ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Numbers 1 and 2,
Winter and Spring 1978

2- AND 3-DIMENSIONAL LOCALIZATION OF RANDOM PHASE WAVES

KUNIOKI MIMA* AND AKIRA HASEGAWA

ABSTRACT. The one-dimensional localization of waves in a nonlinear dispersive medium has been investigated extensively. The stationary solution is well known as the envelope soliton. For the 2- and 3-dimensional localization, however, the stationary solutions have not been found except for some limited cases.

In this paper, 2- and 3-dimensional localization of random phase waves is analyzed by using the wave kinetic equation which is derived from a model nonlinear wave equation (a nonlinear Schrödinger equation). For an arbitrary shape of the envelope profile, the stationary frequency spectrum of trapped waves is obtained by a method analogous to that used by Bernstein, Greene, and Kruskal for determining the trapped particle distribution. As an example the stationary envelope profile is given for Langmuir waves with an appropriate stationary spectrum.

1. **Introduction.** Amplitude modulation instabilities have been investigated for various strongly dispersive waves [1], for example, Langmuir waves [2], electromagnetic waves [3], deep water waves [4], upper hybrid waves [5], and cyclotron waves [6]. Recently the nonlinear stage of these instabilities has been widely studied [7]. It is now recognized that the nonlinear Schrödinger equation describes these phenomena well.

For one-dimensional amplitude modulation, the exact time evolution of nonlinear waves described by the nonlinear Schrödinger equation has been analyzed for coherent waves [8]. One-dimensional modulation of random phase waves (noncoherent waves) has also been investigated [9]. For 2- or 3-dimensional amplitude modulation, the final stage of the instabilities, or the stationary state, is not well understood for both coherent and random phase waves. For instance, as a result of the 2- or 3-dimensional modulation of coherent waves, it is expected that the localized wave shrinks indefinitely; the final state is not known. Saturated nonlinearity was introduced to construct a 3-dimensional envelope soliton [10].

In this proceeding, we investigate the 2- and 3-dimensional localization of random phase waves. When the widths of the frequency

*Permanent address: Institute of Laser Engineering, Faculty of Engineering, Osaka University, Osaka, Japan.

spectrum and the wave-number spectrum are broad enough compared with the characteristic wave number and/or frequency of the amplitude modulation, the wave kinetic equation works well for studying the localization. The system described by the wave kinetic equation is considered to be ensemble of wave packets. Each wave packet behaves like a single particle and the spectral function corresponds to the phase space distribution function. Using the method found by Bernstein, Greene, and Kruskal [11] and developed by others [12], the exact trapped wave packet distribution function is constructed. The localized wave corresponds to the trapped wave packet.

When the nonlinear term of the wave kinetic equation is given in terms of wave intensity, the spectral function is determined. If the nonlinearity does not depend on the envelope profile, the arbitrary envelope profile can be the solution of the wave kinetic equation. Therefore, the instability of the solution and/or the initial condition might determine the realizable envelope profile. When nonlinearity depends on the envelope profile, the envelope profile can be determined for a given stationary wave spectrum.

In § 2, we derive the wave kinetic equation from the nonlinear wave equation. § 3 is devoted to calculating the stationary spectral function for the given nonlinearity. As an example, in § 4, we determine the profile of random-phase Langmuir waves or electromagnetic waves in a nonmagnetized plasma.

2. **Derivation of the Wave Equation.** We will only consider strongly dispersive waves that have amplitude modulation instabilities. When the time scale of the amplitude modulation is much larger than one oscillation period of the waves, the slowly varying complex amplitude can usually be described by the nonlinear Schrödinger equation:

$$(1)\qquad i\frac{\partial}{\partial t}A + \left(\beta_\perp\Delta_\perp + \beta_\parallel\frac{\partial}{\partial z^2}\right)A + N(|A|^2)A = 0.$$

Here we consider a scalar field for simplicity. The coordinate z is arbitrary here but will be defined later. The complex amplitude A is related to a real quantity by

$$E(\mathbf{r}, t) = A(\mathbf{r}, t)e^{i\omega_0 t} + c.c..$$

$N(|A|^2)$ in (1) is an appropriate function of $|A|^2$ which depends on the wave and the medium. Since we are interested in modulationally unstable waves, we only consider the case $\alpha\beta_\parallel > 0$ and $\alpha\beta_\parallel > 0$ [1], where $\alpha = dN/d|A|^2|_{|A|^2=0}$. Note that (1) is valid only when the higher order derivative of A is negligible. The linear dispersion is then given by,

$$\lambda_k = \beta_\perp k_\perp^2 + \beta_{||} k_{||}^2, \tag{2}$$

where $\mathbf{k}$ is a wave number and A is expressed as

$$A(\mathbf{r}, t) = \sum_{\mathbf{k}} \psi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r} - i\lambda_{\mathbf{k}} t}, \tag{3}$$

