ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 3, Summer 1979

ON MODULI OF CONTINUITY OF ANALYTIC AND HARMONIC FUNCTIONS

THAD DANKEL, JR.

ABSTRACT. We consider inequalities relating the modulus of continuity of an analytic or harmonic function in a planar region to its modulus of continuity on the boundary of the region. Using harmonic measure, we give a new proof of such a result for harmonic functions in the unit disc. We also generalize results for both analytic and harmonic functions in the unit disc to such functions defined on a Jordan region G such that ∂G satisfies certain smoothness assumptions.

Introduction. Let G be a region in C, the complex plane. Let $A(G)$ be the algebra of functions which are analytic in G and continuous in $\overline{G}$; similarly, let $a(G)$ be the vector space of functions which are harmonic in G and continuous in $\overline{G}$. If u belongs to $A(G)$ or to $a(G)$, and if $\delta > 0$, put

$$\omega(u, \delta, G) = \sup\{|u(z_1) - u(z_2)| \mid z_1, z_2 \in \overline{G}, |z_1 - z_2| \leqq \delta\},$$

$$\tilde{\omega}(u, \delta, G) = \sup\{|u(z_1) - u(z_2)| : z_1, z_2 \in \partial G, |z_1 - z_2| \leqq \delta\}.$$

When $G = D = \{z \in C : |z| < 1\}$, the following two properties have attracted the attention of a number of analysts:

I. *There exists a constant* $C > 0$ *such that for all* $u \in A(G)$ *and for all* $\delta > 0$, $\omega(u, \delta, G) \leqq C\tilde{\omega}(u, \delta, G)$.

II. *There exists a constant* $C > 0$ *such that for all* $u \in a(G)$, *and for all* $\delta \in (0, 1/2)$, $\omega(u, \delta, G) \leqq C \log(1/\delta)\, \tilde{\omega}(u, \delta, G)$.

(In II, the upper bound $1/2$ for δ is arbitrarily chosen. All that is essential is $\delta \leqq B < 1$, to bound $1/\delta$ away from the zero of the logarithm.)

Proofs of property I for $G = D$ may be found in [1] and [2]. In the latter paper, it is shown that in I necessarily $C > 1$, and that it is sufficient to take $C = 3$; it is also shown that the logarithmic factor in II cannot be dropped. In [3], Shapiro attributes property II for $G = D$ to Hardy and Littlewood [4], and he gives a proof of it based on Fourier analysis.

Received by the editors on July 20, 1977, and in revised form on January 4, 1978.

The purpose of this paper is twofold: to give a new proof of II for the unit disc D based on harmonic measure, and to prove that a broad class of regions G also have properties I and II.

We use "C" to denote an arbitrary positive constant; different appearances of "C" may denote different constants.

Property II for the Unit Disc.

THEOREM 1. *Property* II *holds if* $G = D$, *the unit disc.*

PROOF. Given $\delta_0 > 0$, we may choose an integer n so that $n\delta_0 \geqq 2$. Then, if $u \in a(D)$ and $\delta \geqq \delta_0$, we have

$$\frac{\omega(u, \delta)}{\tilde{\omega}(u, \delta)} \leqq n \frac{\omega(u, \delta_0)}{\tilde{\omega}(u, \delta_0)}.$$

(Here, and in the sequel, we drop the symbol for the region G in ω and $\tilde{\omega}$ when G is clear from the context). Hence, to prove II for D, it suffices to show that for some $C > 0$,

$$\limsup_{\delta \to 0} \frac{\omega(u, \delta)}{\tilde{\omega}(u, \delta)} (\log 1/\delta)^{-1} \leqq C,$$

uniformly for $u \in a(G)$. Fix such a u. By [2],

$$\omega(u, \delta) = \sup\{|u(z) - u(z')| : z \in D, z' \in \partial D, |z - z'| \leqq \delta\}.$$

(In [2], this lemma is stated for u analytic; however, since its proof relies only on the maximum principle, it also holds for u harmonic.) Therefore, we need only show that for sufficiently small $\delta > 0$, and for $z \in D$, $z' \in \partial D$, such that $|z - z'| < \delta$,

$$|u(z) - u(z')| \leqq C(\log 1/\delta)\tilde{\omega}(u, \delta), \tag{1}$$

where C is independent of u, δ, z, z'. Since the class of harmonic functions is invariant under rotations, without loss of generality we assume $z' = 1$.

