ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 5, Number 3, Summer 1975

# HOMOTOPY-ALGEBRAIC STRUCTURES

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In topology, there are many objects of study that consist of a space together with an "operation" on it. One may think of a topological group structure, an H-space structure, a homotopy self-equivalence, etc. One wishes to classify such operations up to homotopy and to consider the possible relations such an operation may satisfy. In this paper we provide a general framework to study these questions in terms of the Postnikov system of the space in question. Our model is the well-known fact that a space is an $H$-space if and only if its Postnikov invariants are primitive, and we are inspired by the work of Stasheff, [7].

The spaces we shall consider will be connected CW-complexes with basepoint. Let $X$ be such a space, with $x_0$ its basepoint. Denote the cartesian product of $n$ copies of $X$ by $X^n$ and let $T_1{}^n(X)$ be the subspace of $X^n$ consisting of all points at least one of whose coordinates is the basepoint.

DEFINITION 1. An *($n$-ary) operation* on $X$ consists of a pointed continuous function $\phi : X^n \to X$.

Let $\mathcal{P}X$ denote the (Moore) free path-space of $X$, i.e., the set of all pairs $(\lambda, r)$ such that $r \geqq 0$ and $\lambda : [0, r] \to X$ is continuous. We have two projections of $\mathcal{P}X$ onto $X$, $\pi_0$ and $\pi_\infty$, given by $\pi_0(\lambda, r) = \lambda(0)$ and $\pi_\infty(\lambda, r) = \lambda(r)$. The basepoint of $\mathcal{P}X$ is taken to be the pair $(\lambda_0, 0)$ such that $\lambda_0(0) = x_0$.

DEFINITION 2. If $\phi, \psi : X^n \to X$ are operations, a *relation* between $\phi$ and $\psi$ is a homotopy $R : X^n \to \mathcal{P}X$ such that $\pi_0 \circ R = \phi$ and $\pi_\infty \circ R = \psi$.

REMARK. Since $T_1{}^n(X)$ is retractile [3] in $X^n$, if $\phi$ and $\psi$ agree on $T_1{}^n(X)$, then $R$ may be chosen to remain fixed on $T_1{}^n(X)$.

DEFINITION 3. Suppose that $\phi : X^n \to X$ and $\phi_1 : X_1{}^n \to X_1$ are operations. A map $f : X \to X_1$ is called a *$(\phi, \phi_1)$-map* provided that there exists a homotopy $H : X^n \to \mathcal{P}X_1$ such that $\pi_0 \circ H = \phi_1 \circ f^n$ and $\pi_\infty \circ H = f \circ \phi$.

Observe that $\mathcal{P}X$ is a functor in $X$, i.e., that given $f : X \to Y$ we may define $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ by $\mathcal{P}f(\lambda)[t] = f(\lambda(t))$.

Received by the editors April 29, 1973 and in revised form September 1, 1973.

DEFINITION 4. Suppose that $R : X^n \to \mathcal{T}X$ and $R_1 : X_1{}^n \to \mathcal{T}X_1$ are relations between operations $\phi, \psi : X^n \to X$ and $\phi_1, \psi_1 = X_1{}^n \to X_1$ respectively. A map $f : X \to X_1$ is called an *$(R, R_1)$-map* provided that there exists a secondary homotopy $D : X^n \to \mathcal{T}(\mathcal{T}X_1)$ such that $\pi_0 \circ D = R_1 \circ f^n$ and $\pi_\infty \circ D = f \circ R$.

Note that if $H = \mathcal{T}\pi_0 \circ D$ and $G = \mathcal{T}\pi_\infty \circ D$, then $H$ and $G$ are homotopies that make $f$ a $(\phi, \phi_1)$-map and a $(\psi, \psi_1)$-map, respectively.

Given $(\lambda, r)$ in $\mathcal{T}X$, define $\lambda(t) = \lambda(r)$ if $t \geqq r$. There is a product $\mu : (\mathcal{T}X)^n \to \mathcal{T}(X^n)$ given by

$$\mu((\lambda_1, r_1), \cdots, (\lambda_n, r_n)) = (\lambda, \max(r_1, \cdots, r_n)),$$

where $\lambda(t) = (\lambda_1(t), \cdots, \lambda_n(t))$. Let $\mathcal{P}X$ be the subset of $\mathcal{T}X$ consisting of all $(\lambda, r)$ such that $\lambda(0) = x_0$ and let $\Omega X$ consist of all $(\lambda, r)$ in $\mathcal{P}X$ such that $\lambda(r) = x_0$. Let $A \subset \mathcal{T}X \times \mathcal{T}X$ consist of all pairs $((\lambda_1, r_1), (\lambda_2, r_2))$ such that $\lambda_1(r_1) = \lambda_2(0)$. Then we obtain an addition, $+ : A \to \mathcal{T}X$, by

$$((\lambda_1, r_1) + (\lambda_2, r_2))[t] = \begin{cases} \lambda_1(t) & (0 \leqq t \leqq r_1) \\ \lambda_2(t - r_1) & (r_1 \leqq t \leqq r_1 + r_2). \end{cases}$$

