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SINGULAR PERTURBATION PROBLEMS USING PROBABILISTIC METHODS

CHARLES J. HOLLAND

In this paper we review some recent results [6], [7], [8] in the theory of boundary layer expansions for second order semilinear elliptic and parabolic partial differential equations that have been obtained using probabilistic methods. This approach depends upon the representation of the solution of the differential equation as the expected value of a functional of an Ito stochastic differential equation. This method was used by Fleming [2] to derive the regular expansion in the theory of singular perturbations. Our results include the validity of the ordinary and parabolic boundary layer expansions. This work appears to be the first theoretical treatment of these expansions for semilinear equations. Even for linear equations this approach has the advantage that it only requires local estimates to prove the expansions. See the Remarks in § II for a discussion of this point.

Although this probabilistic approach may seem unnatural, this approach is sometimes the natural one. It is sometimes appropriate to model physical phenomena by stochastic differential equations containing a small additive noise term. Then the expected value of certain physically important qualities are given by the values of the solutions of the partial differential equations discussed above. The boundary layer expansions describe the effect of the small noise term in regions near the boundary in which the small noise term has an important effect.

Probabilistic approaches have also been successful in treating certain other singular perturbation problems. See the article of Papanicolaou [11] elsewhere in this issue and also the work of Hersh [5], Ventsel-Freidlin [13], and Friedman [4].

I. **The Elliptic Case.** In this section we review some of the results presented in [6] and [7]. For a summary of results using other methods, see [1]. For $\epsilon > 0$ we consider the elliptic equation

$$\begin{aligned}\epsilon(\tilde{a}(z)u_{z_1z_1} + 2\tilde{b}(z)u_{z_1z_2} + \tilde{c}(z)u_{z_2z_2}) +& \\ \tilde{f}(z)u_{z_1} + \tilde{g}(z)u_{z_2} + \tilde{F}(z,u) = 0&\end{aligned} \tag{1}$$

in B, an open subset of $\mathscr{R}^2$, with boundary data $u = \tilde{\Lambda}$ on ∂B, the boundary of B. The regular and boundary layer expansions will be established on certain subsets of B in which the solutions to (1) are uniformly bounded and the method of characteristics yields a C^∞ solution to (1) with $\epsilon = 0$ taking the boundary data $u = \tilde{\Lambda}$ along a certain portion of the boundary of B.

We first treat the case of regular expansions and ordinary boundary layer expansions. For considering this case, the problem (1) may be reduced to studying the following problem. See [6] for details of this conversion.

Let $x = (x_1, x_2) \in \mathcal{R}^2$, and let $S = (0,1) \times (0,1)$. For $\epsilon > 0$ consider the elliptic equation

$$\begin{aligned} &\epsilon \mathcal{L}\phi^\epsilon - \phi^\epsilon_{x_2} + F(x, \phi^\epsilon) = 0, \\ &\qquad \mathcal{L}\phi^\epsilon = a(x)\phi^\epsilon_{x_1x_1} + 2b(x)\phi^\epsilon_{x_1x_2} \\ &\qquad\qquad + c(x)\phi^\epsilon_{x_2x_2} + f(x)\phi^\epsilon_{x_1} + g(x)\phi^\epsilon_{x_2} \end{aligned} \tag{2}$$

in S with the fixed boundary data

$$\begin{aligned} \phi^\epsilon(x_1, 1) &= \Lambda^\epsilon(x_1, 1) = T(x_1) \\ \phi^\epsilon(x_1, 0) &= \Lambda^\epsilon(x_1, 0) = S(x_1) \end{aligned} \qquad 0 \leqq x_1 \leqq 1 \tag{3}$$

along the upper and lower boundaries of S, and the variable boundary data

$$\begin{aligned} \phi^\epsilon(0, x_2) &= \Lambda^\epsilon(0, x_2), \\ \phi^\epsilon(1, x_2) &= \Lambda^\epsilon(1, x_2), \end{aligned} \qquad 0 \leqq x_2 \leqq 1 \tag{4}$$

along the "sides" of S.

Then we have the following theorem.

THEOREM 1. *Let m satisfy $0 < m < 1/2$, and let there exist positive constants ϵ_0, K^* such that the following hold:*

(A1) *S is as defined above.*

(A2) $a > 0$ *and* $ac - b^2 > 0$ *in* $\bar{S}$.

