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ASYMPTOTIC BEHAVIOR OF SOLUTIONS AND THEIR DERIVATIVES FOR LINEAR DIFFERENTIAL EQUATIONS

JAMES E. GEHRMANN AND THOMAS L. SHERMAN

Introduction. In this paper the asymptotic behavior of solutions and derivatives of solutions of linear homogenous equations of the form

$$(1) \qquad L_n y = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0,$$

will be considered. The main tool that will be exploited in this investigation is the relationship between solutions of (1) and solutions of the nonhomogeneous equation

$$(2) \qquad y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x).$$

This result given in Theorem 1 is a generalization of a result of Svec [**6**]. This result is then used to examine a property which Hartmann [2] calls completely monotone. The results presented here are clearly related to those of V. A. Kondrateev [3] and in fact lead to easy generalizations of his results. Also related to these results is the paper of A. C. Lazer [4] one of whose theorems is generalized as the last theorem of the present paper.

RESULTS. In what follows all coefficients $a_1, \cdots, a_n$ as well as $f(x)$ will be assumed continuous on an interval $[a, \infty)$ and unless otherwise noted, all conditions assumed will be assumed on this interval.

DEFINITION 1. A solution of (1) or (2) is oscillatory if it has an infinite number of zeros on $[a, \infty)$. A solution is nonoscillatory if it is not oscillatory.

DEFINITION 2. Equation (1) or (2) is oscillatory if it has at least one oscillatory solution. Otherwise, the equation is nonoscillatory. The equation is strongly oscillatory if every solution is oscillatory.

DEFINITION 3. Equation (1) is said to be disconjugate (on an interval I) if no nontrivial solution of (1) has more than n zeros (on I).

If (1) is disconjugate it may be written in the form

$$(s_n(s_{n-1}(\cdots(s_3(s_2(s_1 y)')')')\cdots)')')' = 0$$

where $s_i(x) > 0, i = 1, 2, \cdots, n$.

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Suppose now that equation (1) has a nonoscillatory solution $\phi(x)$ for $x \in [a, \infty)$ and set $y = \phi z$. Then for x sufficiently large,

$$L_n y = zL_n\phi + \phi[z^{(n)} + \textstyle\sum_{i=1}^{n-1} \hat{a}_i(x)\, z^{(n-i)}] = zL_n\phi + \phi\hat{L}_{n-1}z'$$

where the new coefficients $\hat{a}_i(x)$ may depend on $\phi(x)$.

DEFINITION 4. $\hat{L}_{n-1}$ is called the reduced operator for L_n associated with ϕ.

In Theorem 1 a nonoscillatory condition is given for equation (2). For a related result see M. Medved [5]. Theorem 1 generalizes a result of Svec [6].

THEOREM 1. *Suppose* (1) *is nonoscillatory and, for some solution* ϕ *of* (1), $\hat{L}_{n-1}y = 0$ *is disconjugate for large* x. *Assume* $f(x)$ *is a one-signed function* (*i.e.* $f(x) \geqq 0$ *or* $f(x) \leqq 0$) *which is not identically zero for large* x. *Then* (2) *is also nonoscillatory.*

PROOF. Let ϕ be the solution of (1) given in the hypothesis. Let $y = \phi z$ in (2). Then

$$L_n y = \phi\hat{L}_{n-1}z' = f.$$

However, by assumption, for large x, $\hat{L}_{n-1}$ may be written

$$\hat{L}_{n-1}z' = (s_{n-1}(s_{n-2}(\cdots (s_2(s_1z')')')' \cdots)')')' = f/\phi.$$