Clearly $\Omega X \times \Omega X \subset A$ and $\Omega X + \Omega X \subset \Omega X$.

Henceforth, consider the situation

$$\begin{array}{ccc} \Omega X_1 & = & \Omega X_1 \\ \downarrow & & \downarrow \\ E & \to & \mathcal{P}X_1 \\ \downarrow p & f & \downarrow \pi_\infty \\ X & \to & X_1 \end{array},$$

where the left-hand column is the fibration induced from the right-hand column. Thus $E = \{(x, \lambda) \mid f(x) = \pi_\infty(\lambda)\}$. Suppose that there are operations $\phi : X^n \to X$ and $\phi_1 : X_1{}^n \to X_1$ and that there is a homotopy $H : X^n \to \mathcal{T}X_1$ that makes $f$ a $(\phi, \phi_1)$-map. Construct an operation $\phi_2 : E^n \to E$ by

$$\phi_2((x_1, \lambda_1), \cdots, (x_n, \lambda_n)) = (\phi(x_1, \cdots, x_n), \mathcal{T}\phi_1 \circ \mu(\lambda_1, \cdots, \lambda_n) + H(x_1, \cdots, x_n)).$$

Observe that $\phi_2$ is well-defined and that $P : E \to X$ is a $(\phi_2, \phi_1)$-map. The operation $\phi_2$ is said to be induced by $\phi$, $\phi_1$, and $H$.

Suppose that $\phi, \psi : X^n \to X$ and $\phi_1, \psi_1 : X_1{}^n \to X_1$ are operations and that there are relations $R : X^n \to X$ and $R_1 : X_1{}^n \to X_1$ between the pairs $\phi, \psi$ and $\phi_1, \psi_1$, respectively. Then there is induced in similar fashion a relation $R_2 : E^n \to \vartheta E$.

We wish to consider the existence of operations and relations on a space by examining the stages of its Postnikov system. Thus we examine the situation

$$\begin{array}{l} E \\ \downarrow p \\ X \xrightarrow{\theta} K(G, m+1) \end{array}$$

where $\pi_k(X) = 0 (k \geqq m)$. We want to determine necessary conditions for the existence of operations and relations on $E$ in terms of $X$ and $\theta$.

Suppose we have an operation $\phi_2 : E^n \to E$. By the naturality of Postnikov systems [4] there are induced $\phi : X^n \to X$ and $\phi_1 : K(G, m+1)^n \to K(G, m+1)$ such that $\theta$ is a $(\phi, \phi_1)$-map and $p$ is a $(\phi_2, \phi)$-map. The homotopy classes of $\phi$ and $\phi_1$ are uniquely determined. We need to know more, however, to conclude that $\phi_2$ induces operations on $X$ with prescribed values on $T_1{}^n(X)$.

PROPOSITION 1. *Let $n \geqq 2$. Suppose that $\phi_2 : E^n \to E$ and $\check{\phi} : T_1{}^n(X) \to X$ are such that $p \circ \phi_2 = \check{\phi} \circ p^n$ on $T_1{}^n(X)$. Then there exists an extension $\phi : X^n \to X$ of $\check{\phi}$ such that $p$ is a $(\phi_2, \phi)$-map.*

PROPOSITION 2. *Let $\phi_2, \psi_2 : E^n \to E$, $\phi, \psi : X^n \to X$, and $\phi_1$, $\psi 1 : K(G, m+1)^n \to K(G, m+1)$ be operations such that $p$ is a $(\phi_2, \phi)$- and $(\psi_2, \psi)$-map and $\theta$ is a $(\phi, \phi_1)$- and $(\psi, \psi_1)$-map. Let $R_2 : E^n \to \vartheta E$ be a relation between $\phi_2$ and $\psi_2$. Let $\hat{R} : T_1{}^n(X) \to \vartheta X$ be a homotopy between the restrictions of $\phi$ and $\psi$ to $T_1{}^n(X)$. Then $\hat{R}$ extends to a relation $R$ between $\phi$ and $\psi$ such that $p$ is an $(R_2, R)$-map. Furthermore, there exists a relation $R_1$ between $\phi_1$ and $\psi_1$ such that $\theta$ is an $(R, R_1)$*-map.

