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# $\lambda(n, k)$—CONVEX FUNCTIONS

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**1. Introduction.** Assume $n$ and $k$ are positive integers such that $n \geqq 2$ and $1 \leqq k \leqq n$. Define an Ordered $k$-partition of $n$ (denoted $\lambda(n, k)$) as an ordered $k$-tuple $(n(1), \cdots, n(k))$ of positive integers satisfying $n(1) + \cdots + n(k) = n$. Let $P(n)$ denote the set of all ordered $j$-partitions $\mu(n, j)$ of $n$ with $j$ varying such that $1 \leqq j \leqq n$. Let $F \subset C^r(I)$ and $u \in C^r(I)$ where $I \subset R$ is an interval and $r > 0$ is large enough so that the following definitions make sense.

DEFINITION 1.1. $F$ is a $\lambda(n, k)$-parameter family on $I$ if for every set of $k$ ($k$ fixed) distinct points $x_1 < \cdots < x_k$ in $I$ and every set of $n$ real numbers $y_{ir}$ there exists a unique $f$ in $F$ satisfying

$$f^{(r)}(x_i) = y_{ir}, \; r = 0, \cdots, n(i) - 1, \; i = 1, \cdots, k. \tag{1.1}$$

Given $Q(n)$, a nonempty subset of $P(n)$ we say $F$ is a $Q(n)$-parameter family on $I$ if $F$ is a $\mu(n, j)$-parameter family on $I$ for all $\mu(n, j) \in Q(n)$.

Let $M(i) \equiv n + n(1) + \cdots + n(i)$ for $1 \leqq i \leqq k$, $M(0) = n$ and $F$ be a $\lambda(n, k)$-parameter family on $I$.

DEFINITION 1.2. For $k \geqq 2$, $u$ is $\lambda(n, k)$-convex with respect to $F$ on $I$ if for every set of $k$ points $x_1 < \cdots < x_k$ in $I$ the unique $f$ in $F$ determined by

$$(f - u)^{(r)}(x_i) = 0, \; r = 0, \cdots, n(i) - 1, \; i = 1, \cdots, k \tag{1.2}$$

satisfies

$$(-1)^{M(i)}(f - u)(x) \leqq 0 \text{ on } (x_i, x_{i+1}), \; i = 1, \cdots, k - 1. \tag{1.3}$$

(If in (1.3) strict inequalities are satisfied then we say $u$ is strictly $\lambda(n, k)$-convex.)

DEFINITION 1.3. For $k \geqq 1$, $u$ is $\lambda(n, k)^*$-convex with respect to $F$ on $I$ if for every $x_1 < \cdots < x_k$ in $I$ the function $f$ in $F$ determined by (1.2) satisfies (1.3) for $i = 0, \cdots, k$. ($x_0$ and $x_{k+1}$ are the left and right end points of $I$ respectively).

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Let $u$ be $\lambda(n, k)^*$-convex with respect to $F$ on $I$. We say $u$ has property $P(\lambda(n, k))$ with respect to $F$ on $I$ in case either (i) $u$ is strictly $\lambda(n, k)^*$-convex with respect to $F$ on $I$, or (ii) for every $x_1 < \cdots < x_k$ in $I$ the conditions (1.2) and $f(z) = u(z)$ for some $z \in I$ ($z \neq x_i$, $1 \leqq i \leqq k$) imply $f(x) \equiv u(x)$ on $[\min\{x_1, z\}, \max\{x_k, z\}]$.