An integration of this equation results in the equation

$$s_{n-1}(s_{n-2}(\cdots (s_2(s_1z')')' \cdots)')' = K_1 + \int_{a_1}^{x} \frac{f(t)}{\phi(t)} \tag{3}$$

where a_1 is sufficiently large to guarantee $\phi(x)$ is one signed for $x > a_1$. Since f/ϕ is one signed for $x > a$ it follows that $\int_{a_1}^{x} f(t)/\phi(t)\,dt$ is monotone, hence for any K_1 there is an $a_2(K_1)$ such that $K_1 + \int_{a_1}^{x} f(t)/\phi(t)\,dt$ is one signed for $x > a_2(K_1)$. The above procedure can now be repeated on (3) in the interval $[a_2(K_1), \infty)$ to get

$$s_{n-2}(\cdots (s_2(s_1z')')' \cdots)'$$
$$= K_2 + \int_{a_1}^{x} \frac{1}{s_{n-1}} \left(K_1 + \int_{a_1}^{x_1} \frac{f(s)}{\phi(s)}\, ds \right)dx_1$$

from which the existence of an $a_3(K_1,K_2)$ follows as before. After n repetitions of this argument it is found that

$$z = K_n + \int_{a_1}^{x} \frac{1}{s_1}\Big(K_{n-1} + \int_{a_1}^{x_{n-1}} \frac{1}{s_2}\Big(K_{n-2} + \int_{a_1}^{x_{n-2}} \frac{1}{s_3}\Big(K_{n-3} +$$

$$\cdots + \int_{a_1}^{x_2} \frac{1}{s_{n-1}}\Big(K_1 + \int_{a_1}^{x_1} \frac{f}{\phi}\Big)\cdots\Big)\Big)\Big) ds\, dx_1 \cdots dx_{n-3}dx_{n-2}dx_{n-1}$$

from which it follows that z is one signed for x sufficiently large, and hence $y = \phi z$ is nonoscillatory.

REMARK. Medved [5] claims that his main theorem generalizes the following theorem of Svec [**6**].

THEOREM. *If the equation $y'' + p(x)y = 0$ is nonoscillatory and if $f(x)$ is of constant sign for large x, then the equation $z'' + p(x)z = f(x)$ is also nonoscillatory.*

One of the hypotheses of Medved's theorem is (in the case $n = 2$) that $y'' + p(x)y = 0$ is disconjungate (what he calls 2-nonoscillatory). However $y'' + p(x)y = 0$ being nonoscillatory does not imply that it is disconjugate, hence Medved's theorem does not generalize Svec's result. Our Theorem 1 does however generalize Svec's result since any first order equation is disconjugate. In particular if we start with $L_2y = 0$ then $\hat{L}_1z' = 0$ is disconjugate.

DEFINITION 5. $L_ky = y^{(k)} + \sum_{i=1}^{k} a_i(x)y^{(k-i)}$ where $a_i(x)$ is the coefficient of $y^{(n-i)}$ in (1) ($i = 1, \cdots, k$). The operator $\hat{L}_{k-1}$ is defined as in definition 4.

THEOREM 2. *If the equation $L_{n-1}y = 0$ is nonoscillatory, $a_n(\neq 0)$ is one signed, and, for some solution ϕ of $\hat{L}_{n-1}y = 0$, $\hat{L}_{n-2}y = 0$ is disconjugate for large x, then any nonoscillatory solution of* (1) *has a nonoscillatory first derivative.*

PROOF. Let y be a nonoscillatory solution of (1). Then (1) may be rewritten as

$$y^{(n)} + \sum_{i=1}^{n-1} a_i(x)y^{(n-i)} = -a_n(x)y$$

or $L_{n-1}y' = f(x)$ where $f(x) = -a_n(x)y$ is one-signed for large x. The equations $L_{n-1}z = 0$, $L_{n-1}z = f(x)$ satisfy the conditions of Theorem 1 and the result follows.

COROLLARY. *Under the conditions of Theorem 2, if y is a nonoscillatory solution of* (1), *then y is either bounded* (*in which case it monotonically approaches a finite constant*) *or it tends monotonically to $\pm\infty$.*

EXAMPLE. The condition that a_n be one-signed cannot be removed is shown by the equation

$$L_n y = y'' + (\sin x)y' + (\cos x)y = 0$$

for which $L_{n-1}y = y' + (\sin x)y$ and $L_{n-1}y = 0$ has a solution $y = e^{\cos x}$ which is nonoscillatory. Also $e^{\cos x}$ is a solution of $L_n y = 0$, however $(e^{\cos x})' = -\sin x\, e^{\cos x}$ is oscillatory as is $a_n = \cos x$.