PROPOSITION 3. (cf. [9, pp. 38–40]). *Suppose that $\phi : X^n \to X$ and $\phi_1 : K(G, m+1)^n \to K(G, m+1)$ are operations and that $\bar{H} : X^n \to \vartheta K(G, m+1)$ is a homotopy that makes $\theta$ a $(\phi, \phi_1)$-map. Suppose that $\hat{\phi}_2 : T_1{}^n(E) \to E$ is given by*

$$\hat{\phi}_2((x_1, \lambda_1), \cdots, (x_n, \lambda_n)) = (\phi(x_1, \cdots, x_n), \vartheta\phi_1 \circ \mu(\lambda_1, \cdots, \lambda_n) + \bar{H}(x_1, \cdots, x_n)).$$

*Then any extension $\check{\phi}_2$ of $\hat{\phi}_2$ that makes $p$ a $(\check{\phi}_2, \phi)$-map is homotopic to one of the form*

$$\check{\phi}_2((x_1, \lambda_1), \cdots, (x_n, \lambda_n)) = (\phi(x_1, \cdots, x_n), \vartheta\phi_1 \circ \mu(\lambda_1, \cdots, \lambda_n) + H(x_1, \cdots, x_n)),$$

*for some homotopy $H$ between $\theta \circ \phi$ and $\phi_1 \circ \theta^n$ that agrees with $H$ on $T_1{}^n(X)$.*

PROPOSITION 4. *Suppose $R : X^n \rightarrow \vartheta X$ and $R_1 : K(G, m+1)^n \rightarrow \vartheta K(G, m+1)$ are relations and that $\bar{D} : X^n \rightarrow \vartheta(\vartheta K(G, m+1))$ makes $\bar{\theta}$ an $(R, R_1)$-map. Suppose that $\hat{R}_2 : E^n \rightarrow \vartheta E$ is induced by $R$, $R_1$, and $\bar{D}$. Then any $\check{R}_2 : E^n \rightarrow \vartheta E$ that makes $p$ an $(\check{R}_2, R)$-map is homotopic to one induced by $R$, $R_1$, and $D$, for some $D$ that agrees with $\bar{D}$ on $T_1{}^n(X)$.*

Propositions 1 and 2 are modelled on those of [7]. Propositions 3 and 4 are proved using obstruction theory, cf. [6]. See also [1].

The above techniques have been used to study $H$- and $HAH$-structures in [1], [7], [8] and [12]; $HC$-structures in [10] and [12]; and $QC$-structures in [11]. In order to make calculations we need to examine the image of $[-; \Omega K(G, m+1)] \rightarrow [-; E]$. We illustrate the type of calculation necessary in two examples.

EXAMPLE 1. We enumerate the $H$-equivalence classes of multiplications on real projective 3-space $P_3$. (Two multiplications $m$ and $m'$ are $H$-equivalent if there exists an $H$-map $f : (X, m) \rightarrow (X, m')$ that is a homotopy equivalence. According to [5] there are 768 homotopy classes of $H$-space multiplications on $P_3$. We wish to determine which of these are $H$-equivalent to each other.

Begin by observing that there are two homotopy classes of homotopy equivalences of $P_3$ with itself. For, in the short exact sequence of groups

$$0 \rightarrow [P_3; S^3] \xrightarrow{\pi_*} [P_3; P_3] \rightarrow [P_3; K(Z_2, 1)] \rightarrow 0$$

obtained from the fibration $S^0 \rightarrow S^3 \rightarrow P_3$, we see that $[P_3; P_3]$ is an extension of $Z_2$ by $Z$, and the only elements of $[P_3; P_3]$ that induce isomorphisms of integral cohomology are 1 and $1 - [\pi \circ p]$ for $p$ an appropriately chosen generator of $[P_3; S^3]$.