Fix $\delta > 0$. Let n be the first integer such that δ is at least the length of the side of the regular $2n$ − *gon* inscribed in ∂D; that is, n satisfies

$$2 \sin(\pi/2(n - 1)) > \delta \geqq 2 \sin(\pi/2n).$$

Note that

$$\lim_{\delta \to 0} n\delta = \pi. \tag{2}$$

Let $\sigma(z, d\lambda) \equiv \sigma(z, d\lambda, D)$ be the harmonic measure on ∂D at $z \in D$. (Since we are considering the unit disc, $\sigma(z, d\lambda)$ is $P_r(\theta - t)\,dt$, where $z = re^{i\theta}$, dt is Lebesgue measure on ∂D divided by 2π, and $Pr(\theta - t)$ is the Poisson kernel.) Thus,

$$u(z) = \int_{\partial D} u(\lambda)\, \sigma\,(z, d\lambda), \quad z \in D.$$

If $0 \leqq k \leqq n - 1$, let A_k be the (counterclockwise) arc of ∂D from $e^{ik\pi/n}$ to $e^{i(k+1)\pi/n}$, and let $\tilde{A}_k$ be the symmetrically located arc with respect to the x – axis. Since $\partial D = \cup_{k=0}^{n-1} (A_k \cup \tilde{A}_k)$, and since $\sigma(z, d\lambda)$ is a probability measure,

$$(3) \qquad |u(z) - u(1)| \leqq \sum_{k=0}^{n-1} \int_{A_k \cup \tilde{A}_k} |u(\lambda) - u(1)|\sigma\,(z, d\lambda).$$

As the chord length of A_k is less than δ, the triangle inequality implies

$$|u(\lambda) - u(1)| \leqq (k + 1)\, \tilde{\omega}(u, \delta)$$

provided $\lambda \in A_k \cup \tilde{A}_k$. Then (3) becomes

$$(4) \qquad |u(z) - u(1)| \leqq \left[\sum_{k=0}^{n-1} (k + 1)\, \sigma(z, A_k \cup \tilde{A}_k) \right] \tilde{\omega}(u, \delta).$$

Put $B_k = \cup_{\ell \geqq k} (A_\ell \cup \tilde{A}_\ell)$, $\beta_k(z) = \sigma(z, B_k)$, $0 \leqq k \leqq n - 1$, and $\beta_n(z) = 0$. Because $\sigma(z, A_k \cup \tilde{A}_k) = \beta_k(z) - \beta_{k+1}(z)$, (4) implies

$$(5) \qquad |u(z) - u(1)| \leqq \left(\sum_{k=0}^{n-1} \beta_k(z) \right) \tilde{\omega}(u, \delta).$$

Now the level lies of $\sigma(z, A)$, for A a fixed arc of ∂D, are circular arcs joining the endpoints of A (see, e.g., [5]). This property means that $\beta_k(z)$ is maximized for $|z - 1| \leqq \delta$ at $z = 1 - \delta$. Replacing $\beta_k(z)$ in (5) by $\beta_k \equiv \beta_k(1 - \delta)$, we see that to prove (1) we need only show

$$(6) \qquad \limsup_{\delta \to 0} \left(\sum_{k=0}^{n-1} \beta_k \right) \cdot [\log 1/\delta]^{-1} \leqq C.$$