Now, we consider the deviation from the dispersion relation due to nonlinear effects. When the wave number width of the k-spectrum, $|\Delta\mathbf{k}|$, is much larger than the inverse of the scale length of the amplitude modulation, $|d/d\mathbf{r}\ \ln|\psi_k(\mathbf{r}\ )|^2|$ the wave intensity is approximately given by

$$|A(\mathbf{r}, t)|^2 = |\sum_{\mathbf{k}} \psi\partial e^{i\mathbf{k}\cdot\mathbf{r} - i\lambda_{\mathbf{k}} t}|^2 \simeq \sum_{\mathbf{k}} |\psi_k(\mathbf{r}, t)|^2. \tag{4}$$

Notice that the cross product terms in the right hand side of (4) can be ignored since the cross terms vanish after the summation over k due to phase mixing which comes from the phase factor $e^{i\mathbf{k}\cdot r}$. In such a case, the nonlinear dispersion relation is:

$$\lambda_k = \beta_\perp k_\perp^2 + \beta_{||} k_z^2 - N(I(\mathbf{r}, t)), \tag{5}$$

$$I(\mathbf{r}, t) = \sum_{\mathbf{k}} |\psi_k(\mathbf{r}, t)|^2, \tag{6}$$

within the WKB approximation. Using this dispersion relation, the equation of motion of a wave packet is given by:

$$\begin{aligned} \dot{\mathbf{r}}(t) &= \frac{\partial\lambda_k}{\partial\mathbf{k}} = 2\beta_\perp \mathbf{k}_\perp + \beta_{||} k_z \hat{z} \\ \dot{\mathbf{k}}_\perp &= -\frac{\partial\lambda_k}{\partial\mathbf{r}_\perp} = \frac{\partial N(I(\mathbf{r}, t))}{\partial\mathbf{r}_\perp}, \quad \dot{k}_z = \frac{\partial N}{\partial z}. \end{aligned} \tag{7}$$

We can then derive the wave kinetic equation

$$\left[\frac{\partial}{\partial t} + \frac{\partial\lambda_k}{\partial\mathbf{k}} \cdot \frac{\partial}{\partial\mathbf{r}} - \frac{\partial\lambda_k}{\partial\mathbf{r}} \frac{\partial}{\partial\mathbf{k}} \right] f(\mathbf{k}, \mathbf{r}, t) = 0, \tag{8}$$

where $f(\mathbf{k}, \mathbf{r}, t) \equiv |\psi_k(\mathbf{r}, t)|^2$ is the distribution function of random phase waves. It is shown in Appendix A that the wave kinetic equation (8) is valid for more general dispersion relations than (5) is.

Also in Appendix A, (1) is derived directly under the condition:

(9) $$|\Delta k| \gg \left| \frac{d}{d\mathbf{r}} \ln |\psi_{\mathbf{k}}(r)|^2 \right| ,$$

In the following discussion, we use (8) to determine the spectral function instead of the full wave equation (1).

It is convenient in cylindrically symmetric systems to use cylindrical coordinates and the conjugate wave number. In such a case, the wave amplitude should be expressed by superposition of cylindrical waves:

(10) $$A = \sum_{\mathbf{k}} \psi_{\mathbf{k}}'(\mathbf{r},\, t) H_m^{(1)}(k_r r) e^{i(m\theta + k_z z) - i\lambda_k t},$$

where $H_m^{(1)}(k_r r)$ is the Hankel function of the first kind. The dispersion relation (5) is rewritten as

(11) $$\lambda_k = \beta_\perp \left(k_r^2 + \frac{m^2}{r^2} \right) + \beta_{||} k_z^2 - N(I),$$

and the kinetic equation then becomes

(12) $$\left[\frac{\partial}{\partial t} + 2\beta_\perp \left(k_r \frac{\partial}{\partial r} + \frac{m}{r^2} \frac{\partial}{\partial \theta} \right) + 2\beta_{||} k_z \frac{\partial}{\partial z} - \frac{\partial}{\partial z} \lambda_k \frac{\partial}{\partial k_z} - \frac{\partial \lambda_k}{\partial r} \frac{\partial}{\partial k_r} \right] f(\mathbf{k},\, r,\, t) = 0.$$

Here we assume that the distribution function is cylindrically symmetric,

(13) $$f(\mathbf{k},\, \mathbf{r},\, t) = |\psi_k'(\mathbf{r},\, t)\, H_m^{(1)}(k_r r)|^2 \quad \text{and} \quad I = \sum_{\mathbf{k}} f(\mathbf{k}, \mathbf{r},\, t).$$

Equations (11), (12), and (13) are our basic equations for the following discussions.

