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OMEGA THEOREMS FOR A CLASS OF DIRICHLET SERIES[1]

DON REDMOND

ABSTRACT. The class of Dirichlet series considered in this paper are those satisfying functional equations with multiple gamma factors. We generalize the methods of Gangadharan and Katai and Corradi to obtain omega theorems for the error terms for the summatory functions for the coefficients of these Dirichlet series. As an example we improve known estimates for the Piltz divisor problem for algebraic number fields.

1. **Introduction and Statement of Results.** The problem of determining the size of arithmetical functions is a difficult one and though much effort has been expended on this problem no final results for the general case have been obtained. In this paper we shall obtain an omega theorem for the error term of the summatory function of a clsss of Dirichlet series. The class of Dirichlet series we are concerned with consists of those satisfying a functional equation involving multiple gamma factors such as was considered by Chandrasekharan and Narasimhan in [2].

Let $\{a(n)\}$ and $\{b(n)\}$, $1 \leqq n < +\infty$, be two sequences of complex numbers, not all zero, and let $\{\lambda_n\}$ and $\{\mu_n\}$, $1 \leqq n < +\infty$, be two sequences of positive real numbers increasing to $+\infty$. Suppose that

$$f(s) = \sum_{n=1}^{\infty} a(n)\lambda_n{}^{-s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} b(n)\mu_n{}^{-s}$$

each converge in some half plane with finite abcissas of absolute convergence $\sigma_a(f)$ and $\sigma_a(g)$, respectively. Let

$$\Delta(s) = \prod_{k=1}^{N} \Gamma(\alpha_k + \beta_k), \tag{1.1}$$

where $\alpha_k > 0$ and β_k is complex, $1 \leqq k \leqq N$. Then $f(s)$ and $g(s)$ are said to satisfy the functional equation

$$\Delta(s)f(s) = \Delta(r - s)g(r - s) \tag{1.2}$$

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if there exists in the s-plane a domain D which is the exterior of a compact set S, such that in D a holomorphic function G exists with the properties:

$$(1) \qquad \lim_{|t|\to\infty} G(\sigma + it) = 0,$$

uniformly in every strip $-\infty < \sigma_1 \leqq \sigma \leqq \sigma_2 < +\infty$, and

$$(2) \qquad G(s) = \begin{cases} \Delta(s)f(s) & \text{for } \operatorname{Re}(s) > \sigma_a(f) \\ \Delta(r-s)g(r-s) & \text{for } \operatorname{Re}(s) < r - \sigma_a(g). \end{cases}$$

If

$$(1.3) \qquad Q(x) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s} x^s \, ds$$

where C is a curve enclosing all the singularities of the integrand, let

$$(1.4) \qquad E(x) = \sum_{\lambda_n \leqq x}{}' a(n) - Q(x),$$

where the prime indicates that if $\lambda_n = x$, then we add only $a(n)/2$. $E(x)$ is called the error term for the summatory function of the coefficients of the Dirichlet series $f(s)$.

Let $A = \sum_{k=1}^{N} \alpha_k$. Throughout this paper we shall assume $r > 0$ and $A > 1/2$. Also c_j, $j = 1, 2, \cdots$, will denote a positive absolute constant.

Let P be a set of prime numbers satisfying the estimate

$$(1.5) \qquad B_1 x/\log x \leqq \sum_{p \in P_x} 1 \leqq B_2 x/\log x$$

for all $x \geqq 1$ and some positive constants B_1 and B_2, where $P_x = \{p \in P : p \leqq x\}$. Let Q_x be the set of all square-free integers made up of all the primes in P_x, written in increasing order.

THEOREM 1. *Let Q be a subset of Q_x. If $a(n)$ and $b(n)$ are real, $\mu_n = c_1 n$ for all n and*

$$(1.6) \qquad \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))} \geqq c_2 x^{\alpha} \log^{\beta} x$$

for all $x \geqq 1$ and some nonnegative constants α and β, then

$$E(x) = \Omega \pm \{x^{(r/2)-(1/4A)} (\log\log x)^{\alpha} (\log\log\log x)^{\alpha+\beta}\}.$$

THEOREM 2. *Under the hypotheses of Theorem* 1, *if, instead of* (1.6), *we have*

$$(1.7)\qquad \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))} \geqq c_3 \exp(c_4 x^\alpha / \log x)$$

for all $x \geqq 1$ *and some positive constant* α, *then*

$$E(x) = \Omega \pm \{x^{(r/2)-(1/4A)} \exp(c_5 (\log\log x)^\alpha / (\log\log\log x)^{1-\alpha})\}.$$

