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APPLICATION OF SINGULAR PERTURBATION THEORY TO THE RESTRICTED THREE BODY PROBLEM

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ABSTRACT. This paper describes how the theory of singular perturbations can be applied to establish the existence and asymptotic approximation of those solutions of the restricted three body problem with small mass ratio $\mu > 0$ which reach an $O(\mu)$ neighborhood of the perturbing body. It also describes how singular perturbation theory and the theory of ordinary differential equations can be used to establish the existence of one-parameter families of periodic solutions of the second species of Poincaré for the restricted three body problem.

1. **The restricted problem as a singular perturbation problem.** The equations of motion of the restricted three body problem with mass ratio $0 < \mu \ll 1$ can be written in an inertial, earth-centered coordinate system as

$$\begin{aligned} \ddot{x} &= -\frac{x}{|x|^3} - \mu\left[\frac{x - x_m(t)}{|x - x_m(t)|^3} - \frac{x}{|x|^3} + \frac{x_m(t)}{|x_m(t)|^3}\right] \\ \ddot{x}_m &= -\frac{x_m}{|x_m|^3}, \quad \cdot = \frac{d}{dt} \end{aligned} \tag{1}$$

where $x \in E^3$ is the position vector of the particle and $x_m \in E^3$ is the position vector of the moon. Note that if the particle collides with the moon; i.e., if $x - x_m = 0$, then the perturbation term becomes singular. On the other hand, if the particle reaches an $O(\mu)$ neighborhood of the moon, $|x - x_m| = O(\mu)$,

$$y = \frac{x - x_m}{\mu} = O(1)$$

and the system (1) can be written in the form of a singular perturbation problem with y as the dependent variable:

$$\begin{aligned} \mu^2\ddot{y} &= -\frac{y}{|y|^3} - \mu(1-\mu)\left[\frac{\mu y + x_m(t)}{|\mu y + x_m(t)|^3} - \frac{x_m(t)}{|x_m(t)|^3}\right] \\ \ddot{x} &= -\frac{x_m}{|x_m|^3}. \end{aligned} \tag{2}$$

For $\mu = 0$, (1) has the form

$$\ddot{x}_0 = -\frac{x_0}{|x_0|^3}$$
(3)
$$\ddot{x}_m = -\frac{x_m}{|x_m|^3}$$

Thus, $x_0(t)$ and $x_m(t)$ are earth-centered Keplerian conics. We assume that $x_m(t)$ is a Keplerian ellipse, that $x_0(t) \neq 0$, and that at some time $t_1 > t_0$

(4) $$x_0(t_1) = x_m(t_1)$$

with relative velocity

$$V_1 = \dot{x}_0(t_1) - \dot{x}_m(t_1) \neq 0.$$

We then choose the initial conditions for the restricted problem (1) as

(5)
$$x(t_0, \mu) = x_0(t_0) + \delta x_0(\mu)$$
$$\dot{x}(t_0, \mu) = \dot{x}_0(t_0) + \delta \dot{x}_0(\mu)$$

with $\delta x_0 = O(\mu)$ and $\delta \dot{x}_0 = O(\mu)$. It follows that $|x(t, \mu) - x_m(t)| = O(\mu)$ for $t = t_1 + O(\mu)$ and we have a singular perturbation problem.

2. **The outer approximation and error estimates.** For $t \in [t_0, t_1 - O(\mu^{1-\epsilon})]$, $\epsilon > 0$, the asymptotic expansion for the restricted problem (1), which we write as

$$\ddot{x} = f(x) + \mu g(x, t), \quad f(x) = -\frac{x}{|x|^3}$$

has the form

(6) $$x(t, \mu) = \sum_{j=0}^{N} \mu^j x_j(t) + r_N(t, \mu)$$

where, as in (3), x_0 is the solution of the two body problem $\ddot{x}_0 = f(x_0)$, and the x_j are successively defined as the solutions of the linear equations

$$\ddot{x}_1 = \frac{df}{dx}(x_0)x_1 + g(x_0, t), x_1(t_0) = \frac{\delta x_0}{\mu}, \quad \dot{x}_1(t_0) = \frac{\delta \dot{x}_0}{\mu}$$

(7)
$$\ddot{x}_j = \frac{df}{dx}(x_0)x_j + \sum_{k=2}^{j} \frac{1}{k!}\frac{d^k f}{dx^k}(x_0) \sum_{i_1+\cdots+i_k=j} x_{i_1}\cdots x_{i_k}$$
$$+ \sum_{k=1}^{j-1} \frac{1}{k!}\frac{\partial^k g}{\partial x^k}(x_0, t) \sum_{i_1+\cdots+i_k=j-1} x_{i_1}\cdots x_{i_k},$$

$$x_j(t_0) = \dot{x}_j(t_0) = 0$$

for $j \geqq 2$.

The remainder after N terms, r_N, satisifes a nonlinear equation of the form $\ddot{r}_N = F(r_N, t, \mu)$. Cf. [28, p. 741].

The asymptotic form of the outer approximation as $t \to t_1$ is determined from the Taylor series expansion of $x_0(t)$ about $t = t_1$ and from the solutions of the linear equations (7); cf. [3, 26, 28]:

$$\begin{aligned} x_0(t) - x_m(t) &= -V_1\tau + \frac{1}{6|x_m(t_1)|^3}\left[V_1 - \frac{3(x_m(t_1)\cdot V_1)x_m(t_1)}{|x_m(t_1)|^2}\right]\tau^3 + O(\tau^4) \\ x_1(t) &= \frac{i}{V_1^{\,2}} \ln\left(\frac{\tau_0}{\tau}\right) + x_{1b}(t) \\ x_2(t) &= \frac{i}{V_1^{\,5}}\left(\frac{1}{\tau}\right)\left[\ln\left(\frac{\tau_0}{\tau}\right) + a_1 + b_1\tau\ln\tau\right] + x_{2b}(t) \end{aligned} \tag{8}$$

where $\tau = t_1 - t$, $\tau_0 = t_1 - t_0$, $i = V_1/|V_1|$ and $x_{jb}(t)$, $j \leqq 1$, are analytic functions of t.

Assuming the induction hypotheses that

$$|x_k(t)| \leqq c_k \left| \frac{\ln\tau}{\tau} \right|^{k-1} e^{c_0(t-t_0)}$$

for all $t \in [t_0, t_1]$ and $2 \leqq k < j$, we obtain the following integral inequality for x_j from (7);

$$|x_j(t)| \leqq \int_{t_0}^{t}\int_{t_0}^{t'} \left[c_0|x_j(t'')| + c_j\frac{|\log\tau''|^{j-1}}{(\tau'')^{j+1}}\right] dt''\, dt',$$

and then using a variant of Gronwall's lemma, we prove the following theorem by induction on j; cf. [28, p. 747];

THEOREM 1. *Let $x_0(t)$ and $x_m(t)$ be solutions of* (3) *with $x_0(t) \neq 0$, $x_m(t) \neq 0$, and with the initial conditions $x_0(t_0)$ and $\dot{x}_0(t_0)$ chosen so that conditions* (4) *are satisfied for some $t_1 > t_0$. It follows that there exists positive constants c_j, $j \geqq 1$, such that the solutions to the linear equations* (7) *satisfy*

$$\begin{aligned} |x_1(t)| &\leqq c_1|\ln\tau| e^{c_0(t-t_0)} \\ |x_j(t)| &\leqq c_j \left| \frac{\ln\tau}{\tau}\right|^{j-1} e^{c_0(t-t_0)} \end{aligned} \tag{9}$$

for all $j \geqq 2$ *and* $t \in [t_0, t_1)$.