3. **Stationary Solutions with Cylindrical Symmetry.** We consider the stationary solution of (12) on the moving frame with a velocity V along the z axis. The stationary equation is given by

(14) $$\left[2\beta_\perp k_r \frac{\partial}{\partial r} + (2\beta_{||} k_z - V) \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi} N(I) \frac{\partial}{\partial k_z} + \frac{\partial}{\partial r} N(I) \frac{\partial}{\partial k_r} \right] f(\mathbf{k},\, \mathbf{r},\, \xi) = 0,$$

where $\xi = z - Vt$. The general solution of (14) is an arbitrary function of the constants of motion. Since the effective potential, $-N(I)$, is independent of θ, frequency (or energy) λ_k and azimuthal wavenumber (or angular momentum) m are the constants of motion. Furthermore, k_z becomes constant when $I(\mathbf{k}, \mathbf{r})$ does not depend on z. Therefore, the general solution becomes

$$f = f(\lambda, m, k_z). \tag{15}$$

Taking into account (13), we obtain the integral equation,

$$I = \int d^3\mathbf{k} f[\lambda(\mathbf{k}, \mathbf{r}, I), m]. \tag{16}$$

The same type of equation as (16) has been solved for one dimension by Bernstein, Green, and Kruskal, and for 2 and 3 dimensions by Kato. When f does not depend on m, we can solve (16) by the same method [**12**]. From now on, we assume that f depends only on λ and all the waves are trapped. The partially trapped solutions are not interesting since untrapped waves cause modulational instability and the solution may then become unstable [9]. A fully trapped solution is considered to be a localized envelope wave, since the boundary conditions, $\lim_{r\to\infty} f(\lambda) = 0$ and $\lim_{z\to\infty} f(\lambda) = 0$ are satisfied. The integral equation (13) is rewritten as,

$$I = \frac{1}{2\beta_\perp} \int_{\lambda^-}^{\lambda^+} d\lambda \int_{k_r^-}^{k_r^+} dk_r \int_{k_z^-}^{k_z^+} dk_z \frac{f(\lambda)}{[(\lambda + N - \beta_\parallel k_z^2 + Vk_z - \beta_\perp k_r^2)]},$$

where

$$k_z^\pm = \frac{V}{2\beta_\parallel} \pm \left\{ \frac{V^2}{4\beta_\parallel^2} + \frac{1}{\beta_\parallel} (\lambda - \beta_\perp k_r^2 + N) \right\}^{1/2} \tag{17}$$

$$k_r^\pm = [(\lambda + V^2/4\beta_\parallel + N(I))/\beta_\perp]^{1/2},$$

$$\lambda^+ = -\frac{V^2}{4\beta_\parallel} \quad \text{and} \quad \lambda^- = -N(I) - \frac{V^2}{4\beta_\parallel}.$$

Carrying out the integration over k_r and k_z, we get

$$(= \pi \frac{1}{\beta_\perp} \frac{1}{|\beta_\parallel|^{1/2}} \int_{\lambda^-}^{\lambda^+} d\lambda \left\{ \left| \lambda + \frac{V^2}{4\beta_\parallel} + N(I) \right| \right\}^{1/2} f(\lambda), \tag{18}$$

We can solve (16) easily with respect to $f(\lambda)$ to obtain,

$$f(\lambda) = h_\pm(\lambda + V^2/4\beta_{||}), \tag{19}$$

where

$$\begin{aligned} h_\pm(y) = \frac{2\beta_\perp\sqrt{|\beta_{||}|}}{\pi^2}\Bigg[& \frac{1}{\sqrt{\pm y}}\frac{dI}{dN}\Bigg|_{N=0} \\ & + \int_0^{-y} dN\ \frac{d^2I(N)}{dN^2} \times \{\pm(y+N)\}^{-1/2}\Bigg] \end{aligned} \tag{20}$$

The plus sign corresponds to the case in which $N(I)$, $\beta_\perp$, and $\beta_{||} > 0$; the minus sign corresponds to the case in which $N(I)$, $\beta_\perp$, and $\beta_{||} < 0$. When we take into account only the lowest nonlinearity with respect to I, that is

$$N(I) = \alpha I, \tag{21}$$

the spectral function of the trapped wave then becomes

$$f(\lambda) = \frac{2}{\pi^2}\frac{\beta_\perp}{\alpha}\sqrt{|\beta_{||}|}\ \{\pm(\lambda + V^2/4\beta_{||})\}^{-1/2}. \tag{22}$$

Note that as long as the nonlinearity is given by (21), the envelope profile $I(r, \xi)$ is arbitrary.