COROLLARY. *Suppose* $b(n)$ *is a multiplicative function of* n *which satisfies*

$$(1.8)\qquad |b(p)| \geqq c_6 p^a$$

for all $p \in P$ *and some* $a > (r/2) + (1/4A) - 1$. *Then*

$$E(x) = \Omega \pm \left\{ x^{(r/2)-(1/4A)} \exp\left(c_7 \frac{(\log\log x)^{1+a-(r/2)-(1/4A)}}{(\log\log\log x)^{(r/2)+(1/4A)-a}} \right) \right\}.$$

REMARKS. (1) The method we use to prove these results generalizes the method used by Gangadharan [4], Katai and Corradi [**10**] and Joris [8].

(2) The condition that $a(n)$ and $b(n)$ be real is not essential. With a slight modification of the proof below we can get omega theorems for $\mathrm{Re}(E(x))$ and $\mathrm{Im}(E(x))$.

(3) Theorems 1 and 2 are special cases of the following more general theorem. Under the hypotheses of Theorem 1 suppose that

$$\sum_{q \in \varphi} |b(q)| q^{-((r/2)+(1/4A))} \geqq c_8 t(x)$$

for all $x \geqq 1$, where $t(x)$ is an increasing function of x. Then

$$E(x) = \Omega \pm \{x^{(r/2)-(1/4A)} t(c_9 \log\log x \log\log\log x)\}.$$

Unfortunately, we are not able to give a satisfactory proof of this result. If $t(x)$ satisfies the additional condition

$$t(x)/t(Cx) \longrightarrow D$$

as $x \longrightarrow \infty$ for all constants C, where D is some nonzero constant, then we can prove this more general theorem by a slight modificaton of the proof of Theorem 1.

2. **Lemmas.** We begin with some lemmas on Dirichlet series satisfying the functional equation (1.1).

For $\mathrm{Re}\, v > 0$ let

$$\text{(2.1)}\quad M(v) = \frac{1}{2\pi i}\int_{-(p+1/2)} \frac{\Delta((r-z)/2A)}{\Delta(z/2A)}\Gamma(z)v^{-z}\,dz,$$

where p is an integer satisfying

(1) $p \geqq -1$,

(2) $p + 2Ar + 1/2 > 2A\max(\sigma_a(g),$
$\max\{\mathrm{Re}(-\beta_k/\alpha_k) : 1 \leqq N\})$, and

(3) $p + 1/2 > 2A\max\{\mathrm{Re}(\beta_k - 1)/\alpha_k : 1 \leqq k \leqq N\}$.

For $\mathrm{Re}(s) > 0$ let

$$\text{(2.2)}\qquad R(s) = \frac{1}{2\pi i}\int_{C_1} s^{-z}\Gamma(z)f(z/2A)\,dz,$$

where C_1 is a curve enclosing all the singularities of the integrand which lie to the right of $\mathrm{Re}(z) = -(p + (1/2))$.

LEMMA 1. *Suppose $f(s)$ and $g(s)$ satisfy the functional equation* (1.1). *Then*

$$\text{(2.3)}\qquad \sum_{n=1}^{\infty} a(n)\exp(-s\lambda_n^{1/2A}) = R(s) + \sum_{m=1}^{\infty} b(m)\mu_m^{-r}M(s\mu_m^{1/2A}),$$

where the infinite series on the right hand side converges absolutely for $\mathrm{Re}(s) > 0$.

The essence of this result can be traced back to Hardy [5]. The statement and proof of Lemma 1 can found in [3, Lemma 1, p, 168].

$$\text{(2.4)}\qquad F(s) = \sum_{n=1}^{\infty} a(n)\exp(-s\lambda_n^{1/2A})$$

and

$$\text{(2.5)}\qquad G(s) = \frac{1}{s}\{F(s) - R(s)\}.$$

LEMMA 2. *Let* $\log D = \sum_{k=1}^{N} \alpha_k \log \alpha_k$, $B = \sum_{k=1}^{N} \beta_k$ *and* $H = 2A/D^{1/A}$.