It has been shown (Theorem 4.5 of [3]) that if $F$ is a $P(n)$-parameter family and $u$ is $\lambda(n, n)$-convex with respect to $F$ on $I$ then (i) $u$ is $\mu(n, j)^*$-convex with respect to $F$ on $I$ and (ii) $u$ has property $P(\mu(n, j))$ with respect to $F$ on $I$ for all $\mu(n, j) \in P(n)$, $j \geqq 1$. In the main theorem (Theorem 3.1) of this paper we show under the assumption $\lambda(n, k)$ ($n, k \geqq 3$) has at least two entries equal to 1 that if $F$ is a $P(n)$-parameter family and $u$ is $\mu(n, k-1)^*$-convex with respect to $F$ on $I$ with the property $P(\mu(n, k-1))$ for all $\mu(n, k-1)$ in a certain subset (which depends on $\lambda(n, k)$) of $P(n)$ then $u$ is $\lambda(n, k)^*$-convex with respect to $F$ on $I$ with the property $P(\lambda(n, k))$. It then follows from this theorem that if $F$ is a $P(n)$-parameter family and $u$ is $\mu(n, j)^*$-convex with respect to $F$ on $I$ for all $\mu(n, j) \in P(n)$ which have at most one entry equal to 1 then $u$ is $\lambda(n, n)$-convex with respect to $F$ on $I$. It remains unknown however whether $\lambda(n, 1)^*$-convexity of $u$ together with property $P(\lambda(n, 1))$ implies $\mu(n, j)^*$-convexity of $u$ with property $P(\mu(n, j))$ where $\mu(n, j) \in P(n)$ is arbitrary and $F$ is a $P(n)$-parameter family on $I$.

For earlier results concerning $\lambda(n, k)$-parameter families and associated convex functions or their special cases reference may be made to [**1**, **2**, **3**, **4**] and to the other references mentioned therein. In particular, Theorem 3.1 of [**2**] is analogous to our main theorem in the case $k = n$. Also for the case $k < n$ with the following restrictions on $\lambda(n, k)$ namely, (i) $\max\{n(i) : 1 \leqq i \leqq k\} = 2$ (ii) $n(1) = n(k) = 2$ and (iii) any two entries not equal to 1 are separated by at least two entries equal to 1, an analogous result can be found on page 40 of [**2**].

2. **Preliminary results.** The Lemmas 2.1 and 2.2 stated below are special cases of Theorems 2.1 and 2.2 of [**4**]. We indicate, however, for the sake of reference the proof of one of them, the other being analogous.

LEMMA 2.1. *Suppose $F$ is a $P(n)$-parameter family and $u$ is $\lambda(n, k)^*$-convex with respect to $F$ on $I$. Let $g \in F$ satisfy the condition*

$$(2.1) \qquad (-1)^{M(J)}\,(g-u)^{(n(J)-1)}\,(x_J) > 0$$

*for some $J$, $1 < J \leqq k$ and all the conditions of* (1.2) *except for $i = J$ and $r = n(J) - 1$. Then $g$ satisfies*

$$(-1)^{M(i)}(g-u)(x) < 0 \text{ on } (x_i, x_{i+1}),\ i = 0, \cdots, J-1.$$

LEMMA 2.2. *Suppose $F$ and $u$ are as in Lemma 2.1 and $g \in F$ satisfies the condition*

$$(2.2) \qquad (-1)^{M(J)}(g-u)^{(n(J)-1)}(x_j) < 0$$

*for some $J$, $1 \leqq J < k$ and all the conditions* (1.2) *except for $i = J$ and $r = n(J) - 1$. Then $g$ satisfies $(-1)^{M(i)}(g-u)(x) < 0$ on $(x_i, x_{i+1})$, $i = J, \cdots, k$.*

PROOF OF LEMMA 2.1. Let $f \in F$ be determined by the conditions (1.2). Then the condition (2.1) together with the hypothesis on $F$ implies

$$(2.4) \qquad (-1)^{M(i)}(g-f)(x) < 0 \text{ on } (x_i, x_{i+1}),\ i = 0, \cdots, J-1.$$

Now the conclusion follows by addition of the inequalities (1.3) and (2.4) for $i = 0, \cdots, J-1$.

