving separating closed cover is called a *$\sigma\#$-space* [33].

DEFINITION [1]. A topological space (X, τ) is *submetrizable* provided that there exists a metric topology $\mathcal{U}$ on X such that $\mathcal{U} \subset \tau$. *Subquasi-metrizable,* and *subnon-archimedean quasi-metrizable* are defined analogously.

The space Ψ, due to J. Isbell, given in [19, Example 5I] is a locally compact Moore space that is not submetrizable. [1, Example 2.4]. It follows from Theorem 3.4 of [14] that Ψ is quasi-metrizable. On the other hand, the countable box product of real lines is a simple example of a submetrizable space that is not quasi-metrizable. [32, Example 109].

THEOREM 4.2. *A topological space (X, τ) is a $\sigma\#$-space if and only if it is subnon-Archimedean quasi-metrizable.*

PROOF. Suppose that (X, τ) is a $\sigma\#$-space and let $\mathcal{F} = \bigcup_{i=1}^{\infty}\{\mathcal{F}_i\}$ be a σ-closure preserving separating closed cover. For each $i \geqq 1$, let $\mathcal{F}_i' = \bigcup_{n=1}^{i} \mathcal{F}_n \cup \{\emptyset\}$ and let $\mathcal{B}_i = \{X - F \mid F \in \mathcal{F}_i'\}$. Let $x \in X$. Then for each $i \geqq 1, \bigcap\{B \in \mathcal{B}_i \mid x \in B\} = \bigcap\{X - F \mid F \in \mathcal{F}_i'$ and $x \notin F\} = X - \bigcup\{F \mid F \in \mathcal{F}_i'$ and $x \notin F\} \in \tau$. For each $i \geqq 1$, let $\mathcal{B}_i' = \{A_x^{\mathcal{B}_i} \mid x \in X\}$ and let $\mathcal{B}' = \bigcup \mathcal{B}_i'$. Since $\mathcal{F}$ is a separating closed cover, $\mathcal{B}'$ is a separating open cover; consequently, $\mathcal{B}'$ is a σ-Q-base for a T_1 subtopology τ'. It follows from Theorem 4.1 that (X, τ) is subnon-Archimedean quasi-metrizable. The remaining implication is proved similarly.

It is an open question whether every subquasi-metrizable space is a $\sigma\#$-space; however it follows from [15, Theorem 3.4] that a space (X, τ) with a coarser quasi-metrizable topology τ' is a $\sigma\#$-space if the fine quasi-uniformity for (X, τ) has a transitive base.

The following characterization of semi-stratifiability is given in [21]. A topological space (X, τ) is *semi-stratifiable* provided that there is a function $g : N \times X \to \tau$ such that for each $x \in X$ and $n \in N$,

$x \in g(n, x)$ and such that if $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then x is a cluster point of the sequence $\langle x_n \rangle$. A space (X, τ) is a *$w\Delta$-space* if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of open covers of X such that, for each $x \in X$, if $x_n \in \mathrm{st}(x, \mathcal{G}_n)$ for $n = 1, 2, \cdots$, then the sequence $\langle x_n \rangle$ has a cluster point [7]. The following theorem is based upon the proof of [**21**, Theorem 4.6].

THEOREM 4.3. *Every subquasi-metrizable $w\Delta$-space is semi-stratifiable.*

PROOF. Let (X, τ) be a subquasi-metrizable $w\Delta$-space, let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be a $w\Delta$-sequence for X and let d be a quasi-metric on X such that $\tau_{\mathrm{d}} \subset \tau$. For each $x \in X$ and $n \geqq 1$, let $h(n, x) = \{y \in \mathrm{st}(x, \mathcal{G}_n) \mid \mathrm{d}(x, y) < 1/2^n\}$. Clearly $x \in h(n, x)$. Let $x \in h(n, x_n)$ for $n \geqq 1$. Then for $n \geqq 1$, $x \in \mathrm{st}(x_n, \mathcal{G}_n)$ and so $x_n \in \mathrm{st}(x, \mathcal{G}_n)$. Consequently the sequence $\langle x_n \rangle$ has a cluster point y. Suppose that $y \neq x$. Then there is $k \in N$ such that $\mathrm{d}(y, x) > 1/2^k$. Since y is a cluster point of $\langle x_n \rangle$, there is $m > k + 1$ such that $\mathrm{d}(y, x_m) < 1/2^{k+1}$. Now $x \in h(m, x_m)$ so that $\mathrm{d}(x_m, x) < 1/2^{k+1}$ and $\mathrm{d}(y, x) \leqq \mathrm{d}(y, x_m) + d(x_m, x) < 1/2^{k+1} + 1/2^{k+1} = 1/2^k$ – a contradiction. Therefore x is a cluster point of $\langle x_n \rangle$ and so (X, τ) is semi-stratifiable.