(1) *If* $Y \geqq 2$, $1 \leqq \mu_m \leqq Y$, $0 < \sigma < H$, *then there is a positive constant* B_0 *such that*

$$\sigma^{Ar+(1/2)}G(\sigma \pm i\mu_m{}^{1/2A}H) = B_0\exp\{\pm\pi i(4B - Ar - (3/2))\} \cdot b(m)\mu_m{}^{(r/2)-(1/4A)} + O(\sigma^{1/4}Y^{c_{10}}) \tag{2.6}$$

as $\sigma \to 0+$.

(2) *Let* $R(k, w) = \{s : \mathrm{Re}(s) > 0,\ |s| \leqq k,\ |s \pm i\mu_m{}^{1/2A}H| > w,\ 0 < w < H\}$. *Then for* s *in* $R(k, w)$

$$G(s) = O(w^{-(Ar+(1/2))}k^{c_{11}}), \tag{2.7}$$

as $w \to 0+$.

This lemma is a special case of Lemmas 4 and 5 of [9].

We now take $2A$ to be a rational number. In view of the fact that the statements of the theorems and corollary do not place extra restrictions of λ_n, it suffices to prove these results under the assumption that $\mu_n = n$ for all n.

Let M be the cardinality of P_x and $N = 2^M$ be the cardinality of Q_x. Let Q be a subset of Q_x of cardinality N_1. Then, if $q \in Q$, we have

$$\begin{aligned} \log q &\leqq \log q_N \\ &= \sum_{j=1}^{M} \log p_j \qquad (2.8) \\ &\leqq 2B_2x, \end{aligned}$$

by (1.5).

Let

$$\tilde{\eta}(x) = \inf\left\{\eta : \eta = \left| m^{1/2A} + \sum_{\lambda=1}^{N} r_\lambda q_\lambda{}^{1/2A} \right|, \right.$$

$$\left. m = 1, 2, 3, \cdots, r_\lambda = 0, +1, -1, \sum_{\lambda=1}^{N_1} r_\lambda{}^2 \geqq 2 \right\}.$$

Then as in [5] or [8] one shows that if $q(x) = -\log \tilde{\eta}(x)$, then

$$c_{12}x \leqq q(x) \leqq \exp(c_{13}x/\log x). \tag{2.9}$$

Choose $P(x) = \exp(c_{14}x/\log x)$ such that

$$(2.10)\quad \begin{aligned}&(1)\ c_{14} > 0,\\ &(2)\ q(x) < P(x),\\ &(3)\ P(x)/x \text{ is an increasing function of } x, \text{ and}\\ &(4)\ N^2 \leqq P(x).\end{aligned}$$

That $P(x)$ exists is shown in [8].

If z is real, let $V(x) = 2\cos^2(z/2) = (e^{iz} + e^{-iz})/2 + 1$ and let

$$T(u) = \prod_{q \in Q} V(H_q^{1/2A}u + \rho_q)$$

where

$$(2.11)\qquad \rho_q = \begin{cases} \pi(4B - Ar - (3/2))/2 & \text{if } b(q) \geqq 0 \\ \pi(4B - Ar + (1/2))/2 & \text{if } b(q) < 0. \end{cases}$$

Then, from the definition of $V(z)$, we see that $T(u) \geqq 0$ for all real u. Write $T(u) = T_0(u) + T_1(u) + \overline{T_1}(u) + T_2(u)$, where

$$(2.11)\qquad \begin{aligned} T_0(u) &= 1 \\ T_1(u) &= \frac{1}{2}\sum_{q \in Q} e^{-\pi i \rho_q - Hq^{1/2A}u} \\ \overline{T_1}(u) &= \overline{T_1(u)} \\ T_2(u) &= \sum_m h_m e^{-Hi\eta_m u} \end{aligned}$$

where $|h_m| \leqq 1/4$ and the η_m are real and are the distinct numbers of the form

$$\sum_{q \in Q} r_q q^{1/2A}$$

where $r_q \in \{0, 1, -1\}$ with $\sum r_q^2 \geqq 2$. From the definition of $\tilde{\eta}(x)$, (2.8) and (2) of (2.10), we see that

$$(2.13)\qquad |\eta_j \pm \mathrm{n}^{1/2A}| \geqq e^{-P(x)}$$

for $n \geqq 0$ and every j, $1 \leqq j \leqq N_1$.