We assume hereafter that $n, k \geqq 3$ and $\lambda(n, k)$ is such that $n(p) = 1 = n(m)$ for some fixed $p$, $m$, $1 \leqq p < m \leqq k$. We also let $Q(n) \equiv \{\mu(n, k-1) \in P(n); \mu(n, k-1)$ is obtained from $\lambda(n, k)$ by deleting the entries $n(p) = 1$, $n(m) = 1$ and inserting the integer 2 in exactly one of the possible $k-1$ places in the resulting array$\} \cup \{\mu(n, k-1) \in P(n) : \mu(n, k-1)$ is obtained from $\lambda(n, k)$ by deleting the entries $n(p) = 1$, $n(m) = 1$, replacing $n(i)$ by $n(i) + 1$ for exactly one $i \neq p, m$ and inserting the integer 1 in just one of the possible $k-1$ places in the resulting array$\}$.

LEMMA 2.3. *Suppose $F$ is a $P(n)$-parameter family and $u$ is $\mu(n, k-1)^*$-convex with property $P(\mu(n, k-1))$ with respect to $F$ on $I$ for all $\mu(n, k-1) \in Q(n)$. Let $f \in F$ be determined by the conditions* (1.2) *and assume that $u(x) \not\equiv f(x)$ on $[x_1, x_k]$. Then*

(i) $(-1)^{M(i)}(f-u)^{(n(i))}(x_i) < 0$ *for all $i$, $1 \leqq i \leqq k$*

(ii) $(f-u)(z) = 0$, $z \in (x_i, x_{i+1})$ *implies*

(a) $(-1)^{M(i)}(f-u)'(z) < 0$ *if $m \leqq i \leqq k$ or $i = p$*

(b) $(-1)^{M(i)}(f-u)'(z) > 0$ *if $0 \leqq i \leqq p-1$ or $i = m-1$*

and

(iii) $(f-u)(x) \neq 0$ *for any $x \in (x_i, x_{i+1})$, $p < i < m-1$.*

PROOF. (i) Suppose (A) : $(-1)^{M(J)}(f-u)^{(n(J))}(x_J) \geqq 0$ holds for some $J$. We shall consider two cases. (I) $p < J \leqq k$ and (II) $1 \leqq J \leqq p$.

Case (I). Let $\mu(n, k-1) = (n(1), \cdots, n(p-1), n(p+1), \cdots, n(J)+1, \cdots, n(k))$. (In case $J = p+1$, the entry $n(p+1)$ in $\mu(n, k-1)$ has to be ignored.) If equality holds in (A) then the $\mu(n, k-1)^*$-convexity of $u$ along with property $P(\mu(n, k-1))$ and $(f-u)(x_p) = 0$ implies $f \equiv u$ on $[x_1, x_k]$, a contradiction.

If strict inequality holds in (A) then the $\mu(n, k-1)^*$-convexity of $u$ together with the hypothesis on $F$ implies by Lemma 2.1 that $(-1)^{M(p-1)}(f-u)(x) < 0$ on $(x_{p-1}, x_{p+1})$, a contradiction to $(f-u)(x_p) = 0$.

Case (II). The arguments will be the same as in Case (I) if we interchange the roles of $p$ and $m$ and of Lemmas 2.1 and 2.2 in its proof.

(ii) (a) Suppose (B) : $(-1)^{M(J)}(f-u)'(z) \geqq 0$ for some $J$. We shall consider two cases. (I) $m \leqq J \leqq k$ and (II) $J = p$.

Case I. Let $\mu(n, k-1) = (n(1), \cdots, n(p-1), n(p+1), \cdots, n(m-1), n(m+1), \cdots, n(J), 2, n(J+1), \cdots, n(k))$. (In case $J = m$, the entries $n(m+1), \cdots, n(J)$ are to be ignored.) If equality holds in (B) then the $\mu(n, k-1)^*$-convexity of $u$ together with property $P(\mu(n, k-1))$ and $(f-u)(x_p) = 0$ implies $f \equiv u$, a contradiction.

If strict inequality holds in (B) then the $\mu(n, k-1)^*$-convexity of $u$ and the hypothesis on $F$ imply by Lemma 2.1 that $(-1)^{M(p-1)} (f-u)(x) < 0$ on $(x_{p-1}, x_{p+1})$, a contradiction to $(f-u)(x_p) = 0$.