We now estimate the error term r_N. Using Theorem 1 and the estimate $|x(t) - x_m(t)| > a\tau$, $a > 0$, which follows from (8), it is possible to deduce the following integral inequalities:

$$|r_N| \leqq \int_{t_0}^{t} \int_{t_0}^{t'} F_0(|r_N|, t'', \mu)\, dt''\, dt' \tag{10}$$

$$|\dot{r}_N| \leqq \int_{t_0}^{t} F_0(|r_N|, t', \mu)\, dt'$$

where

$$\begin{aligned} F_0(u, t, \mu) = \; & c_0^2 u + c_N \Bigg\{ \mu^N \frac{|\ln \tau|^{N-1}}{\tau^{N+1}} \\ & + \sum_{k=2}^{N-1} \sum_{j=1}^{k} u^j (\mu|\ln \tau|)^{k-j} + \\ & + \left(\frac{\mu|\ln \tau| + u}{b - u} \right)^N + \frac{\mu(\mu|\ln \tau| + u)^N}{(a\tau - u)^{N+1}} \\ & + \frac{\mu}{\tau^2} \sum_{k=1}^{N-2} \sum_{j=1}^{k} \left| \frac{u}{\tau^j} \right| \left(\frac{\mu|\ln \tau|}{\tau} \right)^{k-j} \Bigg\}. \end{aligned}$$

The above integral inequalities hold on any sub-interval of $[t_0, t_1 - O(\mu^{1-\epsilon})]$ where

$$|r_N(t, \mu)| < K(t) = \min[b, a\tau] \tag{*}$$

provided $\mu > 0$ is sufficiently small. Theorem 1 and this last inequality (*), together with the fact that $|x_0 - x_m| > a\tau$, imply that $|x - x_m| \geqq d_1 > 0$ on $[t_0, t_1 - 0(\mu^{1-\epsilon})]$. Hence, the solution $x(t, \mu)$ exists on $[t_0, t_1 - 0(\mu^{1-\epsilon})]$ for all $\mu > 0$ sufficiently small provided the above inequality (*) is satisfied.

Despite the complicated nature of the integral inequalities (10), estimates for $|r_N|$ and $|\dot{r}_N|$ can be deduced by construction an appropriate majorizing function $M_N(t, \mu)$. This follows from the estimation lemma in [26] and [28] which is a nontrivial generalization of Gronwall's lemma to the nonlinear case when the function $F_0(u, t, \mu)$ is a monotone function of u on its domain of definition. The clue to the proper form of the majorizing function comes from the estimates on $|x_j(t)|$ in Theorem 1. We find that the function

$$M_N(t, \mu) = \mu^N c_N \left(1 + \frac{1}{b^N} + \frac{1}{a^{N+1}}\right) \left| \frac{\ln \tau}{\tau} \right|^{N-1} e^{c_0(t-t_0)}$$

satisfies the differential inequality $\ddot{M}_N \geqq F_0(M_N, t, \mu)$ and the inequality (*) $M_N(t, \mu) < K(t)$ for all $t \in [t_0, t_1 - 0(\mu^{1-\epsilon})]$ and $\mu > 0$ sufficiently small; cf. [28, p. 752–753]. The following theorem then follows from the estimation lemma [28, p. 744].

THEOREM 2. *Under the hypotheses of Theorem* 1, *it follows that for all* $N \geqq 2$ *and* $\epsilon > 0$ *there exists a* $\mu_1 > 0$ *such that for all* $\mu \in [0, \mu_1]$ *there exist constants* C_N *such that* $x(t, \mu)$ exists for $t \in [t_0, t_1 - \mu^{1-\epsilon}]$ *and*

$$\begin{aligned} |r_N(t, \mu)| &\leqq C_N \mu^N \left| \frac{\ln \tau}{\tau} \right|^{N-1} \\ |\dot{r}_N(t, \mu)| &\leqq C_N \mu^N \frac{|\ln \tau|^{N-1}}{\tau^N} \end{aligned} \tag{11}$$

for all $t \in [t_0, t_1 - \mu^{1-\epsilon}]$.

3. **The inner approximation and error estimates.** With the introduction of inner variables

$$Y = \frac{x - x_m}{\mu} \text{ and } s = \frac{t - t_p}{\mu}, \quad t_p = t_1 + O(\mu),$$

the three body equations (1) take the form

$$\begin{aligned} Y'' &= -\frac{Y}{|Y|^3} - \mu(1-\mu)\left[\frac{X_m(s, \mu) + \mu Y}{|X_m(s, \mu) + \mu Y|^3} - \frac{X_m(s, \mu)}{|X_m(s, \mu)|^3}\right] \\ &= f(Y) + \mu G(Y, s, \mu) \qquad (12) \\ X_m'' &= -\frac{\mu^2 X_m}{|X_m|^3} \end{aligned}$$

where capitals are used to denote functions of s.

The inner expansion has the form

$$Y(s, \mu) = \sum_{j=0}^{N-1} \mu^j Y_j(s) + R_N(s, \mu)$$

where

$$
(13)\qquad
\begin{aligned}
Y_0'' &= f(Y_0),\\
Y_1'' &= \frac{df}{dx}(Y_0)Y_1 + G(Y_0, s, 0)\\
Y_j'' &= \frac{df}{dx}(Y_0)Y_j + \sum_{k=2}^{j} \frac{1}{k!}\frac{d^kf}{dx^k}(Y_0) \sum_{i_1+\cdots+i_k=j} Y_{i_1}\cdots Y_{i_k}\\
&\quad + \sum_{k=1}^{j-1} \frac{1}{k!}\frac{\partial^k G}{\partial \bar{x}^k}(\bar{Y}_0, t) \sum_{i_1+\cdots+i_k=j-1} Y_{i_1}\cdots Y_{i_k}
\end{aligned}
$$

for $j \geqq 2$ with $Y_j(0) = \dot{Y}_j(0) = 0$ for $j \geqq 1$, $\bar{x} = (x, \mu) \in E^4$ and $\bar{Y}_0 = (Y_0, 0) \in E^4$. $Y_0(s)$ is a moon centered hyperbola.

If $Y_0(s) \neq 0$, we find that for $s = O(1)$, $|Y_j(s)| \leqq C_j$ and $|Y_j'(s)| \leqq C_j$; and from the linear integral inequalities satisfied by the $Y_j(s)$, similar to those in § 2, we find, using an inductive argument and a variant of Gronwall's Lemma that

$$
|Y_j(s)| \leqq \frac{C_j}{\mu^{1/2}}, \quad |\dot{Y}_j(s)| \leqq C_j \quad \text{for } |s| \leqq O\left(\frac{1}{\mu^{1/2}}\right).
$$

The remainder after N terms R_N satisfies a nonlinear differential equation of the form

$$
R_N'' = F_1(R_N, s, \mu).
$$

By constructing an appropriate majorizing function and using the estimation lemma in [28, p. 744], it is possible to prove the following theorem. The details for the case $N = 2$ are carried out in [26].

