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CAUCHY TRANSFORMS OF MEASURES, AND A CHARACTERIZATION OF SMOOTH PEAK INTERPOLATION SETS FOR THE BALL ALGEBRA

ALEXANDER NAGEL*

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n) \in \mathbf{C}^n$, let $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$, and let $|z| = \langle z, z\rangle^{1/2}$. Let $B_n = \{z \in \mathbf{C}^n | \ |z| < 1\}$ denote the unit ball in $\mathbf{C}^n$, and let $\partial B_n = \{z \in \mathbf{C}^n | \ |z| = 1\}$ denote its boundary. If $F(z)$ is holomorphic on B_n, we say that F belongs to $H^p(B_n)$, $0 < p < \infty$, if

$$\sup_{r<1} \int_{\partial B_n} |F(r\zeta)|^p \, d\sigma(\zeta) < \infty$$

where $d\sigma$ is rotation invariant Lebesgue measure on ∂B_n. We say that $F \in H^\infty(B_n)$ if $\sup_{z \in B_n} |F(z)| < \infty$. If $F \in H^p(B_n)$ for $0 < p \leqq \infty$, then F has radial limits $F^*(\zeta)$ almost everywhere on ∂B_n with respect to $d\sigma$. Moreover, if $1 \leqq p < \infty$, $F(r\zeta)$ converges in L^p to $F^*(\zeta)$. (For a discussion of H^p theory in B_n, see for example Stein [6] or Stout [7].)

Let $d\mu$ be a finite Borel measure on ∂B_n. We shall denote by $C(\mu)$ the *Cauchy transform* of $d\mu$ which is given by

$$C(\mu)(z) = \int_{\partial B_n} [1 - \langle z, \zeta\rangle]^{-n} \, d\mu(\zeta).$$

$C(\mu)(z)$ is holomorphic on B_n, but in general it need not belong to $H^1(B_n)$, for example if $d\mu$ is a point mass.

The object of this paper is twofold. First we study $C(\mu)$ when $d\mu$ is "Lebesgue measure" on a smooth curve $\gamma \subset \partial B_n$. We show that if the tangent to the curve γ does not lie in the maximal complex subspace of the real tangent space to ∂B_n at each point, then $C(\mu)(z)$ does belong to $H^1(B_n)$, and in fact has better behavior depending on the smoothness of γ. (Note that when $n > 1$, it follows that $C(\mu)$ may belong to $H^1(B_n)$ even if $d\mu$ is singular with respect to the surface measure $d\sigma$ on ∂B_n.) Precise statements are given in Theorem 1.

A second object of this paper is to apply Theorem 1 to obtain a necessary condition for a compact set $K \subset \partial B_n$ to be a peak interpolation set for the ball algebra $A(B_n)$ of functions continuous on $\bar{B}_n$ and holomorphic on B_n. (For the definition of peak interpolation set, see section

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2). This condition is simply that the intersection of K with every curve γ satisfying the conditions of Theorem 1 must have zero Lebesgue measure on γ (see Theorem 2). In particular, this, together with the results of [5], leads to a complete characterization of smooth real submanifolds $M \subset \partial B_n$ such that every compact set $K \subset M$ is a peak interpolation set for $A(B_n)$. (See Theorem 3).

Theorem 3 has been announced by Henkin and Tumanov for the more general case of strictly pseudoconvex domains in $\mathbf{C}^n$ (see [3], Chapter VI, § 2, Theorem 15) but no proofs were given there. In a recent Russian preprint [4], Henkin and Tumanov give proofs of generalizations of Theorems 2 and 3. However, their methods are different from those in this paper, and they do not obtain Theorem 1.

1. **Cauchy Transforms of Measures.** Let $\phi : [0, 1] \to \partial B_n$ be a C^k curve, $k = 2, \cdots, \infty$. Suppose that there exists $\delta > 0$ so that

$$|\langle \phi(t), \phi'(t) \rangle| \geqq \delta, \qquad t \in [0, 1]. \tag{1}$$