4. **Example: Envelope Profile for Langmuir Waves.** In the previous section we determined the distribution function when the nonlinearity $N(I)$ is given as a function of I. However, the functional of I, $N(I)$, is usually different when a spatial profile of the envelope is different. When the distribution function (the frequency spectrum) is given and $N(I)$ is given by means of (18), then the spatial profile may be determined.

Langmuir waves can be used as an example. For these waves, the dominant nonlinearity comes from the local density depletion due to the low frequency ponderomotive force of Langmuir waves, when the wave-particle interaction is excluded. In such a case, the parameters in (1) and (14) are given as follows:

$$\beta_\perp = \beta_{||} = 3\lambda_D{}^2/2, \qquad N = -\tfrac{1}{2}\delta n/n_0$$

$$I = \frac{1}{8\pi n_o T_e}|E_k|^2 \text{ and } \omega_0 = \omega_{pe}.$$

The relation between density depletion δn and the intensity I is

$$V^2 \frac{\partial^2 \tilde{n}}{\partial \xi^2} - c_s{}^2\Delta \left(1 + \frac{T_i}{T_e} + \lambda_D{}^2\Delta\right)\tilde{n} = c_s{}^2\Delta(1 + \lambda_D{}^2\Delta)\, I, \tag{23}$$

where $\tilde{n} = \delta n/n_0$ and c_s is the acoustic velocity. Equation (23) is derived in Appendix B. To get an explicit profile of $\tilde{n}$ and I, we assume that the linear relation between N and I is given by (21), i.e., the spectral function is given by (22) in the lowest order approximation with respect to N.

First we consider a static case, that is, $V = 0$. When the dispersive effect [that is, the term proportional to $\Delta^2\tilde{n}$ in (23)] is ignored, we get the relation, $\alpha = T_e/2(T_e + T_i)$; the profile is arbitrary. However, taking into account the dispersive effect, we get

$$\left(1 + \frac{T_i}{T_e} - \frac{1}{2\alpha}\right) \tilde{n} + \lambda_D{}^2\Delta \left(1 - \frac{1}{2\alpha}\right)\tilde{n} = 0. \tag{24}$$

Here we integrated (23) and used a boundary condition $\tilde{n} \to 0$ for $r \to \infty$ or $z \to \infty$. When $1 + T_i/T_e > 1/(2\alpha) > 1$ is satisfied, (24) becomes

$$\Delta\tilde{n} = \kappa^2\tilde{n}, \tag{25}$$

where

$$\kappa^2 = k_D{}^2 \frac{1 + T_i/T_e - 1/2\alpha}{1/(2\alpha) - 1}. \tag{26}$$

The spherically symmetric solution is given by

$$\tilde{n} = a\sqrt{1/\rho}\, e^{-\kappa\rho},$$

where $\rho^2 = z^2 + x^2 + y^2$. Note that $\tilde{n}$ has a singularity at $\rho = 0$, but the nonlinear terms which were neglected in (24) become important around $\rho = 0$. If the nonlinear terms are not neglected the singularity at $\rho = 0$ may be removed. Since $|k| < k_D$ for Langmuir waves and the condition of (9) i.e., $|\Delta k| \gg \kappa$ must be satisfied, $\kappa \ll k_D$, i.e., $1/(2\alpha)$ should be very close to $1 + T_i/T_e$. To determine the difference between $1/(2\alpha)$ and $1 + T_i/T_e$, we probably have to take into account nonlinearity with respect to $\tilde{n}$ in (23).