THEOREM 3. *For $N \geqq 2$ let $|R_N(s, \mu)| = O(\mu^{N-1-\epsilon})$ and $|R_N'(s, \mu)| = O(\mu^{N-1-\epsilon})$ for $s = -\mu^{-1/2}$ and let $Y(s, \mu)$, $Y_j(s)$ and $X_m(s, \mu)$ be solutions of* (12) *and* (13) *with $Y_0(s) \neq 0$ and $X_m(s, \mu) \neq 0$. It follows that for all $N \geqq 2$, and $\epsilon > 0$ there exists a $\mu_1 > 0$ such that for all $\mu \in [0, \mu_1]$ $Y(s, \mu)$ exists for $s \in [-1/\mu^{1/2}, 1/\mu^{\epsilon}]$ and there exist constants C_N such that*

$$
(14)\qquad
\begin{aligned}
|R_N(s, \mu)| &\leqq C_N \mu^{N-1-\epsilon}\\
|R_N'(s, \mu)| &\leqq C_N \mu^{N-1-\epsilon}
\end{aligned}
$$

for all $s \in [-1/\mu^{1/2}, 1/\mu^{\epsilon}]$.

REMARK. In terms of the outer variables, this implies that the inner expansion has the form

$$x(t, \mu) = x_m(t) + \mu y_0(t, \mu) + \sum_{k=1}^{N-1} \mu^{k+1} y_k(t, \mu) + O(\mu^{N-\epsilon})$$
$$\dot{x}(t, \mu) = \dot{x}_m(t) + \mu \dot{y}_0(t, \mu) + \sum_{k=1}^{N-1} \mu^{k+1} \dot{y}_k(t, \mu) + O(\mu^{N-1-\epsilon})$$

for $N \geqq 2$, and for all $t \in [t_p - \mu^{1/2}, t_p + \mu^{1-\epsilon}]$ where $y_j(t, \mu) = y_j(t - t_p/\mu)$.

Thus, if we can establish, via asymptotic matching, that the inner expansion written in terms of outer variables matches the outer expansion to within error terms of $O(\mu^{N-\epsilon})$ for all $t - t_p = O(\mu^{1/2})$, then we will obtain an asymptotic expansion uniformly valid to within error terms of $O(\mu^{N-\epsilon})$ for all $t \in [t_0, t_p + O(\mu^{1-\epsilon})]$ by taking $2N - 1$ terms in the outer expansion and N terms in the inner expansion. Actually, it is not necessary to carry all parts of all of these terms in order to obtain an asymptotic expansion uniformly valid to $O(\mu^{N-\epsilon})$; cf. [4].

4. **The asymptotic matching.** The asymptotic matching for this problem compares the functional forms of the inner expansion expressed in terms of outer variables and the outer asymptotic expansion in the region where $t - t_1 = O(\mu^{1/2})$ as $\mu \to 0$. It determines the parameters of the moon centered parabola $y_0(t, \mu)$ in terms of the initial conditions $x_0(t_0)$, $\dot{x}_0(t_0)$ and the variations in the initial conditions δx_0, $\delta \dot{x}_0$. This matching has been carried out to first order in [3] and in [26] and to second order in [4].

In the remainder of this paper, we find it more convenient to use the outer variables with respect to the earth, $x(t, \mu)$ and with respect to the moon

$$\eta = \mu y = x - x_m.$$

The second order matching then determines the parameters of the moon-centered hyperbola $\eta_0(t, \mu) = \mu y_0(t, \mu)$ correct to within error terms of $O(\mu^{3-\epsilon})$ for any $\epsilon > 0$; i.e., the distance to the asymptote of the hyperbola

$$\begin{aligned} \Delta = {} & \mu j \cdot x_{1b}(t_1) + \mu^2 j \cdot x_{2b}(t_1) \\ & - (\mu^2/V_1)[j \cdot \dot{x}_{1b}(t_1)]\,[i \cdot x_{1b}(t_1)] + O(\mu^{3-\epsilon}) \end{aligned} \tag{15a}$$

the time of perilune passage, t_p, is determined by

$$\begin{aligned} t_1 - t_p = {} & \frac{\mu}{V_\infty} i \cdot x_{1b}(t_1) + \frac{\mu}{|V_\infty|^3}\left[\ln\left(\frac{2V_\infty{}^2\tau_0}{\mu e_1}\right) - 1\right] \\ & + \frac{\mu^2}{V_1{}^2}[j \cdot \dot{x}_{1b}(t_1)]\,[i \cdot x_{1b}(t_1)] + \mu^2 i \cdot x_{2b}(t_1) + O(\mu^{3-\epsilon}) \end{aligned} \tag{15b}$$

where $e_1 = [1 + (V_\infty{}^2\Delta/\mu)^2]^{1/2}$, and the velocity at infinity along the hyperbola, V_∞, is determined by a more complicated formula, the second order terms given in [4] being too lengthy to include here

(15c) $$V_\infty = V_1 + \mu\dot{x}_{1b}(t_1) + \mu^2(\cdot\cdot\cdot) + O(\mu^{3-\epsilon}).$$

We note that the explicit form of $x_{1b}(t_1)$ is given by

$$x_{1b}(t_1) = \Phi_{rr}(t_1, t_0)\left(\frac{\delta x_0}{\mu}\right) + \Phi_{rv}(t_1, t_0)\left(\frac{\delta \dot{x}_0}{\mu}\right) + \int_{t_0}^{t_1} b_1(t_1, t)\, dt$$

where

$$b_1(t_1, t) = \Phi_{rv}(t_1, t)g[x_0(t), t] - \frac{i}{V_1{}^2\tau}, \quad t \neq t_1$$

$$b_1(t_1, t_1) = 0$$

is an analytic function of t and $\Phi(t, t_0)$ is the fundamental matrix solution of

$$\dot{\Phi} = \begin{bmatrix} 0 & I \\ f_x[x_0(t)] & 0 \end{bmatrix} \Phi, \quad \Phi(t_0, t_0) = I.$$

We can then prove the following existence theorem based on Theorems 2 and 3 above and the theory of ordinary differential equations as in [29, p. 206].

THEOREM 4. *Let $x_0(t)$, $x_m(t)$, $x_0(t_0)$ and $x_0(t_0)$ satisfy the conditions of Theorem* 1. *Let the variations δx_0 and $\delta\dot{x}_0$ satisfy the following non-collision condition: given $k_0 > 0$ there are constants $\mu_0 > 0$ and $d_0 > 0$ such that for all $|\delta x_0| \leqq \mu k_0$, $|\delta \dot{x}_0| \leqq \mu k_0$ and $\mu \in (0, \mu_0)$,*

(16) $$|j \cdot \Phi_{rr}(t_1, t_0)\delta x_0 + j \cdot \Phi_{rv}(t_1, t_0)\delta\dot{x}_0 + \mu\int_{t_0}^{t_1} j \cdot b_1(t_1, t)\, dt| \geqq \mu d_0.$$

It then follows that given $\epsilon > 0$ there exists a $\mu_1 > 0$ (with $\mu_1 \leqq \mu_0$) such that for $t \in [t_0, t_1 + \mu^{1-\epsilon}k_0]$ and $\mu \in (0, \mu_1)$ there exists a unique solution $x(t, \delta x_0, \delta\dot{x}_0, \mu)$ of the restricted three body problem (1) *with initial conditions* (5) *which is an anayltic function of its variables and which is approximated uniformly on this interval to within an error of $O(\mu^{3-\epsilon})$ by the outer expansion* (6) *with $N = 5$ on $[t_0, t_1 - \mu^{1/2}k_0]$ and by the inner expansion* (12) *with $N = 3$ on $[t_1 - 2k_0\mu^{1/2}, t_1 + \mu^{1-\epsilon}k_0]$ provided $\eta_0(t, \mu)$ is a moon centered hyperbola with the parameters given by equations* (15).

