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HYPERBOLIC STRUCTURES IN HAMILTONIAN SYSTEMS

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0. **Introduction.** This note considers Hamiltonian systems of two degrees of freedom defined by C^3-Hamiltonians $H : R^4 \to R$. Restricting to regular surfaces of constant energy, computable conditions are presented which guarantee that certain invariant subsets of the resulting flow possess a hyperbolic structure. Our approach is to exploit the parallelizability of the energy surfaces by using a characterization of hyperbolic sets due originally to J. Selgrade [4, 1].

1. **Preliminary Definitions.**

(a) The Frame Field $\{X_i\}_{i=1}^3$.

Define the 2×2 matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and let $A_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Notice that the A_i multiply amongst themselves as do the standard basis quaternions under the identifications $A_0 \sim l$, $A_1 \sim j$, $A_2 \sim k$, $A_3 \sim i$.

Let $H : R^4 \to R$ be a Hamiltonian with gradient H_x, and write Hamilton's equations as

$$\dot{x} = A_3 H_x. \tag{1.1}$$

Fix a regular energy surface M of H, and for each $x \in M$ let $X_0(x) = H_x(x)/|H_x(x)|$, $X_i(x) = A_i X_0(x)$, $i = 1, 2, 3$. Since the A_1, A_2, A_3, are skew-symmetric and orthogonal, the fields $\{X_i\}_{i=1}^3$ form an orthonormal frame field on M.

(b) Chain-recurrent Invariant Sets.

Let (M, d) be a compact metric space, and let $\rho^t : M \to M$ be a flow on M. Given $p, q \in M$ and $\epsilon > 0$, $T > 0$, declare p and q to be (ϵ, T)-*connected* provided there are finite sequences $\{p_0 = p, p_1, \cdots, p_n = q\} \subset M$, $\{t_0, \cdots, t_{n-1}\} \subset [T, \infty)$, such that $d(\rho^{t_j}(p_j), p_{j+1}) < \epsilon$, $j = 0, \cdots, n-1$. Write $p \sim q$ if p is (ϵ, T)-connected to q for all $\epsilon > 0$, $T > 0$, and let $R(\rho^t) = \{p \in X \mid p \sim p\}$ denote the *chain-recurrent set* of ρ^t. A compact invariant set $Y \subset M$ is *chain-recurrent* if $R(\rho^t \mid Y) = Y$. It is not difficult to see that $R(\rho^t)$ contains the non-wandering set C. Conley [2] has shown that $\rho^t \mid R(\rho^t)$ is chain-recurrent, which fact indicates a distinct advantage of the chain-recurrent set over the non-wandering set.

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2. **The Hyperbolicity Theorem.** Let $\langle\ ,\ \rangle$ denote the usual inner product on R^4, and let H_{xx} denote the Hessian matrix of second partials of H. Define 2×2 matrices S, T, U, V, by

$$S = \begin{pmatrix} \langle H_{xx}X_1, X_1\rangle & \langle H_{xx}X_1, X_2\rangle \\ \langle H_{xx}X_1, X_2\rangle & \langle H_{xx}X_2, X_2\rangle \end{pmatrix} + \langle H_{xx}X_3, X_3\rangle I, \tag{2.1}$$

$$T = (T_{ij}) = (\langle \mathrm{grad}(S_{ij}), A_3 H_x\rangle), \tag{2.2}$$

$$U = -(\det S)I + S^2 - JT, \tag{2.3}$$

$$V = (1/2)(U + U^*), \tag{2.4}$$

where U^* denotes the transpose of U, and where the entries of the various matrices are to be evaluated at points of the energy manifold $M = \{x \mid H(x) = h, h \text{ a regular value of } H\}$.

THEOREM 2.5. *Let $\rho^t : M \to M$ be the flow generated by* (1.1), *and let $Y \subset M$ be open. Assume that solutions remaining in Y are defined for all t.*

(a) *If V is negative definite on Y, then there are no invariant sets of ρ^t in Y; and*

(b) *if V is positive definite on Y, and if $K \subset Y$ is a compact chain-recurrent invariant set, then K is hyperbolic.*

The proof will occupy the next two sections.

3. **The Linearized Equations.** The linearized equations for (1.1) are given by

$$\dot{y} = A_3 H_{xx}(x(t))y, \tag{3.1}$$

where $x(t)$ is an arbitrary solution of (1.1). Since M is invariant for the flow of (1.1), $T(M)$ must be invariant for the flow of (3.1), and in this way (3.1) induces a flow $T\rho^t : T(M) \to T(M)$.