If $T(u) = \sum k_v e^{it_v u}$, where k_v are complex and t_v are real and distinct, is any trigonometric polynomial and $U(s)$ is a holomorphic function, define

$$(2.14)\qquad (T \wedge U)(s) = \sum k_v U(s + it_v).$$

Define

$$(2.15)\qquad \begin{aligned} I_\theta(s) &= s^{-\theta}, \\ \sigma_x &= e^{-2P(x)}, \\ \theta_x &= Ar - (1/2) + (1/P(x)), \text{ and} \\ \gamma_x &= \sup_{u\, 0} E(u^{2A})u^{-}x. \end{aligned}$$

Since $E(u) = \Omega_\pm(u^{(r/2)-(1/4A)})$ (see [3, Theorem 3.2]) we see that $\gamma_x > 0$. Also if $\gamma_x = +\infty$ then $E(u) = \Omega_+(u^{\theta_x/2A})$ which is better than the result claimed in Theorems 1 and 2, by (2.9) and (2) of (2.10). Thus we may assume that $0 < \gamma_x < +\infty$. Thus

$$(2.16)\qquad \gamma_x u^{\theta_x} - E(u^{2A}) \geqq 0.$$

Let

$$(2.17)\qquad \begin{aligned} J_x &= \sigma_x^{Ar+(1/2)} \int_0^\infty \{\gamma_x u^{\theta_x} - E(u^{2A})\}\, e^{-\sigma_x u} T(u)\, du \\ &= \sigma_x^{Ar+(1/2)} \{\gamma_x \Gamma(\theta_x+1) T \wedge I_{\theta_x+1}(\sigma_x) \\ &\quad - f(-p/2A) T \wedge I_1(\sigma_x) - T \wedge G(\sigma_x)\}. \end{aligned}$$

Then by (2.16) $J_x \geqq 0$.

Lemma 3. (1) *If $\theta \geqq 0$ and $\sigma < 0$, then as $x \to \infty$, we have*

$$T \wedge I_\theta(\sigma) = \sigma^{-\theta} + O(3^N e^{P(x)}).$$

(2) *With σ_x and θ_x as defined in* (2.15), *we have as $x \to \infty$,*

$$\sigma_x^{Ar+(1/2)}\{T \wedge I_{\theta_x+1}(\sigma_x)\} = e^2 + o(1).$$

(3) *As $x \to \infty$,*

$$\sigma_x^{Ar+(1/2)}\{T \wedge I_1(\sigma_x)\} = o(1).$$

(4) *As $x \to \infty$,*

$$\sigma_x^{Ar+(1/2)}\{T \wedge G(\sigma_x)\} = B_0 \sum_{q\in Q} |b(q)| q^{-((r/2)+(1/4A))} + o(1),$$

where B_0 is the constant defined in (1) *of Lemma* 2.

Proof. (1). We have $T_0 \wedge I_\theta(\sigma) - \sigma^{-\theta} = 0$. Next

$$|T_1 \wedge I_\theta(\sigma)| = \frac{1}{2} \left| \sum_{q\in Q} e^{i_{\rho q}} (\sigma - iHq^{1/2A})^{-\theta} \right|$$

$$\leqq \frac{1}{2} H^{-\theta} \left| \sum_{q \in Q} q^{-\theta/2A} \right|$$

$$\ll N_1 \leqq N,$$

as $x \to \infty$. Similarly $|\overline{T}_1 \wedge I_\theta(\sigma)| \ll N$ as $x \to \infty$.

Since $|h_m| \leqq 1/4$, we have, by (2.13) with $n = 0$, as $x \to \infty$,

$$\begin{aligned} |T_2 \wedge I_\theta(\sigma)| &= |\sum h_m(\sigma + i\eta_m)^{-\theta}| \\ &\leqq \frac{1}{4} \sum |\eta_m|^{-\theta} \\ &\ll 3^{N_1} e^{\theta P(x)} \\ &\ll 3^N E^{\theta P(x)}. \end{aligned}$$

Combining these results gives (1) by (2.12).

(2). We have by (2.12),

$$\begin{aligned} \sigma_x^{Ar+(1/2)}\{T \wedge I_{\theta_x+1}(\sigma_x)\} &= \sigma_x^{Ar+(1/2)}\{T \wedge I_{\theta_x+1}(\sigma_x) \\ &\quad - \sigma_x^{-\theta_x-1}\} + \sigma_x^{-\theta_x-1+Ar+(1/2)} \\ &= \sigma_x^{-1/P(x)} + O(\sigma_x^{Ar+(1/2)} 3^N e^{(\theta_x+1)P(x)}) \\ &= e^2 + O(3^N e^{P(x)((\theta_x+1)-2(Ar+(1/2)))}) \\ &= e^2 + O(3^N e^{-P(x)(Ar+(1/2))}) \\ &= e^2 + o(1) \end{aligned}$$

as $x \to$, by (4) of (2.10).