This analytic condition on ϕ is equivalent to a geometric condition, which we now describe. If $\zeta \in \partial B_n$, let T_ζ be a real tangent space to ∂B_n at ζ, and let $P_\zeta = T_\zeta \cap iT_\zeta$ be the maximal complex subspace of T_ζ. If L_ζ denotes the one-dimensional real subspace of T_ζ generated by $i\zeta$, then

$$T_\zeta = P_\zeta \oplus L_\zeta$$

and this decomposition is orthogonal relative to the usual real inner product on $\mathbf{C}^n$ given by $(z, w) = \operatorname{Re} \langle z, w \rangle$. It is now clear that the tangent to the curve $\phi(t)$ lies in $P_{\phi(t)}$ if and only if $\langle \phi(t)\, \phi'(t) \rangle = 0$. Hence (by continuity) condition (1) is equivalent to

$$\phi'(t) \notin P_{\phi(t)} \qquad \text{for all } t \in [0, 1]. \tag{1'}$$

Next, let $\psi \in C_0^\infty[0, 1]$, the space of real valued infinitely differentiable functions with compact support on $(0, 1)$, and define a measure $d\mu$ on ∂B_n by the equation:

$$\int f d\mu = \int_0^1 f(\phi(t))\psi(t)\, dt \qquad \text{for } f \in C(\partial B).$$

Then $d\mu$ is a finite Borel measure on ∂B_n and its Cauchy transform is given by

$$C(\mu)(z) = \int_0^1 [1 - \langle z, \phi(t) \rangle]^{-n} \psi(t)\, dt \tag{2}$$

THEOREM 1. *Let* $\phi : [0, 1] \rightarrow \partial B_n$ *be a curve of class* $C^k (k \geqq 2)$ *satisfying* (1). *Let* $C(\mu)$ *be defined by* (2). *Let* D^α *be any derivative in* $z_1, \cdots, z_n$ *of total order* $|\alpha|$, *with* $|\alpha| < k - 1$. *Then*

(*a*) *if* $|\alpha| + 1 < k < |\alpha| + 1 + n$, *then* $D^\alpha C(\mu) \in H^p(B_n)$ *for*

$$p < \frac{n}{n - k + |\alpha| + 1};$$

(b) *if* $k = |\alpha| + 1 + n$, *there exists* $K > 0$ *so that*

$$|D^\alpha C(\mu)(z)| \leqq K[|\log \operatorname{dist}(z, \phi[0, 1])| + 1];$$

(c) *if* $k > |\alpha| + 1 + n$, *the* $D^\alpha C(\mu) \in H^\infty(B_n)$.

PROOF. For each $t \in [0, 1]$ there are neighborhoods U_t of t in $[0, 1]$ and V_t of $\phi(t)$ in $\mathbf{C}^n$ so that if $s \in U_t$ and $z \in \bar{B}_n \cap V_t$ then $\phi(s) \in V_t$ and $|\langle z, \phi'(s)\rangle| \geqq \delta/2$. Let $U_1, \cdots, U_p$ be a finite subcover of $\{U_t\}$, let $V_1, \cdots, V_p$ be the corresponding open sets in $\mathbf{C}^n$, and let $\{\theta_1, \cdots, \theta_p\}$ be a C^∞ partition of unity subordinate to $\{U_1, \cdots, U_p\}$. Then

$$C(\mu)(z) = \sum_{j=1}^{p} \int_0^1 [1 - \langle z, \phi(t)\rangle]^{-n} \theta_j(t)\psi(t)\, dt$$

$$= \sum_{j=1}^{p} C_j(z)$$

Each C_j is holomorphic on $\bar{B}_n \setminus V_j$, and hence it suffices to show that each C_j has the required properties in V_j.

If D^α is any derivative in z of total order $|\alpha|$, then we have

$$D^\alpha C_j(z) = \int_0^1 [1 - \langle z, \phi(t)\rangle]^{-n-|\alpha|}\, \Psi_\alpha(z, t)\theta_j(t)\psi(t)\, dt$$

where $\Psi_\alpha : \mathbf{C}^n \times [0, 1] \rightarrow \mathbf{C}$ is holomorphic in z and is of class C^k in t. We wish to integrate by parts to reduce the negative exponent of $[1 - \langle z, \phi(t)\rangle]$.

In general, if $\Phi(z, t)$ is holomorphic in z, and is of class C^ℓ with compact support in U_j in t, then for $z \in V_j$, we have $\langle z, \phi'(t)\rangle \neq 0$ and so if $m > 1$

$$\int_0^1 [1 - \langle z, \phi(t)\rangle]^{-m}\, \Phi(z, t)\, dt$$

$$= \int_0^1 \frac{d}{dt}\left[\, [1 - \langle z, \phi(t)\rangle]^{-m+1} \,\right]$$