NOTE. The appearance of $O(\mu^{3-\epsilon})$ estimates is due to the fact that there are terms of $O(\mu^3|\ln \mu|)$ in the error term. These terms are of $O(\mu^{3-\epsilon})$, for $\epsilon > 0$, but are not of $O(\mu^3)$.

5. **Periodic solutions of the second species of Poincaré.** In this section we will discuss periodic solutions of the planar, circular, restricted three-body problem with the coordinates normalized so that $|x_m(t)| = |\dot{x}_m(t)| = 1$. A periodic solution will refer to a solution of the restricted three body problem (1) which is periodic in rotating coordinates. A periodic solution is called a periodic solution of the second species if it approaches arcs of Keplerian conics joined at corners at the position of the perturbing body as $\mu \to 0$. This type of periodic solution is therefore quite different from periodic solutions of the second kind which approach Keplerian ellipses which do not intersect the position of the perturbing body as $\mu \to 0$. Arenstorf [1] established the existence of one-parameter families of periodic solutions of the second kind using the continuation method of Poincaré to show that a certain periodicity condition is satisfied. The author [29] has established the existence of one-parameter families of periodic solutions of the second species using the boundary layer approximation and error estimates described in the first part of this paper in order to show that this same periodicity condition is satisfied.

In order to describe the families of second species periodic solutions, it is first necessary to describe the limit orbits which are called generating orbits. These are described by the author in [29] and have been studied extensively by Henon [14]. Briefly, a solution $x_0(t)$ of the two body problem (3) is called a generating orbit if for some t_1 and $T_0 > 0$

$$x_0(t_1) = x_m(t_1)$$

and

$$x_0(t_1 + T_0) = x_m(t_1 + T_0).$$

If $x_0(t)$ is an ellipse with $x_0(t_1) = x_{\mathrm{m}}(t_1)$ and with semi-major axis $a_0 = (m/k)^{2/3}$, then $x_0(t)$ is a generating ellipse (of type B) since $x_0(t_1 + T_0) = x_m(t_1 + T_0)$ for $T_0 = ka_0^{2/3}(2\pi) = 2m\pi$. The moon makes m revolutions and the particle k revolutions in the time interval T_0. Figure 1 shows two intersecting generating ellipses of type B with $m = k = 1$. The second species periodic orbit that can be generated from this pair of (type B) intersecting ellipses is shown in both inertial and rotating coordinates.

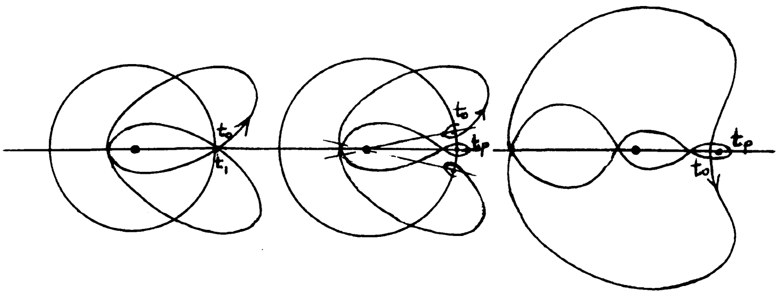

Figure 1

There is another type of generating orbit of periodic solutions of the second species (of type A). This type of generating orbit is illustrated in Figure 2 which indicates that the time for the particle to move along the ellipse from t_0 to t_1 is equal to the time it takes the moon to move along the circle from t_0 to t_1. The timing condition for type A orbits and the existence of families of type A orbits is given in [29, p. 202–203]. These orbits were also studied extensively by Henon in [14]. Figure 2 also shows the type of second species periodic orbit that can be generated from two intersecting (type A) ellipses in both inertial and rotating coordinates.

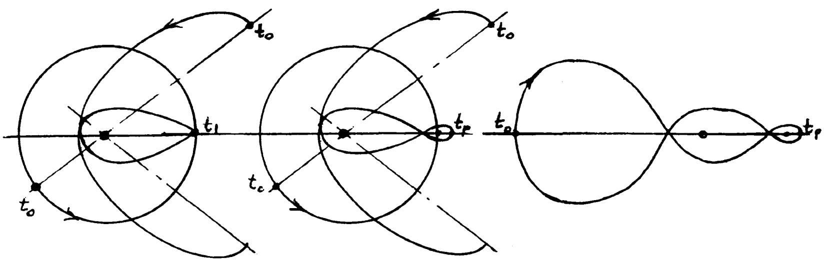

Figure 2

The existence of families of periodic solutions is established by showing that there are solutions of (1) which cross the earth-moon line of centers perpendicularly at two points. *It follows from the symmetry of the three body equations in rotating coordinates that such solutions are periodic in the rotating frame.* This is the periodicity condition that was used by Arenstorf [1] and already by Brikhoff [2].

In order to establish the existence of periodic solutions of the second species of type A, we choose the initial conditions in the form

(5) where $x_0(t)$ is a generating ellipse of type A with $x_0(t_0) \cdot \dot{x}_0(t_0) = 0$ and $x_0(t_0) \times x_m(t_0) = 0$ and with δx_0 and $\delta \dot{x}_0$ parallel to $x_0(t_0)$ and $\dot{x}_0(t_0)$ respectively; i.e., we start at a perpendicular crossing of the earth-moon line of centers at time t_0 as is indicated in Figure 2. We then use the results of the first part of this paper to derive conditions on the variations δx_0 and $\delta \dot{x}_0$, subject to the above constraints, which imply the existence of a second perpendicular crossing of the earth moon line of centers.

We first restrict the apogee and perigee distances to satisfy $a_0(1 - e_0) < 1 < a_0(1 + e_0)$. It then follows that $|V_1| > |\sqrt{a_0(1 - e_0^2)} - 1| \geqq 0$. Cf. [29, p. 206].

We next establish the important result that, just as for the hyperbolic motion $\eta_0(t, \mu)$, the solution of the restricted problem, $x(t, \mu)$, with initial conditions of the form (5), has a unique time, t^*, during each near-moon passage, at which the position and velocity vectors relative to the moon η and $\dot{\eta}$ are perpendicular and at which time the distance to the moon $|\eta|$ is a minimum.