Now, let us consider the case where an envelope moves with finite velocity along the z axis. Neglecting the dispersive term in (23), we get

(27)
$$\left[\left(1+\frac{T_i}{T_e}-\frac{V^2}{c_s^2}-\frac{1}{2\alpha}\right)\frac{\partial^2}{\partial\xi^2}+\left(1+\frac{T_i}{T_e}-\frac{1}{2\alpha}\right)\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\tilde{n}=0.$$

When

$$\frac{1}{2\alpha}>1+\frac{T_i}{T_e}\quad\text{or}\quad\frac{1}{2\alpha}<1+\frac{T_i}{T_e}-\frac{V^2}{c_s^2},$$

we can rewrite (27) as

(28)
$$\left[\frac{\partial^2}{\partial\eta^2}+\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\tilde{n}=0,$$

where

$$\xi^2=\eta^2\left(1+\frac{T_i}{T_e}-\frac{V^2}{c_s^2}-\frac{1}{2\alpha}\right)\Big/\left(1+\frac{T_i}{T_e}-\frac{1}{2\alpha}\right).$$

A symmetric solution of (28) in the space (η, r) is given by

(29)
$$\tilde{n}=1/(r^2+\eta^2)^{1/2}.$$

5. **Concluding Remarks.** We have derived 2- and 3-dimensional localized envelopes of random phase waves, using the wave kinetic equation. The wave kinetic equation is derived from the nonlinear Schrödinger equation as well as from the basic conservation law of quasiparticles. It was shown that unlike the case of an envelope soliton of a monochromatic wave, the shape of the envelope of random phase waves is arbitrary.

There are several unsolved problems. They are the stability of the localized envelope waves which are obtained and dissipation of the envelope waves in the process of their generation. There are two types of stability problems. The first type is related with the stability problem which is pertinent to the particular distribution function $f(\mathbf{k}, \mathbf{r})$. This stability problem arises even for one-dimensional solutions. In particular, if one takes a monochromatic distribution function in $\mathbf{k}$, i.e., $f(\mathbf{k}) = \delta(k - k_0)$, it can be shown [9] that a uniformly distribution of quasiparticles in the coordinate space, $\mathbf{r}$, is always unstable. The dispersion relation is given by $\Omega^2 = -K^2$, for a small amplitude perturbation of the form $\exp i(Kx - \Omega t)$. That is, there exists no stabilization effect in a large K limit. Even if $\delta(k - k_0)$ is not the BGK solution obtained here, this fact has an interesting contrast to the case of the dispersion relation for a modulational instability of a coherent wave described by the nonlinear Schrödinger equation, which is given by

$\Omega^2 = -K^2 + K^4$. In the latter case, the modulation is stable for a large K value. One over the value of this critical $K(=1)$ corresponds to the width of the envelope soliton. The monochromatic distribution function in **k** is not the solution of our case. Hence, there remains an immediate problem of the analysis of the stability of the distribution function obtained in this manuscript at least for a case of uniformly distributed waves in coordinate space.

There is another class of instability problem. This is related with the indefinite shrinkage of the 2- or 3-dimensional localized envelopes which are known to occur in similar solitons obtained from the nonlinear Schrödinger equation [7]. Because the original sets of equations are different in the coherent wave and the random wave cases, the argument that applies to the case of the nonlinear Schrödinger equation does not immediately hold for the case of the envelope waves obtained here. However, we have not proved the stability of our solution with respect to the shrinkage into a line (2-d solution) or a point (3-d solution). One physical process that can eliminate this difficulty is the saturation of the nonlinear effect as discussed by Kaw, et al. [10]. A unique aspect of the envelope waves obtained here may originate from the fact that the WKB approximation breaks down as the envelope size tends to shrink. We leave this problem also as an unsolved problem for future study.

We now discuss the problem of dissipation of the solitons. As was pointed out in the derivation of the wave kinetic equation, the underlying assumption of the equation is the conservation of quasiparticles. When the waves interact with discrete particles in the plasma, however, they exchange energy with the particles and the conservation law breaks down. For example, Landau damping dissipates the number of quasiparticles. We point out that there are three important dissipation processes. One is the linear Landau damping of the wave which essentially cause a linear dissipation of the quasiparticle density, the second is the nonlinear Landau damping which conserves the total number of waves but shifts the spectrum nonlocally in k-space. These dissipative processes should be considered when one solves the nonlinear Schrödinger equation using the inverse scattering technique, because if the dissipation rate is larger than the rate of production of solitons, the solution of the inverse scattering becomes invalid. One might assume that if the amplitude is large enough this problem may not occur because the linear damping rate can arbitrarily be made smaller than the rate of the nonlinear development. However, there exists a third type of instability when the amplitude is made larger. This is the trapping of particles by the ponderomotive force. When the difference of a particle velocity and the average group velocity of the waves becomes smaller than the square root of the ponderomotive potential, the particles are trapped by the ponderomotive potential and their trajectories cross. This trajectory crossing quickly thermalizes the wave energy and the particles are rapidly heated. This type of nonlinear damping of the wave is not included in either nonlinear Schrödinger equation or the wave kinetic equation. Therefore, when the amplitude is large, even if the nonlinear rate of change of the envelope becomes large, this nonlinear dissipation also becomes effective and careful checking of such a nonlinear dissipative process is needed in studying the dynamic development of the envelope.