The next theorem deals with the behavior of functions which have nonoscillatory n-th derivatives. This type of result may easily be applied, for example, to equations of the form $y^{(n)} + a_n(x)y = 0$.

LEMMA 1. *Let $y(x) \in C^k(k > 1)$ and suppose $y^{(k)}(x)y^{(k-1)}(x) > 0$. Then if $y^{(k)}(x) > 0(< 0)$, $y^{(j)}(x) \to +\infty(-\infty)$ $(j = 0, 1, \cdots, k-2)$.*

PROOF. Suppose $y^{(k)}(x) > 0$ on $[x_0, \infty)$ (the case where $y^{(k)}(x) < 0$ is analogous). Then for any $x > x_0$ there is an $x_1 \in [x_0, x]$ such that $y^{(k-2)}(x) - y^{(k-2)}(x_0) = y^{(k-1)}(x_1)(x - x_1) \geqq y^{(k-1)}(x_0)(x - x_1)$ since $y^{(k-1)}(x)$ is increasing. Hence

$$y^{(k-2)}(x) \geqq y^{(k-2)}(x_0) + y^{(k-1)}(x_0)(x - x_0), \tag{3}$$

and, since $y^{(k-1)}(x_0) > 0$, letting $x \to \infty$ on the right hand side of (3), $y^{(k-2)}(x) \to +\infty$. The result for lower order derivatives follows by induction.

THEOREM 3. *Let $y \in C^n(n > 0)$ and satisfy $(-1)^n y^{(n)}(x)y(x) > 0$. If $y > 0$ (< 0), either $y \to +\infty(-\infty)$ or y approaches a non-negative (non-positive) finite constant monotonically from above (below), and the derivatives of y of order less than n approach zero and alternate in sign, with y and y' being of opposite sign.*

PROOF. The proof will be given for the case n odd $(n > 1)$ and $y(x) > 0$, other cases are proved analogously. Suppose $y \not\to +\infty$. The assumptions $y(x) > 0$ and $y^{n)}(x) < 0$ imply $y^{(n-1)}(x) > 0$ for x sufficiently large. For if this were not the case, then, since $y^{(n)}(x) < 0$, there is an x_1 such that for $x \geqq x_1$, $y^{(n-1)}(x) < 0$, and this, by Lemma 1, would contradict $y > 0$. Now for x sufficiently large $y^{(n-2)}(x) < 0$, for if not then $y^{(n-1)}(x) > 0$ implies, for large x, $y^{(n-2)}(x) > 0$ which by Lemma 1 implies $y(x) \to +\infty$ which is a contradiction. Similarly it can be shown that $y^{(n-j)}(x) > 0$ for large x; $j = 1, 3, 5, \cdots, n-2$, and $y^{(n-j)}(x) < 0$ for large x; $j = 2, 4, 6, \cdots, n-1$. To complete the proof, it need only be observed that if $y^{(k)}(x) \not\to 0$ for some k, $0 < k < n$, then $y^{(k)}(x)$ approaches a nonzero constant (since $y^{(k+1)}(x) \neq 0$ for large x) which in turn implies $|y^{(k-j)}(x)| \to \infty$, $j = 1, 2, \cdots, k$, contradicting the assumption $y > 0$ and $y \not\to +\infty$.

COROLLARY. *Let $y \in C^n[x_0, \infty)$, $(n > 0)$ and satisfy $(-1)^n y^n(x)y(x) < 0$ on $[x_0, \infty)$. If $y > 0$ (< 0) either $y \to +\infty$ $(-\infty)$ or y approaches a positive (negative) finite constant monotonically from below (above) and the derivatives of y of order less than n approach zero and alternate in sign with y and y' being of the same sign.*

PROOF. The proof is essentially the same as that of Theorem 3.