THEOREM 5. *Let $x_0(t)$ be a generating ellipse of type A with $a_0(1 - e_0) < 1 < a_0(1 + e_0)$ and let the variations δx_0 and $\delta \dot{x}_0$ satisfy the non-collision condition of Theorem 4. It then follows that given $\epsilon > 0$, there is a $\mu_1 > 0$ such that for $\mu \in (0, \mu_1)$ there exists a unique value of t, $t^* \in [t_1 - \mu^{1-\epsilon}k_0, t_1 + \mu^{1-\epsilon}k_0]$, such that*

$$\eta(t^*, \mu) \cdot \dot{\eta}(t^*, \mu) = 0 \tag{17}$$

and such that $|\eta(t, \mu)|$ assumes its minimum value at $t = t^*$.

The proof of this and the next two theorems is based on the following simple lemma which follows from the intermediate value theorem and the law of the mean.

LEMMA. *If for $\epsilon > 0$, $\alpha \in [1, 3/2]$, $k_0 > 0$, $k_1 > 0$, there exists a $\mu_0 > 0$ and an a_0 such that $z(a, \mu)$ and $z_0(a, \mu)$ are analytic functions of a and μ for all $a \in [a_0 - k_0\mu^\alpha, a_0 + k_0\mu^\alpha]$ and $\mu \in (0, \mu_0)$ which satisfy*

$$z(a, \mu) = z_0(a, \mu) + O(\mu^{2-\epsilon}) \tag{1}$$

$$z_0(a_0, \mu) = 0 \tag{2}$$

$$\partial z_0(a, \mu)/\partial a \geqq k_1 > 0 \tag{3}$$

for all $a \in [a_0 - k_0\mu^\alpha, a_0 + k_0\mu^\alpha]$ and $\mu \in (0, \mu_0)$, then there exists a $\mu_1 > 0$, ($\mu_1 \leqq \mu_0$), and an $a^* = a_0 + O(\mu^{2-\epsilon})$ such that $z(a^*, \mu) = 0$ for all $\mu \in (0, \mu_1)$.

PROOF. By the theorem of the mean, there exists an $a_1 \in (a_0, a_0 + \mu^\alpha k_0)$ such that $z_0(a_0 + \mu^\alpha k_0) = z_0(a_0) + z_0{}'(a_1)\mu^\alpha k_0 \geqq \mu^\alpha k_0 k_1 > 0$. Similarly $z_0(a_0 - \mu^\alpha k_0) \leqq -\mu^\alpha k_1 k_0 < 0$.

Then from (1) it follows that $z(a_0 - \mu^\alpha k_0) < 0$ and that $z(a_0 + \mu^\alpha k_0) > 0$ provided $\mu \in (0, \mu_1) \subset (0, \mu_0)$ and $\mu_1 > 0$ is sufficiently small. Thus, the intermediate value theorem implies that there exists an $a^* \in (a_0 - \mu^\alpha k_0, a_0 + \mu^\alpha k_0)$ such that $z(a^*) = 0$.

Next, from the theorem of the mean, there exists an $a_1 \in (a_0 - \mu^\alpha k_0, a_0 + \mu^\alpha k_0)$ such that $a^* - a_0 = [z_0(a^*) - z_0(a_0)]/z_0{}'(a_1) = O(\mu^{2-\epsilon})$ since $z_0(a_0) = 0$, $z_0(a^*) = z(a^*) + O(\mu^{2-\epsilon})$ and $z_0{}'(a) \geqq k_1 > 0$ for $a \in (a_0 - \mu^\alpha k_0, a_0 + \mu^\alpha k_0)$.

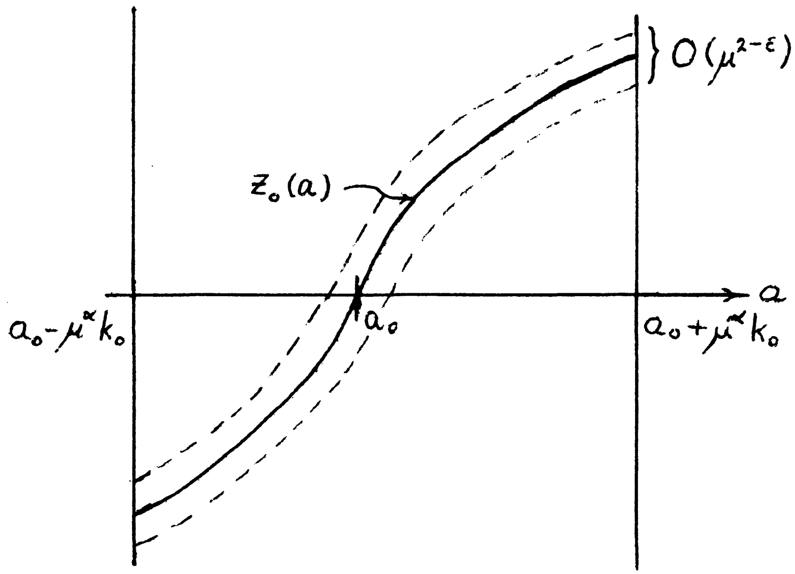

Figure 3

The proof of Theorem 5 is contained in [29, p. 209–212]. Briefly, for $\eta(t, \mu) = x(t, \mu) - x_m(t)$, we note that by Theorem 3 (and the remark following this theorem with $N = 2)\eta(t, \mu) = \eta_0(t, \mu) + O(\mu^{2-\epsilon})$ and $\dot\eta(t, \mu) = \eta_o(t, \mu) + O(\mu^{1-\epsilon})$ for $t \in [t_p - \mu k_0, t_p + \mu k_0]$. Thus, if we let $z(t) = \eta(t) \cdot \dot\eta(t)$ and $z_0(t) = \eta_0(t) \cdot \dot\eta_0(t)$, we have: (1) $z(t) = z_0(t) + O(\mu^{2-\epsilon})$ for $t \in [t_p - \mu k_0, t_p + \mu k_0]$. And since $\eta_0(t)$ is a hyperbolic motion with perilune time t_p, it follows that: (2) $z_0(t_p) = 0$. Also it can be shown, using the energy integral for $\eta_0(t)$, cf. [29, p. 211], that: (3) $\dot z_0(t) \geqq V_1{}^2/2 > 0$. Thus, it follows from the above lemma that for $\mu > 0$ sufficiently small, there exists a $t^* = t_p + O(\mu^{2-\epsilon})$ such that $z(t^*) = 0$.

To establish the uniqueness of t^*, we note that $z(t) = \eta(t) \cdot \dot{\eta}(t) = |\eta|(d|\eta|/dt)$ with $|\eta| > 0$ for $\mu > 0$ sufficiently small and then show that $d^2|\eta|/dt^2 > 0$ using the differential equation of motion for η and the energy integral for η_0; cf. [29, p. 212]. This shows that $d|\eta|/dt$ is a monotone increasing function of t and that $d|\eta|/dt$ and therefore $z(t)$ has a unique zero, t^*, in $[t_p - \mu k_0, t_p + \mu k_0]$.

Since the hyperbolic motion $\eta_0(t, \mu)$ is not uniquely defined by the asymptotic matching, we define $\eta_0(t, \mu)$ uniquely by specifying that $\eta_0(t, \mu)$ is that solution of the two body equations $\ddot{\eta}_0 = \mu f(\eta_0)$ which satisfies

$$\eta_0(t_p) = \eta(t^*) \text{ and } \dot{\eta}_0(t_p) = \dot{\eta}(t^*). \tag{18}$$

Conversely, it then follows that by specifying the parameters of the hyperbolic motion Δ, V_∞ and t_p, we uniquely specify a solution of the restricted three body problem $x(t, \mu) = x_m(t) + \eta(t, \mu)$ through conditions (18).