$$(m-1)^{-1}\langle z, \phi'(t)\rangle^{-1}\, \Phi(z, t)\, dt$$

$$= \int_0^1 [1 - \langle z, \phi(t)\rangle]^{-m+1}\tilde{\Phi}(z, t)\, dt,$$

where $\tilde{\Phi}(z, t)$ is holomorphic in z and of class $C^{\ell'}$ with compact support in U_j in t, where

$$\ell' \geqq \inf(\ell - 1, k - 2).$$

Hence for $z \in V_j$ we have for $r \leqq k - 1$

$$D^\alpha C_j(z) = \int_0^1 [1 - \langle z, \phi(t)\rangle]^{-n-|\alpha|+r}\, \Phi_{\alpha,r}(z, t)\, dt \tag{3}$$

where $\Phi_{\alpha,r}(z, t)$ is holomorphic in z and is of class C^{k-r-1} in t with compact support in U_j.

For each fixed t and $m > 0$, the function $z \to [1 - \langle z, \phi(t)\rangle]^{-m}$ belongs to $H^p(B_n)$ if and only if $p < n/m$. Hence by Stout [7], Corollary III.3, equation (3) shows that $D^\alpha C_j$ belongs to $H^p(B_n)$ if $-n - |\alpha| + r < 0$, $r - |\alpha| > 0$, and $p < n/(n + |\alpha| - r)$. In particular, if $|\alpha| + 1 < k < n + |\alpha| + 1$, we can choose $r = k - 1$ and we obtain $C_j(z) \in H^p(B_n)$ if $p < n/(n - k + |\alpha| + 1)$. This proves part (a) of Theorem 1.

If $k \geqq n + |\alpha| + 1$, we use (3) to write

$$D^\alpha C_j(z) = \int_0^1 [1 - \langle z, \phi(t)\rangle]^{-1}\psi_{\alpha,n+|\alpha|-1}(z, t)\, dt$$

where $\psi_{\alpha,n+|\alpha|-1}(z, t)$ is of class $C^{k-n-|\alpha|}$ in t. Integrating by parts once again we get

$$D^\alpha C_j(z) = \int_0^1 \log[1 - \langle z, \phi(t)\rangle]\, \psi_{\alpha,n+|\alpha|}(z, t)\, dt. \tag{4}$$

and hence $|D^\alpha C_j(z)| \leqq C_1 + C_2|\log \operatorname{dist}(z, \phi[0, 1])|$. This gives part (b).

Finally, if $k > n + |\alpha| + 1$ we can integrate by parts again in (4) to obtain

$$D^\alpha C_j(z) = \int_0^1 [[1 - \langle z, \phi(t)\rangle] \log(1 - \langle z, \phi(t)\rangle)$$

$$- [1 - \langle z, \phi(t)\rangle]]\psi_{\alpha,n+|\alpha|+1}(z, t)\, dt$$

where $\psi_{\alpha,n+|\alpha|+1}(z, t)$ is continuous in t. This shows that $D^{\alpha}C_j(z)$ is uniformly bounded, and gives part (c) of Theorem 1, and completes the proof.

We isolate certain consequences for special notice:

COROLLARY 1. *If* $k \geqq 2$, $C(\mu) \in H^1(B_n)$.

COROLLARY 2. *If* $k = \infty$, $C(\mu) \in A^{\infty}(B_n)$, *the algebra of functions which are* C^{∞} *on* $\bar{B}$ *and holomorphic on* B_n.

2. **Peak Interpolation Sets.** Let $K \subset\subset \partial B_n$ be a compact set. Then the following conditions are known to be equivalent:

(a) $|\mu|(K) = 0$ for all $\mu \in A^{\perp}(B_n)$, the space of Borel measures on B_n which annihilate $A(B_n)$.

(b) If $f \in C(K)$, there exists $F \in A(B_n)$ with $F(z) = f(z)$ for $z \in K$, and $|F(z)| < \|f\|_K$ for $z \in \bar{B}_n \setminus K$.

(c) There exists $F \in A(B_n)$ with $F(z) = 1$ for $z \in K$ and $|F(z)| < 1$ for $z \in \bar{B}_n \setminus K$.

(d) There exists $F \in A(B_n)$ with $F(z) = 0$ for $z \in K$ and $|F(t)| \neq 0$ for $z \in \bar{B}_n \setminus K$.

The equivalence of (a) and (b) is a theorem of Bishop [1]. (b) clearly implies (c), and (c) clearly implies (d). That (d) implies (a) is a special case of a theorem of Val'skii [8].