(3). This follows in the same way since $\sigma_x \to 0$ as $x \to \infty$, by (2.15).

(4). We have

$$\sigma_x^{Ar+(1/2)}\{T_0 \wedge G(\sigma_x)\} = \sigma_x^{Ar+(1/2)} G(\sigma_x) = o(1)$$

as $x \to \infty$, since $G(\sigma_x) = O(1)$ as $\sigma_x \to 0$ by (2.7).

By (2.14),

$$\begin{aligned} \sigma_x^{Ar+(1/2)}\{T_1 \wedge G(\sigma_x)\} &= \frac{1}{2} \sum_{q \in Q} e^{i\rho q} \sigma_x^{Ar+(1/2)} G(\sigma_x - iq^{1/2A}H) \\ &= \frac{1}{2} B_o \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))} \\ &\quad + O(\sigma_x^{(1/4)} N q_N^{c_2/2A}) \end{aligned}$$

as $x \to \infty$, by (2.6) with $Y = q_N$. By (1.5) and (2.8),

$$N \ll e^{c_{14}x/\log x} \quad \text{and} \quad q_N \ll e^{2B_2 x}.$$

Thus by (2.15) nd (4) of (2.10),

$$\sigma_x^{AR+(1/2)}\{T_1 \wedge G(\sigma_x)\} = \frac{1}{2} B_0 \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))}$$

In a similar way we have

$$\sigma_x^{Ar+(1/2)}\{\overline{T}_1 \wedge G(\sigma_x)\} = \frac{1}{2} B_0 \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))} + o(1)$$

as $x \to \infty$.

Finally,

$$\sigma_x^{Ar+(1/2)}\{T_2 \wedge G(\sigma_x)\} = \sigma_x^{Ar+(1/2)} \sum_m h_m G(\sigma_x + i\eta_m).$$

Now

$$|\sigma_x + i\eta_m \pm iHn^{1/2A}| \geqq |\eta_m \pm Hn^{1/2A}| \geqq He^{-P(x)}$$

for $m \geqq 0$ by (2.13). Also, by (2.10) we have, for x sufficiently large,

$$|\sigma_x + i\eta_m| \leqq \sigma_x + Nx^{1/2A} \leqq 2Nx^{1/2A}.$$

Thus by (2.7) with $w = He^{-P(x)}$ and $k = 2Nx^{1/2A}$, we have

$$\begin{aligned} &\sigma_x^{Ar+(1/2)}\{T_2 \wedge G(\sigma_x)\} \\ &\qquad \ll \sigma_x^{Ar+(1/2)} x^{c_{11}/2A} e^{(Ar+(1/2))P(x)} 3^N N^{c_{11}} \\ &\qquad \ll e^{-(Ar+(1/2)P(x)} x^{c_{11}/2A} N^{c_{11}} 3^N \\ &\qquad = o(1) \end{aligned}$$

as $x \to \infty$ by (4) of (2.10) and (2.15). Combining these results gives (4) by (2.12).

This completes the proof of the lemma.

3. **Proof of Theorem 1.** If we combine the results of Lemma 3 with the expansion of J_x, (2.17), we have as $x \to \infty$

$$\begin{aligned} &(e^2 + o(1))\gamma_x^{\Gamma(Ar+(1/2)+(1/P(x))} \\ &\qquad \geqq B_0 \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))} + o(1), \end{aligned} \tag{3.1}$$

since $J_x \geqq 0$. Now $Ar + (1/2) + (1/P(x))$ is positive and bounded away from zero. Also, by (3) of (2.10), $1/P(x) \to 0$ as $x \to \infty$. Thus, by (3.1), we have

$$\gamma_x \geqq c_{15} \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))} + o(1) \tag{3.2}$$

as $x \to \infty$. Thus by the definition of γ_x, (2.15), there is a sequence $u_x \to \infty$ as $x \to \infty$ such that

$$\begin{aligned} E(u_x{}^{2A}) u_x{}^{-\theta x} &\geqq c_{15} \sum_{q \in Q} |b(q)| q^{-((r/2)+(1/4A))} \\ &\geqq c_{16} x^{\alpha} \log^{\beta} x \end{aligned} \tag{3.3}$$

by (1.5).