We now show that $\eta(t^*, \mu)$ is parallel to $x_m(t^*)$; i.e., that there is a second perpendicular crossing of the earth-moon line of centers at t^*, for a particular value of Δ, the distance to the asymptote of the moon centered hyperbola $\eta_0(t, \mu)$.

THEOREM 6. *Under the hypotheses of Theorem* 5, *given* $\epsilon > 0$ *there exists a* $\mu_1 > 0$ *such that for all* $\mu \in (0, \mu_1)$ *there exists a* Δ^* *such that* $\eta(t, \mu)$ *determined by* Δ^*, $V_\infty = V_1 + O(\mu^{1-\epsilon})$ *and* $t_p = t^*$ *through equation* (18) *satisifes*

$$\eta(t^*, \mu) \times x_m(t^*) = 0 \tag{19}$$

where t^* *is defined in Theorem* 5.

The proof of this theorem is contained in [29, p. 214–216]. It follows the same line of reasoning as the proof of Theorem 5. In this case we define $z(\Delta) = \eta(t^*, \Delta) \times x_m(t^*)$ and $z_0(\Delta) = \eta_0(t^*, \mu) \times x_m(t^*)$. It then follows from the error estimates in the first part of this paper that: (1) $z(\Delta) = z_0(\Delta) + O(\mu^{2-\epsilon})$. Also, (2) $z_0(\Delta_1) = 0$ where $\Delta_1 = \mu|\tan\gamma_1|/V_\infty{}^2$ and $\gamma_1 = \alpha[V_\infty, x_{m(tp)}]$. This follows from the elementary formula for hyperbolic motion,

$$\beta_1 = \alpha[V_\infty, V_\infty{}^+] = 2\,\mathrm{Tan}^{-1}\left(\frac{\mu}{\Delta V_\infty{}^2}\right)$$

and the fact that $z_0(\Delta) = 0$ if and only if $\beta_1/2 = \pi/2 + \gamma_1$. Cf. Figure 3. It can also be shown that $z_0{}'(\Delta) \geqq k_1 > 0$ for all $\Delta = \Delta_1 + O(\mu^{3/2})$. The existence of a $\Delta^* = \Delta_1 + O(\mu^{2-\epsilon})$ for all $\mu > 0$ sufficiently small then follows from the above lemma.

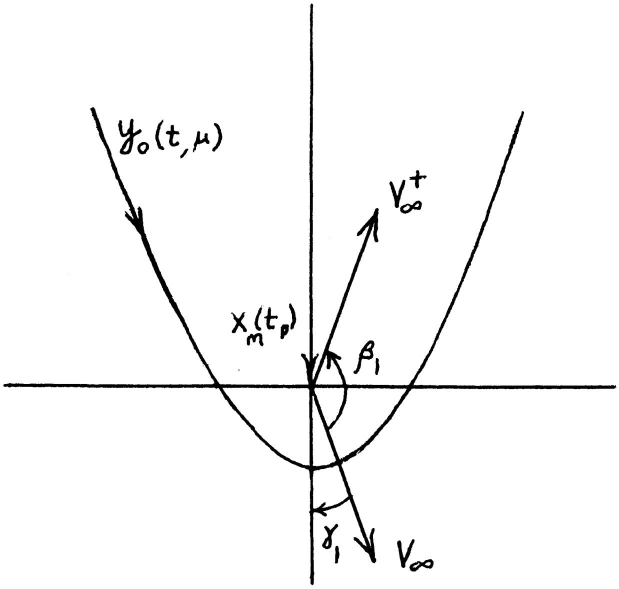

Figure 4

The final step in the existence proof is to show that this value of Δ can be achieved by a particular choice of the variations in initial conditions δx_0 and $\delta\dot{x}_0$. This is accomplished using equation 15(a). That is, we let $z(\delta x_0) = \Delta(\delta x_0) - \Delta^*(\delta x_0)$ and $z_0(\delta x_0) = \Delta_0(\delta x_0) - \Delta_1(\delta x_0)$ where

$$\Delta_0(\delta x_0) = a_{21}\delta x_0 + a_{24}\delta\dot{x}_0 + \int_{t_0}^{t_1} j \cdot b_1(t_1, t)\, dt$$

and a_{pq} are the components of the fundamental solution ΦR where R is a rotation through the angle $\alpha[x_0(t_0), V_1]$ and where Δ_1 is defined above. It then follows that if $a_{21} \neq 0$, then for all $\delta x_0 = O(\mu)$: (1) $z(\delta x_0) = z_0(\delta x_0) + O(\mu^{2-\epsilon})$ since $\Delta = \Delta_0 + O(\mu^{2-\epsilon})$ as in (15a) and since $\Delta^* = \Delta_1 + O(\mu^{2-\epsilon})$ as in Theorem 6. Also, (2) $z_0(\delta x_0) = 0$ if $\delta x_0 = [(\Delta_0 - \Delta_1) - a_{24}\delta\dot{x}_0]/a_{21}$ and (3) $z_0{}'(\delta x_0) = a_{21} \neq 0$. It then follows from the above lemma that if $a_{21} \neq 0$ and $\mu > 0$ is sufficiently small, then there exists a $\delta x_0{}^* = [(\Delta_0 - \Delta_1) - a_{24}\delta\dot{x}_0]/a_{21} + O(\mu^{2-\epsilon})$ such that $\Delta(\delta x_0{}^*) = \Delta^*(\delta x_0{}^*)$.

We then show by direct computation that $a_{21}(a_0, e_0)$ and $a_{24}(a_0, e_0)$ are equal to zero for at most a finite number of points (a_0, e_0) in any compact subset of $\{(a_0, e_0) \mid a_0(1 - e_0) < 1 < a_0(1 + e_0), 0 < e_0 < 1\}$ where a_0 and e_0 are the semi-major axis and eccentricity of the generating ellipse $x_0(t)$ respectively, cf. [29, pp. 219–220].

Finally, since $|\Delta_1| = \mu|\tan\gamma_{10}|/V_1^2 + O(\mu^{2-\epsilon})$ where $\gamma_{10} = \alpha[V_1, x_m(t_1)]$ and since $\tan\gamma_{10} = A_0/\sqrt{V_1^2 - A_0^2}$ with $A_0 = \sqrt{a_0(1-e_0^2)-1}$, we see that the non-collision condition (16) is satisfied if $a_0(1-e_0^2) \neq 1$.

THEOREM 7. *Let* $(a_0, e_0) \in E_0 = \{(a_0, e_0) \mid a_0(1-e_0) < 1 < a_0(1+e_0),\ 0 < e_0 < 1,\ a_0(1-e_0^2) \neq 1\}$ *define a generating ellipse of type A. Then for all but possibly a finite number of* (a_0, e_0) *in any compact subset of* E_0, $a_{21}a_{24} \neq 0$ *and given* $\epsilon > 0$, *there exists a* $\mu_1 > 0$ *such that for all* $\mu \in (0, \mu_1)$ *there exists a one-parameter family of solutions of the restricted three body problem* $x(t, \mu)$, *periodic in rotating coordinates and determined by the initial conditions* (5) *where*

$$a_{21}\delta x_0 + a_{24}\delta\dot{x}_0 = \mu\left[\frac{\tan\gamma_{10}}{V_1^2} - \int_{t_0}^{t_1} j\cdot b_1(t_1, t)\,dt\right] + O(\mu^{2-\epsilon})$$

and the period $T = 2(t_p - t_0)$.