THEOREM 2. *Let* $K \subset \partial B_n$ *be compact. In order for* K *to satisfy conditions* (a)–(d) *it is necessary that for every* C^2 *curve* $\phi : [0, 1] \to \partial B_n$ *satisfying* (1) *or* (1'), $\phi^{-1}(K)$ *have Lebesgue measure zero in* [0, 1].

PROOF. Let $\phi : [0, 1] \to \partial B_n$ satisfy (1), let $\psi \in C_0^{\infty}[0, 1]$, and define a measure $d\mu$ on ∂B_n by

$$\int f d\mu = \int_0^1 f(\phi(t))\psi(t)\, dt.$$

By Corollary 1, $C(\mu) \in H^1(B_n)$ where $C(\mu)$ is the Cauchy transform of $d\mu$ as defined by (2). Let $F \in A(B_n)$. Then

$$\int F d\mu = \int_0^1 F(\phi(t))\psi(t)\, dt$$

$$= \lim_{r \to 1} \int_0^1 F(r\phi(t))\psi(t)\, dt.$$

Since $r\phi(t) \in B_n$, we have by the Cauchy integral formula for B_n

$$F(r\phi(t)) = \int_{\partial B_n} F(\zeta)[1 - \langle r\phi(t), \zeta\rangle]^{-n}\, d\sigma(\zeta).$$

Thus using Fubini's theorem, we obtain

$$\int F\, d\mu = \lim_{r\to 1} \int_{\partial B_n} F(\zeta) \left[\int_0^1 [1 - \langle r\phi(t), \zeta\rangle]^{-n}\psi(t)\, dt \right] d\sigma(\zeta)$$

$$= \lim_{r\to 1} \int_{\partial B_n} F(\zeta)\, \overline{C(\mu)(r\zeta)}\, d\sigma(\zeta).$$

Since $C(\mu) \in H^1B_n)$ we can put the limit under the integral sign and obtain

$$\int F\, d\mu - \int F(\zeta)\, \overline{C(\mu)^*(\zeta)}\, d\sigma(\zeta) = 0$$

It follows that if we let $d\nu = d\mu - \overline{C(\mu)^*}\, d\sigma$ then $d\nu \in A(B_n)^\perp$.

Now if K is a set satisfying (a)–(d), so is $K \cap \phi[0, 1]$. But $K \cap \phi[0, 1]$ has zero measure with respect to $d\sigma$. Since we must have $|\nu|(K \cap \phi[0, 1]) = 0$, it follows that measure $(\phi^{-1}(K)) = 0$.

Recall that a measure $d\mu$ on ∂B_n is called an A-measure if for every uniformly bounded sequence $\{F_n\}$ in $A(B_n)$ with $\lim_{n\to\infty} F_n(z) = 0$ for all $z \in B_n$, it follows that $\int F_n\, d\mu \to 0$. (see Henkin [2]).

Corollary 3. *If $\phi : [0, 1] \to \partial B_n$ is a curve satisfying* (1), *if $\psi \in C_0^\infty[0, 1]$, and if $d\mu$ is defined by $\int f\, d\mu = \int_0^1 f(\phi(t))\psi(t)\, dt$,* then $d\mu$ is an A-measure.

Proof. By Theorem 2, $\int f\, d\mu = \int f(\zeta)\, \overline{C(\mu)^*(\zeta)}\, d\sigma(\zeta) + \int f\, d\nu$ where $d\nu \in A(B_n)^\perp$. $d\nu$ is clearly an A-measure, and it follows from Henkin [2], that so is $C(\mu)(\zeta)\, d\sigma(\zeta)$.

Now let $M \subset \partial B_n$ be a not necessarily closed real submanifold of class C^3. For $\rho \in M$ we let $T_\rho M$ be the real tangent space to M at ρ.

Theorem 3. *Let M be a real submanifold at ∂B_n of class C^3. Then every compact set $K \subset M$ satisfies* (a)–(d) *if and only if $T_\rho M \subset P_\rho$ for all $\rho \in M$.* (Recall that P_ρ is the maximal complex subspace of $T_\rho(\partial B_n)$.)

Proof. A proof of the sufficiency appears in [5]. As indicated in the introduction, this was also stated by Henkin and Tumanov in [3] and a proof appears in [4]. The necessity follows from Theorem 2, since if $T_\rho M \not\subset P_\rho$ for some ρ, we can clearly find a curve $\phi : [0, 1] \to M$ which satisfies (1). Hence $\phi[0, 1]$ does not satisfy (a)–(d), and hence neither does M.

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08540
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISEN, WI 53706