Case II. Let $\mu(n, k-1) = (n(1), \cdots, n(p-1), 2, n(p+1), \cdots, n(m-1), n(m+1), \cdots, n(k))$. If equality holds in (B) then the $\mu(n, k-1)^*$-convexity of $u$ along with property $P(\mu(n, k-1))$ and $(f-u)(x_m) = 0$ implies $f \equiv u$ on $[x_1, x_k]$, a contradiction. If strict inequality holds in (B) then the $\mu(u, k-1)^*$-convexity of $u$ with the hypothesis on $F$ yields, by Lemma 2.2 that $(-1)^{M(m)}(f-u)(x_m) < 0$, a contradiction to $(f-u)(x_m) = 0$.

(ii) (b). The arguments will be similar to those of (ii) (a) if we interchange the roles of $p$ and $m$ and of the Lemmas 2.1 and 2.2 in its proof.

(iii) Suppose $(f-u)(z) = 0$ for some $z \in (x_J, x_{J+1})$ where $p < J < m-1$. Let $\mu(n, k-1) = (n(1), \cdots, n(p-1), n(p+1), \cdots, n(J), 2, n(J+1), \cdots, n(m-1), n(m+1), \cdots, n(k))$. If $(-1)^{M(J)}(f-u)'(z) > 0$ $(<0)$ then the $\mu(n, k-1)^*$-convexity of $u$ implies by Lemma 2.2 (2.1) that $(-1)^{M(m)}(f-u)(x_m) < 0$ $((-1)^{M(p-1)}(f-u)(x_p) < 0)$, a contradiction. If $(f-u)'(z) = 0$ then the $\mu(n, k-1)^*$-convexity of $u$ with the property $P(\mu(n, k-1))$ and $(f-u)(x_p) = 0$ implies $f \equiv u$, a contradiction.

## 3. Main results.

THEOREM 3.1. *Let $\lambda(n, k)$ be a given ordered k-partition of the type referred to above and let $Q(n)$ be the corresponding subset of $P(n)$ as defined above. Then, if $F$ is a $P(n)$-parameter family and if $u$ is $\mu(n, k-1)^*$-convex and has property $P(\mu(n, k-1))$ with respect to $F$ on $I$ for all $\mu(n, k-1) \in Q(n)$, it follows that $u$ is $\lambda(n, k)^*$-convex and has property $P(\lambda(n, k))$ with respect to $F$ on $I$.*

PROOF. Let $f \in F$ be determined by the conditions (1.2). We will show

$$(3.1) \qquad (-1)^{M(i)}(f-u)(x) \leqq 0 \text{ on } (x_i, x_{i+1}), \; i = 0, \cdots, k.$$

If $f \equiv u$ on some subinterval of $(x_1, x_k)$ then by virtue of our hypothesis on $u$ we will have that $f \equiv u$ on $[x_1, x_k]$, the inequality (3.1) holds for $i = 0, k$ and $u$ has property $P(\lambda(n, k))$ with respect to $F$ on $I$. Hence without loss of generality we can assume $f \not\equiv u$ on any sub-interval of $(x_1, x_k)$.

We will first show that the inequality (3.1) holds for $i = k$. By (i) of Lemma 2.3 we have $(f-u)^{(n(k))}(x_k) < 0$. If the inequality (3.1) does not hold for $i = k$ we can assume there exists a smallest number $z(x_k < z \leqq x_{k+1})$ such that $f(z) = u(z)$ and $(f-u)(x) < 0$ on $(x_k, z)$. Consequently we must have $(f-u)'(z) \geqq 0$, which is a contradiction to (ii) (a) of Lemma 2.3 for $i = k$. Hence the inequality (3.1) holds for $i = k$.