Note that, by symmetry, $\beta_k = 2\gamma_k$, where γ_k is the harmonic measure at $1 - \delta$ of the half of B_k which lies in the upper half plane. Since $\beta_0 = 1$, we have

$$\sum_{k=0}^{n-1} \beta_k = 1 + 2 \sum_{k=1}^{n-1} \gamma_k.$$

To calculate γ_k, we map D conformally onto the upper half plane via the linear fractional transformation

$$\zeta = (z - 1)/i(z + 1), \quad z \in D.$$

Now, a short calculation shows that $\zeta(e^{ik\pi/n}) = \tan(k\pi/2n)$; moreover, $\zeta(-1) = \infty$. Since harmonic measure is conformally invariant [5], γ_k is the harmonic measure at $\zeta(1 - \delta) = i\delta/(2 - \delta)$ of the interval $(\tan k\pi/2n, +\infty)$ on the real axis with respect to the upper-half plane. Since the function $1 - 1/\pi \, Arg\,(z - \tan k\pi/2n)$ is harmonic in the upper half-plane and assumes at $x \in R$ the boundary value 1 if $x > \tan(k\pi/2n)$ and 0 if $x < \tan(k\pi/2n)$, γ_k is the value of this function at $z = i\delta/(2 - \delta)$, namely

$$\gamma_k = \frac{1}{\pi} \tan^{-1}\left(\frac{\delta \cot(k\pi/2n)}{2 - \delta}\right).$$

Now, for $\delta < 1$,

$$\begin{aligned}\gamma_k &\leqq \frac{\delta}{\pi} \cot\left(\frac{k\pi}{2n}\right) \\ &\leqq \frac{\delta}{\pi} \csc\left(\frac{k\pi}{2n}\right) \\ &\leqq \frac{\delta n}{k\pi}.\end{aligned}$$

Therefore, since by (2) $n\delta < 4$ if δ is sufficiently small,

$$\sum_{k=1}^{n-1} \gamma_k \leqq \frac{4}{\pi} C \log n.$$

for such δ. Using (2) again, we see that (6) holds, and hence that II holds for D.

More General Regions. In this section G is a Jordan region—a simply-connected region such that ∂G is a Jordan curve—and $f: G \rightarrow D$ is a one-to-one, conformal mapping of G onto D. By [6, vol. 2, p. 96], f extends to a one-to-one, continuous mapping (also denoted by "f") of $\overline{G}$ onto $\overline{D}$; moreover, $f^{-1}: \overline{D} \rightarrow \overline{G}$ is continuous.

THEOREM 2. *Suppose f and f^{-1} satisfy (global) Lipschitz conditions on $\overline{G}$ and $\overline{D}$, respectively. Then G. has properties* I *and* II.

PROOF. Suppose

(7) $$|f(\zeta_1) - f(\zeta_2)| \leqq K|\zeta_1 - \zeta_2|, \; \zeta_1, \zeta_2 \in \overline{G}$$

and

$$|f^{-1}(z_1) - f^{-1}(z_2)| \leqq K' |z_1 - z_2|, \ z_1, z_2 \in \overline{D}. \tag{8}$$

We assume $K' \geqq 1$. For $u \in a(G)$, $\bar{u} \equiv u \circ f^{-1} \in a(D)$; and for $u \in A(G)$, $\bar{u} \in A(D)$. From (7) and the definitions of ω and $\tilde{\omega}$, we have

$$\omega(u, \delta, G) \leqq \omega(\bar{u}, K\delta, D). \tag{9}$$

Similarly, using (8),

$$\tilde{\omega}(u, \delta, G) \geqq \tilde{\omega}(\bar{u}, \delta/K', D). \tag{10}$$

Combining, (9) and (10), we obtain

$$\frac{\omega(u, \delta, G)}{\tilde{\omega}(u, \delta, G)} \leqq \frac{\omega(\bar{u}, K\delta, D)}{\tilde{\omega}(\bar{u}, \delta/K', D)} \tag{11}$$

But if n is the first integer greater than or equal to KK', the triangle inequality implies

$$\omega(\bar{u}, K\delta, D) \leqq n\omega(\bar{u}, \delta/K', D).$$

Thus, (11) becomes

$$\frac{\omega(u, \delta, G)}{\tilde{\omega}(u, \delta, G)} \leqq n \frac{\omega(\bar{u}, \delta/K', D)}{\tilde{\omega}(\bar{u}, \delta/K', D)}. \tag{12}$$

Since D has property I, (12) implies that G does also. If $u \in a(G)$ and $\delta \leqq 1/2$, than $\delta/K' \leqq 1/2$, so the right-hand side of (12) is $\leqq n \cdot C \log(K'/\delta) \leqq n \cdot C' \log(1/\delta)$, by property II for D. Hence, G has property II, and the proof is complete.