ADDED IN PROOF. A method has been found for constructing the shape of the envelope solitons [**15**].

Appendix A. Here we derive the nonlinear wave kinetic equation from the basic conservation law of the quasiparticles and discuss the validity of the WKB approximation which serves as the basis of the quasiparticle concept.

The quasiparticle density $f(\mathbf{k}, \mathbf{r}, t)$ is conserved in the 6-dimensional phase space $(\mathbf{k}, \mathbf{r})$ in the absence of waveparticle interactions. The conservation law can be written

$$\text{(A1)} \qquad \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial f}{\partial \mathbf{k}} \frac{d\mathbf{k}}{dt} = 0.$$

If the Hamiltonian $H(\mathbf{k}, \mathbf{r})$ describing the kinematics of the quasiparticles is known, the Hamilton's equation of motion becomes,

$$\text{(A2)} \qquad \frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{k}},$$

$$\text{(A3)} \qquad \frac{d\mathbf{k}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} = -\frac{\partial H}{\partial I}\frac{\partial I}{\partial \mathbf{r}},$$

where I is defined in (6) and given by

$$\text{(A4)} \qquad \mathrm{I}(\mathbf{r}, t) = \int f d\mathbf{k}.$$

In the wave-kinematics, the Hamiltonian is the frequency ω of the wave, which is given by the nonlinear dispersion relation,

$$\text{(A5)} \qquad H = \omega[\mathbf{k}, \mathbf{r}, N(I)].$$

Substitution of (A2), (A3) and (A5) into (A1) gives

$$\text{(A6)} \qquad \frac{\partial f}{\partial t} + \frac{\partial \omega}{\partial \mathbf{k}}\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \omega}{\partial I}\frac{\partial I}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{k}} = 0.$$

Now, we present a direct derivation of the wave kinetic equation from the nonlinear Schrödinger equation and discuss the validity of the wave kinetic equation. Let us consider one wave packet with a certain wave number $\mathbf{k}$. We can then write A as

$$\begin{aligned} A &= \psi_k(\mathbf{r}, t) \exp(i\mathbf{K} \cdot \mathbf{r} - i\lambda_k t) \\ &= \exp(i\mathbf{k} \cdot \mathbf{r} - i\lambda_k t) \int d\mathbf{K}\, \psi_{\mathbf{k}+\mathbf{K}}(t) e^{i\mathbf{K}\cdot\mathbf{r}}. \end{aligned} \tag{A7}$$

Substituting (A7) into (1), we get

$$\begin{aligned} \Big[I \frac{\partial}{\partial t} &+ \{\lambda_k - \beta_\perp(\mathbf{k}_\perp + \mathbf{K}_\perp)^2 \\ &- \beta_{\|}(k_{\|} + K_{\|})^2 \} \Big] \psi_{k+K} \\ &+ \sum_{\mathbf{K}'} N_{\mathbf{K}'} \psi_{\mathbf{k}+\mathbf{K}-\mathbf{K}'} = 0. \end{aligned} \tag{A8}$$

Multiplying by $\psi^*_{\mathbf{k}-\mathbf{K}} e^{2i\mathbf{K}\cdot\mathbf{r}}$, taking the imaginary part and summing the resultant equation over $\mathbf{K}$ we have

$$\begin{aligned} \frac{\partial}{\partial t} f(\mathbf{k}, \mathbf{r}, t) &+ 2(\beta_\perp \mathbf{k}_\perp \cdot \nabla + \beta_{\|} \mathbf{k}_{\|} \cdot \nabla) f(\mathbf{k}, \mathbf{r}, t) \\ &+ \Lambda(\mathbf{k}, \mathbf{r}, t) = 0, \end{aligned} \tag{A9}$$

where

$$\Lambda(\mathbf{k}, \mathbf{r}, t) = 2\Delta_k \mathrm{Im} \left\{ \int \int d\mathbf{K} d\mathbf{K}'\, \psi^*_{\mathbf{k}-\mathbf{K}} N_{\mathbf{K}'} \psi_{\mathbf{k}+\mathbf{K}-\mathbf{K}'} e^{2i\mathbf{K}\cdot\mathbf{r}} \right\}, \tag{A10}$$

and use is made of the following relation;