Now consider the bottom stage of a Postnikov system for $P_3$. We have

$$\begin{array}{c} E_1 \\ \downarrow \\ K(Z_2, 1) \xrightarrow{\theta_1} K(Z, 4). \end{array}$$

There is one self-equivalence on $K(Z_2, 1)$, there are two on $K(Z, 4)$, and $\theta_1$ is a map for each pair of these, since $\theta_1{}^*$ takes both generators of $H^4(Z, 4; Z)$ to the non-zero element of $H^4(Z_2, 1; Z)$. Differences in homotopies $H : K(Z_2, 1) \rightarrow \Omega K(Z, 4)$ for $\theta_1$ lie in

$$[K(Z_2, 1); \Omega K(Z, 4)] \approx H^3(Z_2, 1; Z) = 0.$$

Thus we obtain two classes of self-equivalences for $E_1$, which are easily seen to lift to the two classes on $P_3$.

We now count the multiplications on $E_1$. There are unique multiplications on $K(Z_2, 1)$ and $K(Z, 4)$, respectively, and $\theta_1$ must be an $H$-map with respect to these. The classes of multiplications on $E_1$, therefore, are determined by elements of the group $H^4(K(Z_2, 1) \wedge K(Z_2, 1); Z) \approx Z_2$, so there are at most two classes of multiplications on $E_1$.

We may regard $P_3$ as a loop space $\Omega BSO(3)$, and consequently may consider the spaces and maps in its Postnikov system to be loop spaces and loop maps. Let $m$ denote the loop addition on $E_1$. If we can show that $m$ is not homotopy-commutative, then the two classes of multiplications on $E_1$ must be those determined by $m$ and $m \circ T$.

Let us write $E_1 = \Omega E_1'$, $K(Z_2, 1) = \Omega K(Z_2, 2)$, $K(Z, 4) = \Omega K(Z, 5)$, and $\theta_1 = \Omega \theta_1'$. For any space $Y$ let $\epsilon : \Sigma \Omega Y \rightarrow Y$ denote the evaluation map. It is easy to see, cf. [12], that the composition

$$H^5(K(Z_2, 2) \wedge K(Z_2, 2)) \xrightarrow{(\epsilon \wedge \epsilon)^*} H^5(\Sigma K(Z_2, 1) \wedge \Sigma K(Z_2, 1)) \approx H^3(K(Z_2, 1) \wedge K(Z_2, 1))$$

takes the obstruction to $\theta_1'$ being an $H$-map to an element of the obstruction set to $\theta_1$ being an $HCH$-map. This latter obstruction set is a coset of the subgroup $(T^* - 1^*)(H^3(K(Z_2, 1) \wedge K(Z_2, 1)))$. By use of the Künneth theorem we see that this subgroup is trivial and that $(\epsilon \wedge \epsilon)^*$ is an isomorphism in this dimension. Thus $\theta_1'$ is an $H$-map if and only if $\theta_1$ is an $HCH$-map. But it is shown in [2] that $\theta_1'$ is not an $H$-map. Thus no multiplication on $E_1$ can be homotopy-commutative.

Let $\phi$ denote the non-identity self homotopy-equivalence on $E_1$. We may represent $\phi$ by $\phi(\alpha) = -\alpha$. (Here $-(\alpha, r) = (-\alpha, r)$ where $-\alpha(t) = \alpha(r - t)$.) Then $\phi \circ m(\alpha, \beta) = -(\alpha + \beta) = (-\beta) + (-\alpha)$ whereas $m \circ (\phi \times \phi)\ (\alpha, \beta) = m(-\alpha, -\beta) = (-\alpha) + (-\beta)$. Thus $m \circ (\phi \times \phi)$ is not homotopic to $\phi \circ m$. Let $\phi_1$ denote the nonidentity self homotopy-equivalence of $P_3$. Since any multiplication $m_1$ on $P_3$ is a lifting of either $m$ or $m \circ T$, then $\phi_1^{-1} \circ m_1 \circ (\phi_1 \times \phi_1)$ must be a lifting of the other. Hence $\phi_1$ is not an $H$-map between any multiplication and itself, so the 768 homotopy classes of multiplications on $P_3$ reduce to exactly 384 $H$-equivalence classes.