The following theorems are proved by showing in each case that Theorem 2 applies.

THEOREM 3. *If ∂G is an analytic curve, then G has properties* I *and* II.

PROOF. By [**6**, vol. 2, p. 102], f may be analytically continued across ∂G to a univalent function defined on a larger simply-connected region Ω. Hence, by the argument principle and the compactness of G, there exist m, M such that

$$0 < m < |f'(\zeta)| < M < \infty, \quad \zeta \in \overline{G}. \tag{13}$$

Thus, $|(f^{-1})'| < 1/m$ on $\overline{D}$. Now, as $\overline{D}$ is convex, if two points of $\overline{D}$ are given, we may integrate $(f^{-1})'$ along the line segment joining them. Therefore, f^{-1} satisfies a global Lipschitz condition on $\overline{D}$. To see that f satisfies a Lipschitz condition on $\overline{G}$, let Γ be a rectifiable Jordan curve in Ω such that $\overline{G}$ lies in the interior of Γ. An application of the Cauchy integral formula to f and Γ yields the desired result. An application of Theorem 2 completes the proof.

Rather than the strong assumption of analyticity in Theorem 3, it is desirable to obtain "C^1-type" conditions on ∂G sufficient for the hypthesis of Theorem 2; i.e., conditions about the existence and smoothness of tangents to ∂G. To this end, we recall some definitions from the theory of conformal mapping ([7], chapter 10). The Jordan curve Γ, represented by $w(t)$, has a *tangent* at $w_0 = w(t_0)$ if $\arg(w(t) - w_0) \to \theta$ as $t \downarrow t_0$ and $\arg(w(t) - w_0) \to \theta + \pi$ as $t \uparrow t_0$, for some $\theta \equiv \theta(t_0) \in R$. Γ is *smooth* if it has a tangent at each of its points and $\theta(t)$ is continuous in t. The curve Γ is *Dini-smooth* if the angle $\theta(s)$ of the tangent, considered as a function of the arc length s, satisfies

$$|\theta(s_2) - \theta(s_1)| < \omega(s_2 - s_1),\ s_1 < s_2,$$

where ω is an increasing function such that $\int_0^1 \omega(x)/x\, dx < \infty$.

THEOREM 4. *Suppose that ∂G is Dini-smooth and that ∂G has bounded arc-chord ratio: there exists $C > 0$ such that for every pair of points $\zeta_1, \zeta_2 \in \partial G$,*

$$|s_2 - s_1| \leqq C|\zeta_2 - \zeta_1|,$$

where $|s_2 - s_1|$ is the arc length along ∂G between ζ_1 and ζ_2. Then G has properties I *and* II.

PROOF. By a theorem of Warschawski [7, 8], $(f^{-1})'$ extends continuously to $\overline{D}$, and there exist m, M such that

$$0 < m < |(f^{-1})'(z)| < M < \infty, \quad z \in \overline{D}. \tag{13}$$

As $\overline{D}$ is convex, integration of the right-hand inequality along line segments in $\overline{D}$ implies that f^{-1} satisfies a Lipschitz condition on $\overline{D}$. Warschawski's theorem also implies that $(f^{-1})'$ may be calculated on ∂D by differentiating along ∂D. Therefore, by the chain rule,

$$\frac{df}{ds} = \frac{df}{d\zeta}\frac{d\zeta}{ds} = \frac{df}{d\zeta}\,\frac{\dfrac{df^{-1}}{dt}}{\left|\dfrac{df^{-1}}{dt}\right|},\ t \in [0, 2\pi),$$

exists and is continuous on ∂G. But this means f satisfies a Lipschitz condition with respect to s on ∂G; and, therefore, by the boundedness of the arc-chord ratio, f satisfies a Lipschitz condition on ∂G. Now, [2, Theorem 2.7] implies that if $u \in A(G)$ and $\tilde{\omega}(u, \delta, G) \leqq K\delta$, there exists a $C > 0$ such that $\omega(u, \delta, G) \leqq C \cdot \delta$. Letting $u = f$, we see from this result that f satisfies a Lipschitz condition on G. By an application of Theorem 2, the proof is complete.