THEOREM 4. *Suppose in equation* (1) *the coefficients alternate in sign* (i.e., $\operatorname{sign} a_i \neq \operatorname{sign} a_{i+1}$, $i = 1, 2, \cdots, n-1$) *and* $\lim_{x\to\infty} a_i(x) = 0$ $(i = 1, 2, \cdots, n-1)$ *and* $|a_n(x)| > d$, *d a positive constant. Suppose also $L_k y = 0$, $k = 1, 2, \cdots, n-1$ are nonoscillatory, and there are solutions ϕ_k of $L_k y = 0$ for which $\hat{L}_{k-1} y = 0$ are disconjugate for large x $(k = 2, 3, \cdots, n-2)$. Then any nonoscillatory solution $y(x)$ of* (1) *is either unbounded and tends monotonically to $\pm\infty$ or is bounded and satisfies*

(i) $\operatorname{sgn} y' = \operatorname{sgn} y^{(3)} = \operatorname{sgn} y^{(5)} = \cdots,$
$\operatorname{sgn} y'' = \operatorname{sgn} y^{(4)} = \operatorname{sgn} y^{(6)} = \cdots,$
$\operatorname{sgn} y' \neq \operatorname{sgn} y''$ *and* $\operatorname{sgn} y^{(n-1)} \neq \operatorname{sgn} y^{(n)}$
(ii) $\lim_{x\to\infty} y^{(k)}(x) = 0, k = 1, 2, \cdots, n-1$
(iii) $\lim_{x\to\infty} y(x) =$ *finite constant.*

PROOF. Assume n odd and $y > 0$ (other cases are similar). Assume also y is bounded (if it were unbounded the corollary to Theorem 2 gives the result). By Theorem 2, y' is nonoscillatory, and since y is bounded so is y' [1, page 141]. Hence $a_{n-1}y' \to 0$ as $x \to \infty$. If $y' < 0$ then $a_{n-1}y' + a_n y \neq 0$ since y and y' are of opposite sign as are a_{n-1} and a_n. On the other hand if $y' > 0$ then $y \not\to 0$ and hence $a_n y$ is bounded away from zero and consequently for large x so is $a_{n-1}y' + a_n y$. Since $a_{n-1}y' + a_n y \neq 0$, Theorem 2 can now be applied again to give y' nonoscillatory. Since y is bounded and y' is nonoscillatory $y' \to 0$ as $x \to \infty$ and hence $y'y'' < 0$ and y' is bounded [1]. This in turn implies $a_{n-2}y'' + a_{n-1}y' \to 0$ giving again as above, $a_{n-2}y'' + a_n y' + a_n y \neq 0$ for large x. The above steps need only be repeated to give the result. The following corollary involves only minor modifications of the proof of Theorem 4.

COROLLARY 1. *Suppose in equation* (1) *the coefficients alternate in sign and $a_n \neq 0$ for large x. Suppose also $L_k y = 0$ $(k = 1, 2, \cdots, n-1)$ are nonoscillatory, and there are solutions ϕ_k of $L_k y = 0$ for which $L_{k-1} y = 0$ are disconjugate for large x $(k = 2, 3, \cdots, n-2)$. Then any nonoscillatory solution $y(x)$ of* (1) *is either unbounded and tends monotonically to $\pm\infty$ or is bounded. If y is nonoscillatory, bounded* *and* $y'(x)y(x) < 0$ *then*

(i) $\operatorname{sgn} y' = \operatorname{sgn} y^{(3)} = \cdots$, $\operatorname{sgn} y = \operatorname{sgn} y'' = \cdots$, *and* $\operatorname{sgn} y \neq \operatorname{sgn} y'$

and (ii) *and* (iii) *of Theorem* 4 *hold.*

COROLLARY 2. *Suppose in addition to the hypothesis of Corollary* 1 *it is assumed that* $(-1)^n a_n \geqq 0$. *Then* (1) *has no nonoscillatory solutions which are asymptotic to zero.*

PROOF. Suppose y is a positive nonoscillatory solution and $y > 0$ as $x \to \infty$. Then $y' < 0$ and by the above theorem $y'' > 0$, $y''' < 0$, etc. In particular if n is even $y^{(n)} > 0$. However, $y^{(n)} = -\sum_{i=0}^{n-1} a_{n-i} y^{(i)} \leqq 0$ which is a contradiction. A similar contradiction is arrived at for n odd.