It was also shown in [29] that a denumerable number of the periodic solutions of Theorem 7 are periodic in both rotating and inertial coordinates; cf. Theorem 2, p. 224. Periodic solutions which approach arcs of generating ellipses of type B were also established in [29]; cf. Theorem 3, p. 230.

6. **Second species solutions with near-moon passages.** As was noted in section 5, for $O(\mu)$ variations δx_0 and $\delta\dot{x}_0$ in the initial conditions (5), the particle passes within an $O(\mu)$ neighborhood of the moon, the minimum distance occurring at a unique time $t_p = t_1 + O(\mu^{1-\epsilon})$, $\epsilon > 0$, which is defined to second order by equation (15b). At $t = t_p$ we have $\eta(t_p)\cdot\dot{\eta}(t_p) = 0$. If the point $\eta(t_p)$ lies on the earth-moon line of centers, we have a second perpendicular crossing of the earth-moon line of centers and a periodic orbit in the rotating frame. However, if $\eta(t_p)$ does not lie on the earth-moon line of centers, we have what is referred to as a *near-moon passage*.

It is possible to have periodic solutions of the second species with n near-moon passages between two perpendicular crossing of the earth-moon line of centers. However, since the angle through which the velocity vector turns at a near-moon passage

$$\beta = 2\tan^{-1}\left(\frac{\mu}{\Delta V_\infty^2}\right),$$

it follows that if Δ is determined to Nth order, then β is only determined to $(N-1)$st order; i.e., an order of accuracy is lost with each near-moon passage. And since it is necessary to know β to O(1) in order to establish a second perpendicular crossing of the earth-moon line of centers, it follows that to establish the existence and 1st order asymptotic approximation of a second species peridoic solution with n near-moon passages, it is necessary to carry out $(n+1)$st order asymptotic matching over the first arc, (n)th order matching over the second arc. . . . Second species periodic solutions with one near-moon passage have been established in [30] based on the results of second order matching [4]. Figure 4 shows an example of a second species solution with one near-moon passage in both inertial and rotating coordinates.

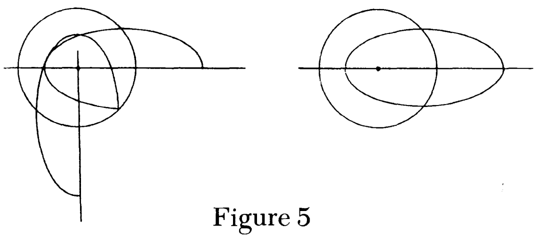

Figure 5

7. **Second species bifurcation.** Guillaume [13] extended the Breakwell-Perko asymptotic matching in the restricted three body problem to include variations in the initial conditions δx_0 and $\delta \dot{x}_0$ of $O(\mu^\alpha)$, $1/3 \leqq \alpha \leqq 1$. The second species periodic solutions with these "large" variations in the initial conditions possess the interesting property that for $\alpha = 1$ the angle through which the velocity vector turns at a near-moon passage

$$\beta = 2 \operatorname{Tan}^{-1}\left(\frac{\mu}{\Delta V_\infty^2}\right) = O(1)$$

as $\mu \to 0$ since $\Delta = O(\mu)$ and $V_\infty = O(1)$, but for $0 < \alpha < 1$, the angle

$$\beta = 2 \operatorname{Tan}^{-1}\left(\frac{\mu}{\Delta V_\infty^2}\right) = O(\mu^{1-\alpha}) \to 0$$

as $\mu \to 0$ since $\Delta = O(\mu^\alpha)$. Thus, we have two entirely different types of limiting orbits as $\mu \to 0$ for $\alpha = 1$ and for $0 < \alpha < 1$; cf. Fig. 5. Guillaume in [12, p. 254] refers to this phenomena as a second species bifurcation.

Another type of bifurcation phenomena referred to by Guillaume [11, p. 114] occurs at the intersection of two characteristics describing generating orbits of type A. These characteristics are shown in Henon's work [14] in the (a_0, e_0) plane on page 389 and in an equivalent system on page 384. This latter figure is reproduced below in Figure 6. At these points of intersection or "bifurcation points," a second order analysis in $(\delta x_0, \delta \dot{x}_0)$ or equivalently in $(\delta a_0, \delta e_0)$ yields a quadratic form (Equation IV-13 on p. 114 in [11])

$$(*) \qquad (W_0 \delta a_0 + W_1 \delta e_0)\delta a_0 = \frac{4\mu a_0^2}{3V_1 T_0}(A_0 - A_1 W_1) + O(\mu^{2-\epsilon})$$

which describes a hyperbola with the tangents to the characteristics of the generating ellipse at the bifurcation point as asymptotes; cf. Figure 7.

Equation (*) above was compared to the numerical work of Deprit [8] which studies the Hecuba gap and the Hilda group in the asteroid belt between Mars and Jupiter. The qualitative agreement is excellent cf. Figure IV-8 in [11, p. 116]. And as Guillaume points out, a better quantitative agreement could be obtained by replacing the linear equations for the tangents to the characteristics of the generating ellipse by a non-linear approximation valid far from the bifurcation point.

Finally, we note that the numerical work of Colombo et al. [7] on families of periodic orbits of the restricted problem for the asteroids at least appears to have a form similiar to what one would expect to obtain from the second species bifurcations; cf. Figures 6 and 8. It would indeed be interesting if the theory of singular perturbations could be used to describe the gaps in the asteroid belt and the stability of the Hilda and Hecuba groups of asteroids.

8. **Historical Notes.** Lagerstrom and Kevorkian [19] performed the first asymptotic matching in the restricted three body problem for those trajectories which are near the earth-moon line of centers; i.e., those trajectories with an initial angular momentum $h_0 = O(\mu^{1/2})$. The author [27] then carried out the asymptotic matching for those trajectories with $h_0 = (1)$. Further work by Lagerstrom, Kevorkian and Lancaster [19, 20, 21, 23] and by Breakwell and Perko [3, 4, 26] generalized this matching to apply to all cases for which $V_1 \neq 0$. In addition, [3] includes the case of earth-to-Venus trajectories in the restricted four body problem. Second order matching has been carried out in [4] and in [23]. As was previously mentioned, Guillaume [13] extended the Breakwell-Perko matching theory to include $O(\mu^\alpha)$, $1/3 \leqq \alpha \leqq 1$, variations in the initial conditions. It is interesting to note that the restricted three body problem with $|x - x_m| = O(\mu^\alpha)$ has the form of a singular perturbation problem only if $\alpha > 1/3$; i.e., $\alpha = 1/3$ is the lower limit for which singular perturbation theory can be, or need be for that matter, applied.