EXAMPLE 2. We compute the number of classes of homotopy self-equivalences of the special unitary group, SU(3). The first stages of a Postnikov system for SU(3) may be written

$$\begin{array}{rcl} & E_8 & \\ & \big\downarrow{\scriptstyle P_8} & \\ E_7 = & E_6 & \xrightarrow{\theta_8} K(Z_{12}, 9) \\ & \big\downarrow{\scriptstyle P_6} & \\ & E_5 & \xrightarrow{\theta_6} K(Z_6, 7) \\ & \big\downarrow{\scriptstyle P_5} & \\ E_3 = E_4 = & K(Z, 3) & \xrightarrow{\theta_5} K(Z, 6) \quad . \end{array}$$

To construct and classify the equivalences on $E_n$, we consider the Serre exact sequence (coefficients in $\pi_n(SU(3))$):

$$\cdots \leftarrow H^k(K(\pi_n, n)) \leftarrow H^k(E_n) \leftarrow H^k(E_{n-1}) \xleftarrow{\theta_n} H^{k-1}(K(\pi_n, n)).$$

Note that $H^{n-1}(E_{n-1}) \to H^{n-1}(E_n)$ is isomorphic and $H^n(E_{n-1}) \to H^n(E_n)$ is monomorphic. Thus, $H^n(E_n) \approx H^n(SU(3))$ and $H^{n+1}(E_n)$ injects monomorphically into $H^{n+1}(SU(3))$. Since $H^n(SU(3))$ is an exterior algebra with two generators in dimensions 3 and 5, $H^n(SU(3)) = 0$ ($n \neq 0$, 3, 5 and 8). Thus if $n \geqq 8$, $H^n(E_n) = 0 = H^{n+1}(E_n)$. We observe further that if $n \geqq 8$, $\theta_n^* \circ \sigma : H^n(K(\pi_n, n)) \to H^{n+1}(E_{n-1})$ is an isomorphism, whence $\theta_n^* : H^{n+1}(K(\pi_n, n+1)) \to H^{n+1}(E_{n-1})$ is isomorphic. Further examination reveals that $\theta_n^*$ is an isomorphism in dimension $n+1$ for all $n > 5$. Thus any self-equivalence of $E_{n-1}$ ($n > 5$), induces a unique one of $K(\pi_n, n+1)$ such that $\theta_n$ is a map of these structures. There are two self-equivalences each on $K(Z, 3)$ and $K(Z, 6)$ and $\theta_5$ is a map for each of the four pairings of these, since $\theta_5^* : H^6(K(Z, 6) : Z) = Z \to H^6(K(Z, 3) : Z) = Z_2$. We now need to count the various liftings of these structures from $E_{n-1}$ to $E_n$.

According to Proposition 3, we need to look at elements of $H^n(E_{n-1}; \pi_n(SU(3)))$ and determine which of them define different operations on $E_n$; precisely, we examine the image of the composition

$$H^n(E_{n-1}; \pi_n(SU(3))) \xrightarrow{P_n^*} [E_n; \Omega K(\pi_n(SU(3)), n+1)] \xrightarrow{i_*} [E_n; E_n]$$

where $i : \Omega K(\pi_n(SU(3)); n+1) \to E_n$ is the inclusion of the fiber. We have already seen that $p_n^*$ is monomorphic in this dimension.

Let $n = 5$. Then $H^5(E_n; Z) = H^5(K(Z, 3); Z) = 0$. Consequently each pair of equivalences on $K(Z, 3)$ and $K(Z, 6)$ determines a unique equivalence of $E_5$. Thus $E_5$ has four self-equivalences.

The group $H^6(E_5; Z_6)$ injects into $H^6(\mathrm{SU}(3); Z_6) = 0$, so that $E_6$ possesses four self-equivalences.

Finally let $n = 8$. We observe, by use of the cohomology ring structure, that $p_8{}^* : H^8(E_7; Z_{12}) \to H^8(E_9; Z_{12}) \approx Z_{12}$ is an isomorphism. Consider the diagram

$$\begin{array}{ccccc} & & [E_8; \Omega K(Z_6, 6)] = H^5(E_8; Z_6) & & \\ & & \downarrow j_* & & \\ & & [E_8; \Omega E_7] \to H^8(E_8; 12) & \xrightarrow{i_*} & [E_8; E_8] \\ & & \downarrow & & \\ 0 = H^4(E_8; Z) & \to & [E_8; \Omega E_5] \to H^2(E_8; Z) = 0. & & \end{array}$$

We see that $i_*$ is onto. Consequently we may look at the compositions $H^5(E_8; Z_6) \to [E_8; \Omega E_7] \to H^8(E_8; Z_{12})$. This is induced by a cohomology operation $K(Z_6, 5) \to K(Z_{12}, 8)$. Any such operation is zero in the cohomology of SU(3) (it must be "essentially" $\mathrm{Sq}^3$) and so must also be zero in $E_8$. Thus $i_*$ is injective and so each equivalence of $E_7$ lifts to twelve of $E_8$. We conclude that $E_8$ (and consequently SU(3)) possesses 48 classes of homotopy self-equivalences.

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