COROLLARY. *Every countably compact regular subquasi-metric space is a compact metric space.*

PROOF. Clearly every countably compact space is a $w\Delta$-space, and it is known that a regular $w\Delta$-space is a Moore space if and only if it is semi-stratifiable [**21**, Corollary 2.6].

5. **Spaces with a (weak) G_δ-diagonal.** In [8] Ceder obtained the following useful characterization of spaces with a G_δ-diagonal; this characterization also motivates the definition of a weak G_δ-diagonal.

PROPOSITION 5.1. [8, Lemma 5.4]. *A space (X, τ) has a G_δ-diagonal. if and only if there is a sequence $\langle \mathcal{G}_i \rangle$ of open covers of X such that for each $x \in X$, $\bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathcal{G}_i) = \{x\}$.*

DEFINITION [33]. A space (X, τ) has a *weak G_δ-diagonal* if there is a sequence $\langle \mathcal{G}_i \rangle$ such that for each $i, \mathcal{G}_i \subset \tau$ and for each $x \in X$, $\bigcap \{\mathrm{st}(x, \mathcal{G}_i) \mid x \in \mathrm{st}(x, \mathcal{G}_i)\} = \{x\}$.

We note that every quasi-developable space has a weak G_δ-diagonal (For the definition of a quasi-developable space see [**5**]).

THEOREM 5.2. *Let (X, τ) be a topological space with a G_δ-diagonal and suppose that (X, τ) admits a compatible quasi-uniformity $\mathcal{U}$ with the Lebesgue property. Then (X, τ) is subquasi-metrizable.*

PROOF. Let $\langle \mathcal{G}_i \rangle$ be a sequence of open covers as given in Proposition 5.1. For each i, let $U_i \in \mathcal{U}$ such that $\mathcal{U}_i = \{U_i(x) \mid x \in X\}$ refines $\mathcal{G}_i$ and such that for each i, $U_{i+1}^2 \subset U_i$. Let $\mathcal{V}$ be the quasi-uniformity generated by $\{U_i\}_{i=1}^{\infty}$. Then $\mathcal{V} \subset \mathcal{U}$ so that $\tau_{\mathcal{V}} \subset \tau_{\mathcal{U}} = \tau$ and since for each $i \geqq 1$, $\mathrm{st}(x, \mathcal{U}_i) \subset \mathrm{st}(x, \mathcal{G}_i)$, $\Delta = \bigcap \{V \in \mathcal{V}\}$. It follows from Proposition 2.2 that $(X, \tau_{\mathcal{V}})$ is quasi-metrizable.

DEFINITION [31]. A topological space (X, τ) is *(σ-)orthocompact* provided that if $\mathcal{C}$ is an open cover of X, then there is an open refinement $\mathcal{R}$ of $\mathcal{C}$ such that $\mathcal{R}$ is a Q-cover ($\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{R}_i$ where each $\mathcal{R}_i$ is a Q-collection).

THEOREM 5.3. *Every σ-orthocompact space with a G_δ-diagonal is a $\sigma^{\#}$-space.*

PROOF. Let (X, τ) be a σ-orthocompact space with a G_δ-diagonal and let $\langle \mathcal{G}_i \rangle$ be a sequence of open covers as given in Proposition 5.1. For each $i \geqq 1$ let $\mathcal{R}_i = \bigcup_{d=1}^{\infty} \mathcal{R}(i, d)$ be an open refinement of $\mathcal{G}_i$ such that each $\mathcal{R}(i, d)$ is a Q-collection. Let $\mathcal{R}'(i, d) = \{A_x^{\mathcal{R}(i,d)} \mid x \in X\}$. Then $\mathcal{B} = \bigcup_{i=1}^{\infty} {}_{d=1}^{\infty} \mathcal{R}'(i, d)$ is a σ-Q-subbase for a T_1 subtopology τ'. It follows that τ' has a σ-Q-base so that by Theorem 4.1, (X, τ) is a $\sigma\#$-space.