COROLLARY. *Suppose ∂G is Dini-smooth and that G is convex. Then G has properties* I *and* II.

PROOF. The quickest proof of this result is to integrate the reciprocal of the left-hand inequality in (13) along line segments in $\overline{G}$ to conclude that f satisfies a Lipschitz condition on $\overline{G}$, and then appeal to the first part of the proof of Theorem 4 and to Theorem 2. However, we will show that when ∂G is smooth and rectifiable (in particular, when it is Dini-smooth), convexity of G implies bounded arc-chord ratio on ∂G—a result perhaps of some interest itself.

Suppose ∂G does not have bounded arc-chord ratio, so that there are sequences $\{\zeta_n\}$ $\{\zeta_n'\}$ in ∂G with

$$|s_n - s_n'| \geqq n\,|\zeta_n - \zeta_n'|. \tag{14}$$

Passing to a subsequence, if necessary, we may assume that $\zeta_n \to \zeta_0 \in \partial G$. Since ∂G is rectifiable, (14) implies that $\zeta_n' \to \zeta_0$ also.

Since G is convex, we may enclose ∂G in a rectangle whose sides are tangent to ∂G such that none of the four points of tangency is ζ_0. The four points of tangency divide ∂G into four arcs. Suppose ζ and ζ' lie on the same one of these arcs, and define a Cartesian coordinte system whose axes pass through the end points of this arc and are perpendicular to the sides of the rectangle. If we partition the subarc from ζ to ζ' and inscribe chords, then

$$\begin{aligned}\sum_i \sqrt{(\Delta x)_i^2 + (\Delta y)_i^2} &\leqq \sqrt{2} \sum_i (|\Delta x_i| + |\Delta y_i|) \\ &\leqq \sqrt{2}\left(\left|\sum_i \Delta x_i\right| + \left|\sum_i \Delta y_i\right|\right) \\ &\leqq 2\,|\zeta - \zeta'|,\end{aligned}$$

where the second inequality follows from the constancy in sign of Δx_i and Δy_i, a consequence of G's convexity. This means

$$|s - s'| \leqq 2\,|\zeta - \zeta'|.$$

Since this is contrary to (14), we conclude that ∂G must have bounded-arc chord ratio.

CONCLUSION. It would be desirable to discover the weakest possible geometric assumptions on G and/or ∂G which would guarantee that G has properties I and II. Is there a simple counterexample among, say, the polygonal regions G, which shows the failure of I or II?

It would also be desirable to know the best values of the constants C in I and II. One possible approach to this problem might be to prove—in the case of I, say—that

$$\limsup_{\delta \to 0} \frac{\omega(u, \delta, G)}{\tilde{\omega}(u, \delta, G)} \leqq C'$$

then show (possibly by a kind of dilation argument?) exactly how much C' must be increased to serve as a bound for all $\delta > 0$.

This work was carried out while the author was a visiting faculty member at the University of North Carolina at Chapel Hill. It is a pleasure to thank Professors Cima, Petersen, Wogen, and Roberts for many valuable conversations and suggestions. Thanks are also extended to the referee for several valuable comments.

BIBLIOGRAPHY

1. P. M. Tamrazov, Russian Math. Surveys **28** (1973), 141–173; and other papers cited therein.

2. L. A. Rubel, A. L. Shields, and B. A. Taylor, Journal of Approximation Theory **15** (1975), 23–40.

3. H. S. Shapiro, preprint of his lecture at the conference "Complex and Functional Analysis," Kristiansand, Norway, 1975.

4. G. H. Hardy and J. E. Littlewood, Math. Zeit. **34** (1931), 403–439.

5. R. Nevanlinna, *Analytic Functions*, Springer, Berlin, 1970.

6. C. Caratheodory, *Theory of Functions of a Complex Variable*, 2 vols., New York, 1954.

7. C. Pommerenke, *Univalent Functions*, Goettingen, 1975.

8. S. Warschawski, Proc. Amer. Math. Soc. **12** (1961), 614–620.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL AND AT WILMINGTON, WILMINGTON, NC 28401