$$\begin{aligned} f(\mathbf{k}, \mathbf{r}, t) &= |\psi_{\mathbf{k}}(\mathbf{r}, t)|^2 \\ &= \int dK \int d\mathbf{K}'\, \psi_{\mathbf{k}+\mathbf{K}} \psi^*_{\mathbf{k}+\mathbf{K}'} e^{i(\mathbf{K}-\mathbf{K}')\cdot\mathbf{r}} \\ &= \int d\mathbf{m} \int d\mathbf{n}\, \psi_{\mathbf{k}+\mathbf{m}+\mathbf{n}} \psi^*_{\mathbf{k}+\mathbf{m}-\mathbf{n}} e^{2i\mathbf{n}\cdot\mathbf{r}} \\ &= \Delta_k \int d\mathbf{n}\, \psi_{\mathbf{k}+\mathbf{n}} \psi^*_{\mathbf{k}-\mathbf{n}} e^{2i\mathbf{n}\cdot\mathbf{r}}, \end{aligned} \tag{A11}$$

in which Δ_k means a characteristic k-space volume occupied by the wave packet. When $N(\mathbf{r}, t)$ is a slowly varying function of $\mathbf{r}$, i.e., $N_{\mathbf{K}'} \neq 0$ only for $|\mathbf{K}'| \ll |\mathbf{k}|$, equation (A10) rewritten as,

$$\begin{aligned}\Lambda(\mathbf{k}, \mathbf{r}, t) &= 2\Delta_k \operatorname{im} \left\{ \int d\mathbf{m} \int d\mathbf{n}\, \psi_{\mathbf{k}-\mathbf{n}/2+\mathbf{m}} \psi^*_{\mathbf{k}-\mathbf{n}/2-\mathbf{m}} N_{\mathbf{n}} \right. \\ &\qquad \times e^{2i\mathbf{m}\cdot\mathbf{r}+i\mathbf{n}\cdot\mathbf{r}} \\ &= 2 \operatorname{Im} \int d\mathbf{n}\, f(\mathbf{k} - \mathbf{n}/2, \mathbf{r}, t) N_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{r}} \\ &= 2 \operatorname{Im} \left\{ f(\mathbf{k}, \mathbf{r}, t) N(\mathbf{r}) + \frac{i}{2} (\nabla \cdot N(\mathbf{r}, t)) \cdot \frac{\partial}{\partial \mathbf{k}} \right. \\ &\qquad \left. \times f(\mathbf{k}, \mathbf{r}, t) + \cdots \right\} \\ &\simeq \nabla \cdot N(\mathbf{r}, t) \cdot \frac{\partial}{\partial k} f(\mathbf{k}, \mathbf{r}, t)). \end{aligned} \tag{A12}$$

It is easily shown that (A12) is valid when $|\partial/\partial\mathbf{k} \ln|f(\mathbf{k}, \mathbf{r}, t)||^{-1} \simeq \Delta k$ (which is a width of the distribution function) is much larger than $|\nabla n|N(\mathbf{r}, t)||$ (which is a characteristic wave number of imhomogeneity). From (A9) and (A12), we get (7).

Appendix B: Derivation of the Relation Between Density Depletion and the Intensity of Langmuir Waves. The starting equations are the equations for ions;

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{e}{M} \nabla\Phi - \frac{T_i}{Mn_i} \nabla n_i, \tag{B1}$$

$$\frac{\partial n_i}{\partial t} + n_i \nabla \cdot \mathbf{v} = 0; \tag{B2}$$

the pressure balance equation for electrons;

$$0 = e\nabla\Phi - \frac{T_e}{n_e} \nabla n_e - \frac{\nabla P_r}{n_e}, \tag{B3}$$

and the Poisson equation;

$$\Delta\Phi = 4\pi e(n_e - n_i). \tag{B4}$$

Here, Φ is an electrostatic potential, n_i and n_e are electron and ion densities, respectively. P_r is the ponderomotive potential, and T_i and T_e are electron and ion temperatures respectively. Taking the time derivative of (B2) and substituting (B1) to eliminate **v**, we get the linear equation

$$\frac{\partial^2}{\partial t^2} n_i - n_0 \frac{e}{M} \Delta\Phi - \frac{T_i}{M} \Delta n_i = 0. \tag{B5}$$