COROLLARY. *A countably compact space with a G_δ-diagonal is compact if and only if it is orthocompact.*

PROOF. Let (X, τ) be a countably compact orthocompact space with a G_δ-diagonal. Since (X, τ) has a G_δ-diagonal, (X, τ) is a T_1 space. By Corollary 4.5 of [9], in order to show that (X, τ) is compact it suffices to show that (X, τ) is semi-stratifiable. By the previous theorem (X, τ) is $\sigma\#$ and so by Theorem 4.3 (X, τ) is compact. The converse is evident.

Every orthocompact space has a compatible quasi-uniformity with the Lebesgue property [14, Theorem 2.2]. We do not know if every topological space that admits a compatible quasi-uniformity with the Lebesgue property is orthocompact; however, this equivalence holds in any space whose fine quasi-uniformity has a transitive base. It is evident that every metacompact space is orthocompact so that in light of the well-known result of R. Arens and J. Dugundji that every metacompact countably compact space is compact, it might seem reasonable to conjecture that every countably compact orthocompact space is compact. It is known, however, that every linearly ordered space is orthocompact in its order topology [12], [13] and [24]. Consequently, the space W of all ordinals less than the first uncountable ordinal is an example of a normal countably compact orthocompact space in which each point is a G_δ-point that is not compact. (But this space is also not perfectly normal since the set of all limit ordinals is a closed set that is not a G_δ-set. Hence this example does not provide a solution to Problem 6 as claimed in [**32**].)

THEOREM 5.5. *Let (X, τ) be a hereditarily orthocompact space with a weak G_δ-diagonal. Then (X, τ) is a $\sigma\#$-space.*

PROOF. Let $\langle \mathcal{G}_i \rangle$ be a sequence of open collections such that for each i, $\mathcal{G}_i \subset \tau$ and for each $x \in X$, $\bigcap \{\mathrm{st}(x, \mathcal{G}_i) \mid x \in \mathrm{st}(x, G_i)\} = \{x\}$. For each i let $\mathcal{G}_i^* = \bigcup \mathcal{G}_i$ and let $\mathcal{R}_i$ be a Q-refinement of $\mathcal{G}_i$. Note that since $\mathcal{G}_i^*$ is open, $\mathcal{R}_i \subset \tau$ and for each $x \in \mathcal{G}_i^*$, $A_x^{\mathcal{R}_i} \in \tau$. For each $i \geqq 1$ let $\mathcal{B}_i = \{A_x^{\mathcal{R}_i} \mid x \in \mathcal{G}_i^*\}$ and let $\mathcal{B} = \bigcup \mathcal{B}_i$. Then $\mathcal{B}$ is a σ-Q-subbase for a T_1 subtopology τ'. It follows that τ' has a σ-Q-base and so the result follows from Theorems 4.1 and 4.2.

COROLLARY. *Every hereditarily orthocompact countably compact space with a weak G_δ-diagonal is compact.*

COROLLARY. *Every hereditarily orthocompact countably compact quasi-developable space is compact.*

It is known that every countably metacompact σ-orthocompact space is orthocompact and that any orthocompact space in which every closed set is a G_δ is hereditarily orthocompact [**16**, Proposition 3.1 and Theorem 3.3]. Consequently, we have the following corollary, which sheds some light on Problem 6.

COROLLARY. *Let (X, τ) be a countably compact perfectly normal space. Then the following statements are equivalent.*

(a) *(X, τ) is quasi-metrizable.*

(b) *(X, τ) is sub-quasi-metrizable.*

(c) *(X, τ) is σ-orthocompact and has a weak G_δ-diagonal.*

(d) *(X, τ) is a compact metric space.*

6. **Addendum.** Some of the foregoing corollaries were, for simplicity, not formulated in their most general setting. Their generalizations are evident. Furthermore, although our primary concern is the study of countably compact spaces, some of our results bear upon other areas. For example, it is clear that the above results bear upon some of the problems raised in [**30**]. We also believe that the following two problems, which we wish to add to the list of problems concerning countably compact spaces that are not compact, are well-motivated.

(8) Does every countably compact Hausdorff space admit a compatible quasi-uniformity with the Lebesgue property?

(9) If $\mathcal{U}$ is a quasi-uniformity that is compatible with a countably compact Hausdorff space, is $\mathcal{U}$ necessarily weakly locally symmetric?

ADDED IN PROOF. Recently Adam Ostaszewski has used $\Diamond$ (or to be exact $\clubsuit$ and the continuum hypothesis) to obtain negative answers to questions 1 and 6. It is unknown if these questions can be answered negatively without employing extra set theoretic conditions. Moreover, it is still plausible that these questions could be answered affirmatively with the aid of different, extra, set-theoretic hypotheses.

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