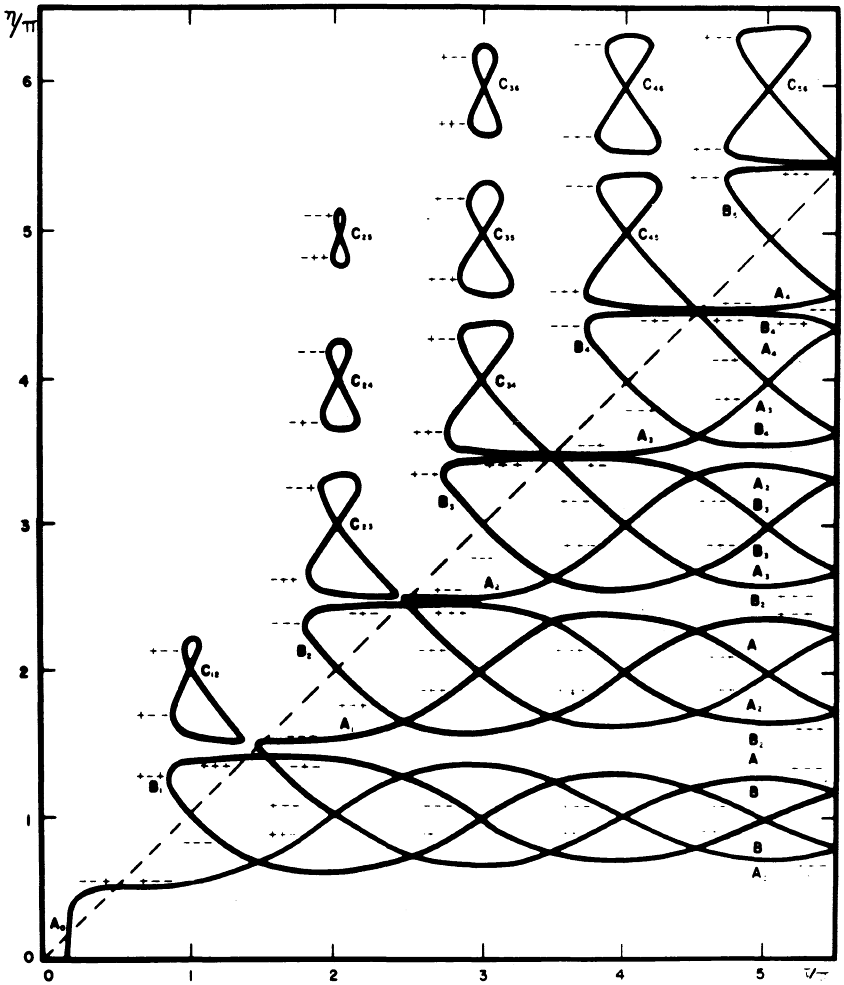

Figure 6

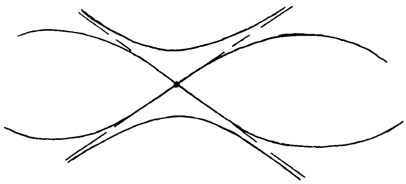

Figure 7

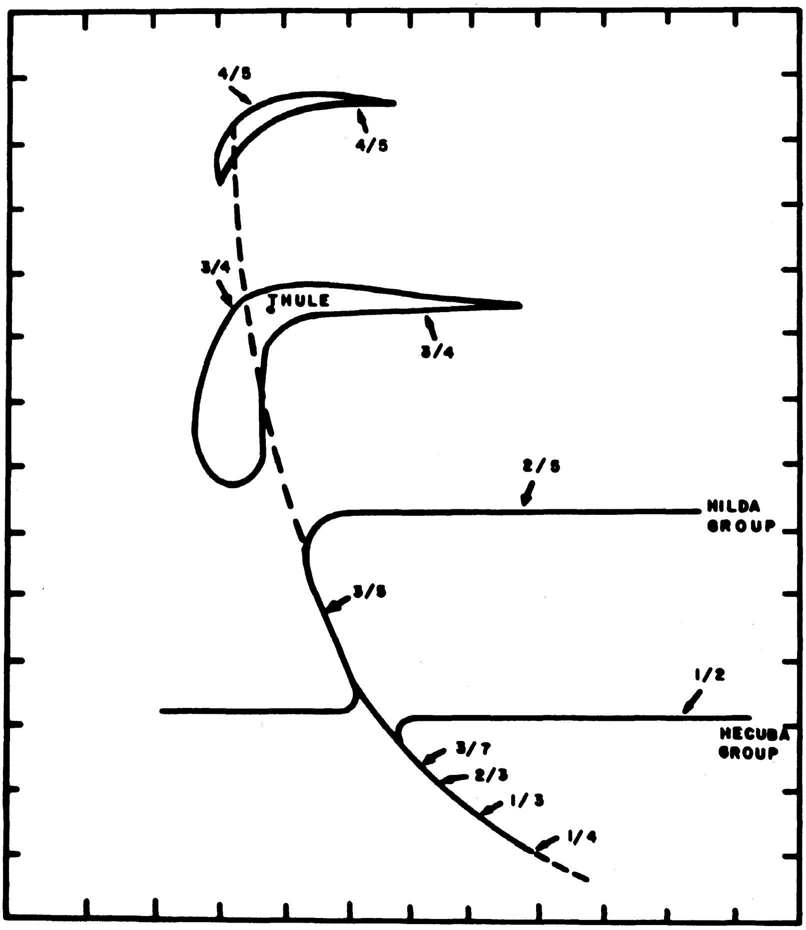

Figure 8

The error estimates for the first order boundary layer approximation to the restricted problem, uniformly valid for $t \in [t_0, t_p]$; i.e., during one near-moon passage, were derived by the author [26] and [28] using differential inequalities and the concept of a majorizing function.

There has been an extensive amount of work on periodic orbits in the restricted three body problem of both a numerical and analytical nature. Poincare [31] described the following classes of periodic solutions: periodic solutions of the first kind which approach Keplerian circular orbits as $\mu \to 0$; periodic solutions of the second kind which approach Keplerian elliptical orbits as $\mu \to 0$; and periodic solutions of the second species which approach arcs of Keplerian conics, joined at corners, as $\mu \to 0$. For small $\mu > 0$ existence of periodic solutions of the first kind was established by Poincare [31] and by Birkhoff [2]; existence of periodic solutions of the second kind was established by Arenstorf [1]; and existence of periodic solutions of the second species was established by the author [29]. The limit as $\mu \to 0$ is uniform for periodic solutions of the first and second kinds and non-uniform for periodic solutions of the second species.

Periodic solutions, using asymptotic methods, have also been studied by Kevorkian and Lancaster; cf. [17] and [22]. Periodic solutions of small period near either the earth or the moon have been established by Hill [16] and Siegel [32]. And periodic solutions of large period far from the earth and moon have been established by Koopman [18]. Solutions for small $\mu > 0$ which close only after many revolutions have been established by Birkhoff [2] and Moser [24].

Numerical studies of periodic orbits in the restricted three body problem have been extensive. Starting with some of the early work of Stromgren [33], many different facets of periodic solutions have been studied for both large and small values of μ in [5, 8, 9, 10, 14, 15, 33]. These numerical studies, combined with the theory of ordinary differential equations and a computational error analysis, also serve to establish the existence of periodic orbits in the restricted three body problem; however, to the author's knowledge, no numerical study of second species solutions with near-moon passages, as described in § 6, has been made due to the difficult nature of this problem.

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