Let $v_x = u_x{}^{1/P(x)}$. Then $(2 \log u_x)/P(x) = 2 \log v_x$. If $2 \log v_x \leqq 1$, then $2 \log u_x \leqq P(x)$. Since $P(x)$ is increasing by (3) of (2.10), we have $P^{-1}(2 \log u_x) \leqq x$, where P^{-1} denotes the functional inverse of P. If $2 \log v_x \geqq 1$, then by (3) of (2.10),

$$\frac{P(x)}{x} \cdot \frac{P(2x \log v_x)}{2 \log u_x} = \frac{P(2x \log v_x)}{2 \log v_v} \cdot \frac{P(x)}{x}.$$

Thus $P(2x \log v_x) \geqq 2 \log u_x$, for x sufficiently large and hence $P^{-1}(2 \log u_x) \leqq 2x \log v_x$. If we let $w_x = \max(1, 2 \log v_x)$, we may write these last two results as

$$P^{-1}(2 \log u_x) \leqq x w_x. \tag{3.4}$$

For x sufficiently large, we have

$$w_x \leqq c_{17} v_x{}^{1/(\alpha+\beta+\epsilon)}, \tag{3.5}$$

where $\alpha + \beta + \epsilon > 0$ and $\epsilon > 0$. Finally, for x sufficiently large we have

$$\begin{aligned} \log P^{-1}(2 \log u_x) &\leqq \log x + \log w_x \\ &< \log x + w_x{}^{(1/2)} \\ &\leqq 3 w_x{}^{(1/2)} \log x. \end{aligned} \tag{3.6}$$

Thus by (3.4)–(3.6) we have

$$
\begin{aligned}
&\frac{E(u_x^{2A})}{u_x^{Ar-(1/2)}} \cdot \frac{1}{\{P^{-1}(2\log u_x)\}^{\alpha}\{\log P^{-1}(\log u_x)\}^{\beta}} \\
&= \frac{E(u_x^{2A})}{u_x^{\theta_x}} \cdot \frac{u_x^{1/P(x)}}{\{P^{-1}(2\log u_x)\}^{\alpha}\{\log P^{-1}(2\log u_x)\}^{\beta}} \\
(3.7)\qquad &> c_{17}x^{\alpha}\log^{\beta}x \quad \frac{v_x}{(xw_x)^{\alpha}(w_x^{(1/2)}\log x)^{\beta}} \\
&= c_{17}v_x w_x^{-(\alpha+(\beta/2))} \\
&> c_{18}v_x^{1-(\alpha+(\beta/2))(\alpha+\beta+\epsilon)} \\
&> c_{19},
\end{aligned}
$$

since $q_x \ll 1$. This gives

$$
(3.8)\qquad E(x) = \Omega_{\pm}(x^{(r/2)-(1/4A)}\{P^{-1}(\log x)\}^{\alpha}\{\log P^{-1}(\log x)\}^{\beta}).
$$

As in [5] one shows that there exist constants c_{20} and c_{21} such that, for x sufficiently large, we have

$$
(3.9)\qquad e_{20}\log x \log\log x < P^{-1}(x) < c_{21}\log x \log\log x.
$$

Combining (3.8) and (3.9) gives

$$
E(x) = \omega_{+}(x^{(r/2)-(1/4A)}(\log\log x)^{\alpha}(\log\log\log x)^{\alpha+\beta}).
$$

The Ω_{-} result is proved similarly.
This completes the proof of Theorem 1.

4. **Proof of Theorem 2.** We proceed as in the proof of Theorem 1 up to (3.3), where we assert the existence of the sequence u_x. This is replaced by

$$
\begin{aligned}
&u_x^{(1/2)-Qr}E(u_x^{2A}) \geqq u_x^{1/P(x)} \sum_{q\in Q} |b(q)| q^{-((r/2)+(1/4A))} \\
(4.1)\qquad &\geqq c_{22}\exp\{c_4 x^{\alpha}\log x + (\log u_x)/P(x)\}
\end{aligned}
$$

by (1.7).
Suppose

$$
(4.2)\qquad (\log u_x)/P(x) \leqq c_4 x^{\alpha}/\log x.
$$

Then by the definition of $P(x)$, we have

$$
(4.3)\qquad \log\log u_u \leqq c_{23}x/\log x.
$$

Then, since $y^a\log^b y$ is an increasing function of y for $a > 0$ and y sufficiently large, we have from (4.3), with $a = \alpha$ and $b = \alpha - 1$,

$$\frac{(\log\log u_x)^{\alpha}}{(\log\log\log u_x)^{1-\alpha}} \leqq c_{24}\,\frac{x^{\alpha}}{\log x}. \tag{4.4}$$