It should be noted that the conditions of the above theorem and corollary are satisfied by $y^{(n)} + p(x)y = 0$ where $(-1)^n p(x) > d > 0$. In a paper of V. A. Kondrateev [3] some properties of the equation $y^{(n)} + py = 0$ were investigated where the equation was assumed to have property A.

DEFINITION 6. A differential equation has property A if all solutions are either oscillatory or are asymptotic to zero.

It can be easily seen that the results of Kondrateev hold equally well for the equation $L_n y = 0$ under conditions similar to those described above. In fact, Corollary 2 of Theorem 4 is such a generalization. The next theorem also generalizes a result of Kondrateev.

First a lemma will be stated which is an immediate consequence of writing (1) in vector matrix form (see Hartman [2, pages 506–508]).

LEMMA 2. *Suppose in* (1) $(-1)^i a_i \leqq 0$ $(i = 1, 2, \cdots, n)$. *If* $y(x)$ *is a solution of* (1) *satisfying* $y(c) \geqq 0$, $y'(c) \leqq 0$, $y''(c) \geqq 0, \cdots$, $(-1)^{n-1} y^{(n-1)}(c) > 0$, *then* $y(x) > 0$, $y'(x) < 0$, $y''(x) > 0, \cdots$, $(-1)^{n-1} y^{(n-1)}(x) > 0$ *for* $x < c$.

THEOREM 5. *If* (1) *has property* A, $(-1)^i a_i(x) \geqq 0$ *or* $(-1)^i a_i(x) \leqq 0$ $(i = 1, 2, \cdots, n^{-1})$, $a_n(x) \not\equiv 0$ *for large* x, $L_k y = 0$ $(k = 1, 2, \cdots, n-1)$ *are nonoscillatory, and there are solutions* ϕ_k *of* $L_k y = 0$ *for which* $L_{k-1} y = 0$ *are disconjugate for large* x $(k = 2, 3, \cdots, n-2)$; *then any solution of* (1) *which has at least one zero is oscillatory.*

PROOF. If $(-1)^n a_n \geqq 0$ the result is immediate from Corollary 2 of Theorem 4. If $(-1)^n a_n \leqq 0$ then, since property A holds, if y is a nonoscillatory solution, there is a c such that y (or possibly $-y$) satisfies $y > 0$, $y' < 0, \cdots, (-1)^{n-1} y^{(n-1)} > 0$ for $x \geqq c$, which, by Lemma 2, means it could not have vanished for $x < c$.

The final lemma and theorem generalize a result of Lazer [4] and give a sufficient condition that a nonoscillatory solution of (1) be asymptotic to zero.

DEFINITION 7. Let $W_{n-k}[y_1, \cdots, y_{n-k}](x) = \det(y_i^{(j)})$, $i = 1, \cdots, n-k; j = 0, \cdots, n-k-1$.

LEMMA 3. *Suppose in* (1) $(-1)^i a_i \leqq 0$ $(i = 1, 2, \cdots, n)$. *If* $y_1(x), \cdots, y_{n-1}(x)$ *are linearly independent solutions of* (1) *with* $y_i^{(j)}(x_0) = 0$ *for some fixed* j $(0 \leqq j \leqq n-2)$ *and* $i = 1, \cdots, n-1$; *then* $W_{n-1}(x) = W_{n-1}[y_1, \cdots, y_{n-1}](x)$ *is not zero for* $x > x_0$.

PROOF. Suppose there is an $x_1 > x_0$ such that $W_{n-1}(x_1) = 0$. Then there are constants $c_1, c_2, \cdots, c_{n-1}$ such that

$$\sum_{i=1}^{n-1} c_i y_i^{(k)}(x_1) = 0 \quad (k = 0, 1, \cdots, n-2)$$

with $c_1^2 + c_2^2 + \cdots + c_{n-1}^2 \neq 0$. Let $z(x) = \sum_{i=1}^{n-1} c_i y_i(x)$. Then $z(x)$ is a solution of (1) which satisfies $z^{(k)}(x_1) = 0$ $(k = 0, 1, \cdots, n-2)$ and by hypothesis $z^{(j)}(x_0) = 0$. However, $z(x)$ may be made, by multiplication by -1, if necessary, to fit the hypotheses of Lemma 2, which, in particular, implies $z^{(j)}(x_0) \neq 0$. This contradiction gives the desired result.