Nonlinear terms with respect to density or potential fluctuation are negligible. From (B3) and (B4), we get

$$\Delta \frac{e\Phi}{T_e} = k_D{}^2 \left(- \frac{P_r}{n_0 T_e} + \frac{e\Phi}{T_e} - n_i/n_0 \right), \tag{B6}$$

where k_D is the Debye wave number and we again neglected nonlinear terms. Eliminating Φ from (B5) and (B6), we get

$$\begin{aligned} \frac{\partial^2 \tilde{n}}{\partial t^2} - c_s{}^2 \Delta \Big\{ & (I + \tilde{n}) + \lambda_D{}^2 \Delta (I + \tilde{n}) \\ & + \frac{T_i}{T_e} \tilde{n} \Big\} = 0, \end{aligned} \tag{B7}$$

where $\tilde{n} = \delta n_i / n_0$ i.e., normalized density fluctuation,

$$\lambda_D = 1/k_D,\ I \equiv \frac{P_r}{n_0 T_e} = \frac{\Sigma_k |E_k|^2}{8\pi n_0 T_e}$$

(E_k is an electric field of a Langmuir wave), and the higher order terms with respect to $\lambda_D{}^2\,\Delta$ were neglected. Finally, we assume the stationary state on the frame moving with a velocity V and get (23).

References

1. T. Taniuti (Ed.), *Reductive perturbation method for nonlinear wave propagation*, Prog. Theor. Phys. Suppl. **55** (1974); and B. B. Kadomtsev and V. I. Karpman, *Nonlinear waves*, Sov. Phys.-Usp. **14** (1971), 40–60.

2. V. I. Karpman and E. M. Krushkal', *Zh. Eksp. Teor. Fiz.* **55** (1968), 530–538 (translation in Sov. Phys.-JETP **28** (1969), 277–281).

3. R. Y. Chiao, E. Garmine, and C. H. Townes, *Self-trapping of optical beams*, Phys. Rev. Lett. **13** (1964), 479–480; P. L. Kelly, *Self-focusing of optical beams*, Phys. Rev. Lett. **15** (1965), 1005–1008; A. Hasegawa and F. Tappert, *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion*, Appl. Phys. Lett. **23** (1973), 142–144; ———, *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers, II. Normal dispersion*, Appl. Phys. Lett. **23** (1973), 171–172.

4. M. J. Lighthill, *Contributions to the theory of waves in non-linear dispersive systems*, J. Inst. math. and Appl. **1** (1965), 269–306; ———, *Some special cases treated by the Whitham theory*, Proc. Roy. Soc. London Ser. A **299** (1967), 28–53; H. Hasimoto and H. Ono, *Nonlinear modulation of gravity waves*, J. Phys. Soc. Japan **33** (1972), 805–811; T. B. Benjamin and J. E. Feir, *The disintegration of wave trains on deep water. Part* 1. *Theory*, J. Fluid Mech. **27** (1967), 417–430.

5. M. Porkolab and M. V. Goldman, *Upper-hybrid solitons and oscillating-two-stream instabilities*, Phys. Fluids 19 (1976), 872–881.

6. T. Taniuti and H. Washimi, *Self-trapping and instability of hydromagnetic waves along the magnetic field in a cold plasma*, Phys. Rev. Lett. **21** (1968), 209–212; A Hasegawa, *Theory and computer experiment on self-trapping instability of plasma cyclotron waves*, Phys. Fluids **15** (1972), 870–881.

7. V. E. Zakharov, *Collapse of Langmuir waves,* Zh. Eksp. Teor. Fiz. **62** (1972), 1745–1759 (translation in Sov. Phys.-JETP **35** (1972), 908–914).

8. V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media,* Zh. Eksp. Teor. Fiz. **61** (1971), 118–134 (translation in Sov. Phys.-JETP **34** (1972), 62–69).

9. P. Kaw, K. Mima, and K. Nishikawa, *Envelope solutions for random-phase phasmons,* Phys. Rev. Lett. **34** (1975), 803–806; A. Hasegawa, *Dynamics of an ensemble of plane waves in nonlinear dispersive media,* Phys. Fluids **18** (1975), 7–79.

10. P. K. Kaw, K. Nishikawa, Y. Yoshida, and A. Hasegawa, *Two-dimensional and three-dimensional envelope solitons, Phys. Rev. Lett.* **35** (1975), 88–91.

11. I. B. Bernstein, J. M. Greene, and M. D. Kruskal, *Exact nonlinear plasma oscillations,* Phys. Rev. **108** (1957), 546–550.

12. K. Kato, *J. Phys. Soc. Japan* (to be published).

13. E. S. Weibel and L. Chen, preprint.

14. R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory,* Chapter 1, W. A. Benjamin, New York, 1969.

15. A. Hasegawa, *Envelope soliton of random phase waves,* Phys. Fluids **20** (1977), 2155–2156.

BELL LABORATORIES, 600 MOUNTAIN AVENUE, MURRAY HILL, NEW JERSEY, 07974