Now we will show that (3.1) holds for $i = k-1$. Again by (i) of Lemma 2.3 there exists a largest number $z(x_{k-1} \leqq z < x_k)$ such that $(f(z) = u(z)$ and $(-1)^{M(k-1)}(f-u)(x) < 0$ on $(z, x_k)$. Now we claim $z = x_{k-1}$. If not by (ii) (a) of Lemma 2.3 we must have $(-1)^{M(k-1)}(f-u)'(z) < 0$. Consequently there must exist a largest number $w(x_{k-1} \leqq w < z)$ such that $f(w) = u(w)$ and

$$(3.2) \qquad (-1)^{M(k-1)}(f-u)(x) > 0 \text{ on } (w, z).$$

If $x_{k-1} = w$ then (i) of Lemma 2.3 for $i = k-1$ yields a contradiction to (3.2). If $x_{k-1} < w$ then by (ii) (a) of Lemma 2.3 we have $(-1)^{M(k-1)}(f-u)'(w) < 0$. This again yields a contradiction to (3.2). This proves our claim.

The argument to show that (3.1) holds for $i = m, \cdots, k-2$ is similar and hence is omitted.

Now we will show (3.1) holds for $i = m-1$.

By (i) of Lemma 2.3 we have $(-1)^{M(m)}(f-u)'(x_m) < 0$. Hence there exists a largest number $z\,(x_{m-1} \leqq z < x_m)$ such that $f(z) = u(z)$ and

$$(3.3) \qquad (-1)^{M(m-1)}(f-u)(x) < 0 \text{ on } (z, x_m).$$

If $x_{m-1} < z$ then by (ii) (b) of Lemma 2.3 we have $(-1)^{M(m-1)}(f - u)'(z) > 0$ which yields a contradiction to (3.3). Hence $x_{m-1} = z$ and (3.1) holds for $i = m - 1$.

That (3.1) holds for all $i$, $p < i < m - 1$ follows at once from (i) and (iii) of Lemma 2.3.

The arguments for the cases $i = 0$, $p$ and $1 \leqq i \leqq p - 1$ are respectively analogous to those for the cases $i = k$ and $i = m - 1$ and hence are omitted.

COROLLARY 3.2. *Suppose* $\lambda(n, k)$, $Q(n)$ *and* $F$ *are as in Theorem* 3.1 *and* $u$ *is strictly* $\mu(n, k - 1)^*$*-convex with respect to* $F$ *on* $I$ *for all* $\mu(n, k - 1) \in Q(n)$. *Then* $u$ *is strictly* $\lambda(n, k)^*$*-convex with respect to* $F$ *on* $I$.

THEOREM 3.3. *Suppose* $F$ *is a* $P(n)$*-parameter family and* $u$ *is* $\mu(n, j)^*$*-convex with respect to* $F$ *on* $I$ *with the property* $P(\mu(n, j))$ *for all* $\mu(n, j) \in P(n)$ *which have at most one entry equal to* 1. *Then* $u$ *is* $\lambda(n, n)^*$*-convex with respect to* $F$ *on* $I$ *with the property* $P(\lambda(n, n))$.

PROOF. Let $v(n, r) \in P(n)$ be any $r$-tuple ($r \geqq 3$, arbitrary) having exactly two entries equal to 1. Then by our hypothesis and Theorem 3.1 it follows that $u$ is $v(n, r)^*$-convex with property $P(v(n, r))$. Since $v(n, r)$ is arbitrary using the above result with Theorem 3.2 again we can show that $u$ is $\mu(n, j)^*$-convex ($j \geqq 3$, arbitrary) with property $P(\mu(u, j))$ for all $j$-tuples $\mu(n, j)$ having exactly three entries equal to 1. Repeating the above argument a finite member times we arrive at the conclusion of the theorem.

Thus if $F$ and $u$ are as in Theorem 3.3, on combining the conclusions of Theorem 3.3 and Theorem 4.5 of [3] we obtain that $u$ is $\lambda(n, k)^*$-convex with respect to $F$ on $I$ with property $P(\lambda(n, k))$ for all $\lambda(n, k) \in P(n)$, $k \geqq 1$.

To illustrate the above remark, in the case $n = 4$ we have that if $u$ is strictly $(1,3)^*$, $(3,1)^*$ and $(2,2)^*$-convex then $u$ is strictly $(2,1,1)^*$, $(1,2,1)^*$, $(1,1,2)^*$, $(1,1,1,1)^*$ and $(4)^*$-convex.

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