If $y(t)$ is a motion of $T\rho^t$, then in terms of our frame field we have

$$y(t) = \sum_{i=1}^{3} \alpha^i(t) X_i(x(t)), \tag{3.2}$$

where again $x(t)$ is a solution of (1.1). Substituting (3.2) into (3.1) and using the multiplicative properties of the matrices A_i, it is not difficult to verify that the equations governing $\alpha = \binom{\alpha^1}{\alpha^2}$ and α^3 are

$$\dot{\alpha} = -JS\alpha, \tag{3.3a}$$

$$\dot{\alpha}^3 = \sum_{j=1}^{3} \alpha^j \langle H_{xx}X_j, X_0\rangle - \alpha^1 \langle H_{xx}X_3, X_2\rangle + \alpha^2 \langle H_{xx}X_3, X_1\rangle, \tag{3.3b}$$

where S is the matrix of (2.1), and where all functions are evaluated on $x(t)$.

4. **Proof of Theorem 2.5.** By (3.3a) we have

$$\left(\frac{1}{2}\right)\frac{d}{dt}\langle\alpha,\alpha\rangle = -\langle JS\alpha,\alpha\rangle, \qquad \alpha = \begin{pmatrix}\alpha^1\\ \alpha^2\end{pmatrix}, \tag{4.1}$$

for any solution $y(t) = \sum_{i=1}^{3} \alpha^i(t)X_i(x(t))$ of (3.1). Letting $N = (1/2)(JS + (JS)^*)$ denote the symmetrization of JS, we have

$$\langle JS\alpha,\alpha\rangle = \langle N\alpha,\alpha\rangle \tag{4.2}$$

for all α.

LEMMA 4.3. *The matrix N is indefinite.*

PROOF. S is of the form $\begin{pmatrix}a & b\\ b & c\end{pmatrix}$, hence N is of the form $\begin{pmatrix}b & (1/2)(c-a)\\ (1/2)(c-a) & -b\end{pmatrix}$ with $\det(N) = -b^2 - (1/4)(c-a)^2 \leqq 0$. The claim follows.

Together with (4.1) and (4.2), the lemma implies the following result.

COROLLARY 4.4. *Let $x(t)$ be any solution of* (1.1), *let t_0 be an arbitrary element of the domain of $x(t)$, and let $x_0 = x(t_0)$. Then one can find initial conditions $\alpha^1(t_0)$, $\alpha^2(t_0)$, $\alpha^3(t_0)$, for* (3.3a,b) *at x_0 such that $d/dt\,\langle\alpha,\alpha\rangle \leqq 0$ at $t = t_0$.*

Observe that $T = \dot{S}$, hence by (4.1) we have

$$\left(\frac{1}{2}\right)\frac{d^2}{dt^2}\langle\alpha,\alpha\rangle = \langle V\alpha,\alpha\rangle. \tag{4.5}$$

Now assume V is negative definite on Y, and that there is a solution $x(t)$ of (1.1) which remains in Y for all time. If we choose $\alpha^1(0)$, $\alpha^2(0)$, $\alpha^3(0)$ as in Corollary 4.4, then (4.5) forces $|\alpha|^2 < 0$ for large t, an obvious impossibility.

On the other hand, assume V is positive definite on Y, and let $y(t) = \sum \alpha^i X_i(x(t))$ be any solution of (3.1) with $\alpha = \begin{pmatrix}\alpha^1\\ \alpha^2\end{pmatrix} \neq 0$. Then by (4.5), $|\alpha|$ will be unbounded if $x(t)$ remains in Y for all t. However, under the assumption of chain-recurrence, the unboundedness of $|\alpha|$ is equivalent to hyperbolicity; this is a recent result of J. Selgrade, J. Franke, and R. Churchill [1]. This completes the proof of Theorem 2.5.

5. **Geometric Significance of the Matrix S.** We follow the notation of [3, Chapter 2]. In particular, D will denote the standard connection on R^4, and $L : T(M) \to T(M)$ the Weingarten map.

Since $X_i(|H_x|^{-1}) = -X_i(|H_x|)|H_x|^{-2}$, and $D_{X_i}(H_x) = H_{xx}X_i$, we have, for $i = 1, 2, 3$,

$$L(X_i) = |H_x|^{-1}(-X_i(|H_x|)X_0 + H_{xx}X_i).$$

For $1 \leqq i, j \leqq 3$ this implies $\langle H_{xx}X_i, X_j\rangle = |H_x|\langle LX_i, X_j\rangle$, where $\langle LX_i, X_j\rangle$ is the second fundamental form of M evaluated on X_i, X_j. Note that $\langle LX_3, X_3\rangle$ is also the normal curvature of $x(t)$. Using the self-adjointness of L, we can then write (3.3a) as

$$(5.1)\quad \dot{\alpha} = -|H_x|J\left[\begin{pmatrix} \langle LX_1, X_1\rangle & \langle LX_1, X_2\rangle \\ \langle LX_1, X_2\rangle & \langle LX_2, X_2\rangle \end{pmatrix} + \langle LX_3, X_3\rangle I\right]\alpha.$$