THEOREM 6. *If in the equation*

$$y^{(n)} + a_{n-1}(x)y' + a_n(x)y = 0 \qquad (n \geqq 3) \tag{4}$$

n *is odd*, $a_{n-1}(x) \leqq 0$, $a_n(x) > 0$, $\int_a^\infty x^{n-1} a_n(x)dx = \infty$, *then any nonoscillatory solution* $y(x)$ *of* (4) *which satisfies* $y(x)y'(x) < 0$ *for large* x *is asymptotic to zero.*

PROOF. Let $y_i(x)$, $i = 1, 2, \cdots, n-1$ be linearly independent solutions of (4) satisfying for some x_0, $y_i^{(j)}(x_0) = \delta_{ij}$, $i = 1, 2, \cdots, n-1$; $j = 0, 1, \cdots, n-2$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Since $y_i(x_0) = 0$, $i = 1, 2, \cdots, n-1$, Lemma 3 shows $W_{n-1}(x) = W_{n-1}[y_1, \cdots, y_{n-1}](x) \neq 0$ for $x > x_0$, and since $W_{n-1}(x_0) = W'_{n-1}(x_0) = \cdots = W_{n-1}^{(n-2)}(x_0) = 0$ and $W_{n-1}^{(n-1)}(x_0) = 1$, then $W_{n-1}^{(j)}(x) > 0$ for $x > x_0$ and $0 \leqq j \leqq n-2$. Let $D(x) = \det(y_i^{j})$, $i, j = 1, 2, \cdots, n-1$. Then $D'(x) = a_n(x)W_{n-1}(x)$, and hence $D(x) = 1 + \int_{x_0}^x a_n(t)W_{n-1}(t)\, dt$. However,

$$W_{n-1}^{(n-1)}(x) = 1 - a_{n-1}(x)W_{n-1}(x) + \int_{x_0}^x a_n(t)W_{n-1}(t)\, dt \geqq 1,$$

for $x \geqq x_0$, since $W_{n-1}(x) > 0$, $a_{n-1} \leqq 0$, and $a_n \geqq 0$. Repeated integration of this last inequality gives $W_{n-1}^{(n-1)}(x) \geqq 1$, $W_{n-1}^{(n-2)}(x) \geqq x - x_0$, $W_{n-1}^{(n-3)}(x) \geqq (1/2)(x - x_0)^2, \cdots, W_{n-1}(x) \geqq (1/(n-1)!)(x - x_0)^{n-1}$. Substituting this last estimate for W_{n-1} into the expression for D it is found that $D(x) \geqq t + \int_{x_0}^{x} ((t - x_0)^{n-1}/(n-1)!)\ a_n(t)\,dt$, and hence $\lim_{x\to\infty} D(x) = \infty$. Let $y(x)$ be any solution of (4) satisfying $y(x)y'(x) < 0$ for $x \geqq x_0$. It can be assumed without loss of generality that $y(x) > 0$, $y'(x) < 0$ for $x \geqq x_0$, and hence by Theorem 3, $y(x) > 0$, $y'(x) < 0, y''(x) > 0, \cdots, y^{(n-1)}(x) > 0, y^{(n)}(x) < 0$ for $x \geqq x_0$.

Now by Liouville's Identity $W(x) = W_n[y, y_1, y_2, \cdots, y_{n-1}](x) = W(x_0) = y(x_0)$. Also by expanding $W(x)$ by minors along the first column it is found that $y(x_0) = y(x)D(x) - y'(x)W_{n-1}^{(n-2)}(x) + y''(x)W_{n-1}^{(n-3)}(x) - \cdots + y^{(n-1)}(x)W_{n-1}(x) > y(x)D(x)$. Hence, since $D(x) \to \infty$ as $x \to \infty$, it follows that $y(x) \to 0$ as $x \to \infty$, and the proof is complete.

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U.S. International University, San Diego, California 92106
Arizona State University, Tempe, Arizona 85281