(A3) a, b, c, f, g *are C^∞ functions on $\bar{S}$; F is a C^∞ function on $\bar{S} \times (-\infty, \infty)$.*

(A4) Λ^ϵ *is a continuous function of x on ∂S for $0 < \epsilon < \epsilon_0$, and $S(x_1)$, $T(x_1)$ are C^∞ functions on $[0, 1]$.*

(A5) *The method of characteristics defines a C^∞ solution ϕ^0 to (2) with $\epsilon = 0$ on $\bar{S}$ taking boundary values $\phi^0(x_1, 0) = S(x_1)$ on the lower boundary.*

(A6) *For $0 < \epsilon < \epsilon_0$, there exists a C^2 solution ϕ^ϵ to (2) on S, continuous on $S \cup \partial S$ with $\phi^\epsilon = \Lambda^\epsilon$ on ∂S such that $|\phi^\epsilon| < K^*$ on S.*

(A7) $\delta(\epsilon) = \epsilon^m$.

Then there exist functions $\theta_1, \theta_2 \cdots$ *bounded on* S *and functions* $\psi^0, \chi_1, \chi_2, \cdots$ *bounded on* $[0,1] \times [0,\infty]$ *and satisfying an exponential decay in their second argument such that for any positive integer* n,

$$\begin{aligned}\phi^\epsilon(x) &= \phi^0(x) + \epsilon\,\theta_1(x) + \cdots + \epsilon^n\,\theta_n(x)\\ &+ \psi^0\left(x_1, \frac{1-x_2}{\epsilon}\right) + \epsilon\chi_1\left(x_1, \frac{1-x_2}{\epsilon}\right)\\ &+ \cdots \epsilon^n\chi_n\left(x_1, \frac{1-x_2}{\epsilon}\right) + o(\epsilon^n)\end{aligned} \tag{5}$$

uniformly on $[\delta(\epsilon), 1-\delta(\epsilon)] \times [0,1]$.

Further

$$\phi^\epsilon(x) = \phi^0(x) + \epsilon\,\theta_1(x) + \cdots \epsilon^n\,\theta_n(x) + o(\epsilon^n) \tag{6}$$

uniformly on $[\delta(\epsilon), 1-\delta(\epsilon)] \times [0, 1-\delta(\epsilon)]$.

Expansion (6) is the regular expansion in the theory of singular perturbations. The coefficients ϕ^0, θ_k in (6) satisfy the equations to make the coefficient of ϵ^k identically zero in the formal expansion of (2) in powers of ϵ. By direct calculation ϕ^0 satisfies (2) with $\epsilon = 0$, and boundary data $\phi^0 = S$ on the lower boundary while θ_k, $k = 1, 2, \cdots$ satisfies

$$-(\theta_k)_{x_2} + F_\phi(x, \phi^0(x))\,\theta_k + \Gamma_k + \mathcal{L}(\theta_{k-1}) = 0 \tag{7}$$

with boundary data $\theta_k = 0$ on the lower boundary of S. $\Gamma_1 = 0$ and in general Γ_k is a polynomial in $\theta_1, \cdots \theta_{k-1}$ of degree k, with coefficients $F_{\phi\phi}$, $F_{\phi\phi\phi}$, $\cdots$ evaluated at $(x, \phi^0(x))$. If $F(x,\phi)$ is linear in ϕ, then $\Gamma_k = 0$ for any k.

Along the upper boundary, in general $\phi^\epsilon \neq \phi^0$, and hence the expansion (6) cannot be correct on the upper boundary. "Near" the upper boundary one expects a boundary layer region to occur where the values of ϕ^ϵ change rapidly from the given boundary values to values near those given by the regular expansion. This region whose size depends upon ϵ is called an ordinary boundary layer since the terms representing the difference of the actual value of ϕ^ϵ from the regular expansion value satisfy ordinary differential equations. Recall the boundary layer expansion (5). Equations for the functions ψ^0, χ_1, $\chi_2, \cdots \chi_n$ are found by formally substituting the expansion (5) into (2). In Appendix A of [6] we indicate how this is done. For example $\psi^0 = \psi^0(x)$ satisfies the equation

$$c(x_1, 1)\psi^0_{x_2x_2} + \psi^0_{x_2} = 0 \tag{8}$$

on $[0, 1] \times [0, \infty]$ with the boundary conditions $\psi^0(x_1, 0) = T(x_1) - \phi^0(x_1, 1)$, $0 \leqq x_1 \leqq 1$, and $\psi^0(x_1, \infty) = 0$. Note that for fixed x_1, equation (8) is an ordinary differential equation in the variable x_2 with solution