From (5.1) we compute

$$\det S = |H_x|^2\left\{\det\begin{pmatrix} \langle LX_1, X_1\rangle & \langle LX_1, X_2\rangle \\ \langle LX_1, X_2\rangle & \langle LX_2, X_2\rangle \end{pmatrix} + \langle LX_3, X_3\rangle(\text{trace}\,(L))\right\}$$

$$= |H_x|^2\{\langle R(X_1, X_2)X_2, X_1\rangle + \langle LX_3, X_3\rangle\,(\text{trace}\,(L))\},$$

where $\langle R(X_1, X_2)X_2, X_1\rangle$ is the sectional curvature of X_1, X_2, in M (see also [3, p. 78]), and trace (L) is the mean curvature of M along $x(t)$.

6. **Remarks on Applications.** Except in a few simple cases, calculation of the matrix V of Theorem 2.5, as well as its leading principal minors, is best done using a computer. For a Hamiltonian of the form

$$(6.1)\qquad H(x, y) = (1/2)|y|^2 + W(x), x = (x_1, x_2), y = (y_1, y_2),$$

one picks a grid $\{p_i\}$ of points in the x-plane with $W(p_i) \leqq h$, and then calculates V as y varies around the circle $|y|^2 = 2(h - W(p_i))$. One obtains "wedges" of velocity directions with $V < 0$ corresponding to regions in the energy manifold M having no invariant sets. Similarly, for those "wedges" of y for which $V > 0$ one has regions in which all compact chain-recurrent invariant sets are hyperbolic. Such a study was made for the "monkey saddle" potential $W(x) = (1/3)x_1^3 - x_1x_2^2$, and near the level curves $W = h$ the matrix V was found to be positive definite in all directions, generally indefinite elsewhere, and was never found to be negative definite. Moreover, V was not positive definite on the entirety of a known hyperbolic periodic orbit (although it was positive definite along most of the orbit), indicating that in some instances Theorem 2.5b must be combined with other techniques which relax the requirement that $(1/2)\, d^2/dt^2\, \langle\alpha, \alpha\rangle > 0$ along the entire orbit length.

A simple example, where only a slight refinement of Theorem 2.5b is needed, occurs in the two-saddle potential $W(x) = (1/2)(x_2^2 - x_1^2)$, which for energies $h > 0$ admits a periodic orbit with x-plane projection running along the x_2-axis and connecting the two branches of $W = h$. Here $V > 0$ along the orbit in phase space except at points $(0, y)$, where $V = 0$.

An easy direct application of Theorem 2.5b can be made for $H(x, y) = (1/2)|y|^2 + \cos(x_1) + \cos(x_2)$. It is easy to see that for energies $0 < h < 2$ the associated equations admit periodic solutions with x-plane projections running along the x_1 and x_2-axes (and others parallel to these axes) connecting distinct branches of $W = h$. Along these the matrix V is identically $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, proving hyperbolicity. Of course here, as in the two-saddle example, hyperbolicity is easy to verify by other means.

It is well-known that for a Hamiltonian of the form (6.1) the x-plane projection of the solutions of energy h may be regarded as reparameterizations of the geodesics in the Jacobi metric on the projected energy surface, excluding the boundary $W = h$. In cases where the resulting Gaussian curvature in the Jacobi metric is negative, one can again use [1] to prove hyperbolicity of compact chain-recurrent invariant sets which avoid the boundary. It should be mentioned, however, that the matrix V of Theorem 2.5 can be positive definite in regions where this Gaussian curvature is *positive,* as for example in the above mentioned "monkey saddle" potential.

For Hamiltonians of the form (6.1), one can relate the entries of S to the geometry of the surface $W = h$ in R^3, taking into account the direction in which one is moving. This leads to the hope that the regions of hyperbolicity for such systems might be apparent simply by examining these surfaces. But consider the Hamiltonian $H(x, y) = (1/2)|y|^2 + (1/2)|x|^2 - x_1 x_2^2$ whose equations admit a periodic solution with x-plane projection running along the x_1-axis for all energies $h > 0$. For $h > 1/8$ the corresponding surfaces are geometrically very similar. Nevertheless, by integrating the linearized Poincaré map and using standard results on Mathieu functions, we can show that this orbit oscillates between being hyperbolic and being elliptic infinitely often as $h \uparrow \infty$.

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