$$\psi^0(x_1, x_2) = (T(x_1) - \phi^0(x_1, 1)) \exp(-x_2/c(x_1, 1)).$$

We now treat the case of parabolic boundary layers. These boundary layers occur when a characteristic of the reduced equation to (1) with $\epsilon = 0$ is tangent to a portion of the boundary of B. For studying this case, the problem may be reduced to the previous case (2) – (4) except that we assume in addition that the left boundary data is also fixed. Define $\Lambda^\epsilon(0, x_2) = U(x_2)$ for $0 \leqq x_2 \leqq 1$.

We seek an expansion of ϕ^ϵ in the region $[0, \delta(\epsilon)] \times [0, 1]$ which was not treated by Theorem 1. Let W^0 be a solution of the parabolic equation

$$\begin{aligned} &a(0, x_2) W^0_{x_1 x_1} - W^0_{x_2} \\ (9) \qquad &+ \left[\int_0^1 F_u(0, x_2, \phi^0(0, x_2) + \lambda W^0(x))\, d\lambda \right] W^0(x) = 0 \end{aligned}$$

with boundary conditions

$$W^0(x_1, 0) = 0, 0 \leqq x_1 \leqq \infty,$$

and

$$W^0(0, x_2) = R(x_2), 0 \leqq x_2 \leqq 1,$$

where we have defined $R(x_2) = U(x_2) - \phi^0(0, x_2)$.

Then we have the following theorem.

THEOREM 2. *Let the following hold:* (A1)–(A7) *and* (A8) *There exists a C^2 solution W^0 to* (9) *in* $(0, \infty) \times (0, 1)$, *continuous in* $[0, \infty] \times [0, 1]$, *with uniformly bounded derivatives* $W^0_{x_1}$, $W^0_{x_1 x_1}$, $W^0_{x_1 x_2}$, $W^0_{x_2 x_2}$, $W^0_{x_2}$ *in* $(0, \infty) \times (0, 1)$. *Then, for any α with $0 < \alpha < 1/2$ and $\alpha < m < 1/2$,*

$$\begin{aligned} &\phi^\epsilon(x) = \phi^0(x) + \psi^0(x_1, (1 - x_2)/\epsilon) \\ (10) \qquad &+ [1 - \exp[-(1 - x_2)/(\epsilon c(x_1, 1)]]\, W^0(x_1 \epsilon^{-1/2}, x_2) + o(\epsilon^n) \end{aligned}$$

uniformly on $[0, \epsilon^m] \times [0, 1]$.

If $F(x, u)$ is not linear in u, then (9) is a non-linear parabolic equation for W^0. Unless $R'(0) = 0$, then the derivatives $W_{x_1 x_2}$, $W^0_{x_2 x_2}$ are not uniformly bounded and assumption (A8) is not satisfied. When the compatibility condition $R'(0) = 0$ is not satisfied, then we have the following result for the linear case, if U is Lipschitz.

THEOREM 3. *Let the following hold*: (A1)–(A7) *and* (A9) $F(x, u) = m(x)u + n(x)$.

Then, for any α, m *with* $0 < \alpha < 1/3$ *and* $\alpha < m < 1/2$, (10) *holds uniformly on* $[0, \epsilon^m] \times [0, 1]$.

II. **Remarks.** Consider again equation (1) in a domain B. Let D be a subdomain of B in which the regular and boundary layer expansions are being derived. Outside $\bar{D}$ it is not necessary that the boundary data be of Dirichlet type. We only need to know that there exists smooth solutions such that an a priori bound of the type (A6) is satisfied in $\bar{D}$.

We have not yet treated the case of free boundary layers or turning point problems in partial differential equations. It would appear interesting to treat these using probabilistic methods.

It would also be of interest to treat boundary layer expansions in problems where, in the subdomain of interest, either Neumann or mixed boundary data is prescribed. Oleinik [9], [**10**], by non-probabilistic methods, and Freidlin [3], by probabilistic methods, have derived some results for singular perturbation problems with these types of boundary data. Probabilistic representations of solutions to these equations exist: see Freidlin [3] and the references there and Stroock-Varadhan [**12**].

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PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907