Then, under the assumption of (4.2), we have by (4.1) and (4.4),

$$u_x^{-(Ar-(1/2)E(u_x^{2A})} \geqq c_{22}\exp\left\{ c_{25}\,\frac{(\log\log u_x)^{\alpha}}{(\log\log\log u_x)^{1-\alpha}} \right\}.$$

This gives the Ω_+ result of Theorem 2 under the assumption (4.2).

Suppose

$$(\log u_x)/P(x) \geqq c_4 x^{\alpha}/\log x. \tag{4.5}$$

If

$$\frac{(\log\log u_x)^{\alpha}}{(\log\log\log u_x)^{1-\alpha}} \leqq c_{26}\,\frac{\log u_x}{P(x)}, \tag{4.6}$$

then by (4.5) we again obtain the Ω_+ result of Theorem 2. If not, then take logs once on both sides of (4.6) with the inequality reversed. This gives

$$\log\log u_x \leqq c_{27} x/\log x. \tag{4.7}$$

Then, as above for (4.4), (4.7) implies

$$\frac{(\log\log u_x)^{\alpha}}{(\log\log\log u_x)^{1-\alpha}} \leqq c_{28}\,\frac{x^{\alpha}}{\log x}. \tag{4.8}$$

This, by (4.5), yields the Ω_+ result of Theorem 2.

The Ω_- result is proved similarly.

This proves Theorem 2.

5. **Proof of the Corollary.** We have, by (1.8),

$$\begin{aligned}
&\sum_{q\in Q_x} |b(q)|q^{-((r/2)+(1/4A))}\\
&= \prod_{p\in P_x} (1 + |b(p)|p^{-((r/2)+(1/4A))}),\\
&= \exp\left\{ \sum_{p\in P_x} \log(1 + |b(p)|p^{-((r/2)+(1/4A))}) \right\}\\
&\geqq \exp\left\{ \sum_{p\in P_x} \log(1 + c_6 p^{\alpha-(r/2)-(1/4A)}) \right\}
\end{aligned}$$

$$\geqq c_{29}\exp\left\{\sum_{p\in P_x}\left(c_6 p^{a-(r/2)-(1/4A)}-\frac{1}{2}c_6{}^2 p^{2a-rx(1/2A)}\right)\right\}$$

$$\geqq c_{29}\exp\left\{c_{30}\sum_{p\in P_x} p^{-((r/2)+(1/4A)-a)}\right\}$$

$$\geqq c_{31}\exp\{(c_{32}x^{1-((r/2)+(1/4A)-a)}/\log x)\}$$

by (1.5).

The result of the corollary then follows from Theorem 2 by taking $\alpha = 1 + a - (r/2) - (1/4A)$.

This completes the proof of the corollary.

6. **Examples.** We give four examples of Theorems 1 and 2 and the Corollary.

EXAMPLE 1. The Piltz divisor problem in algebraic number fields. Let K be an algebraic number field of degree $n = r_1 1 + 2r_2$, where r_1 is the number of real conjugates and $2r_2$ is the number of imaginary conjugates of K. Let $a(m)$ be the number of integral ideals of K with norm exactly m. For $\mathrm{Re}(s) > 1$, let

$$\zeta_K(s) = \sum_{m=1}^{\infty} a(m)m^{-s}$$

be the Dedekind zeta function associated with K. If g is a positive integer, let $g_1 = gr_1$, $g_2 = gr_2$ and $\varphi(s) = \{\zeta_K(s)\}^g$. Then it is known [11, p. 22] that there is a positive constant B, depending only on K, such that $f(s) = B^{gs}\,\varphi(s)$ satisfies the functional equation

$$\Delta(s)f(s) = \Delta(1-s)f(1-s),$$

where

$$\Delta(s) = \{\Gamma(s/2)\}^{g_1}\{\Gamma(s)\}^{g_2}.$$

We have

$$f(s) = B^{gs}\sum_{m=1}^{\infty} a^{*g}(m)m^{-s},$$

where $a^{*g}(m)$ denotes the g^{th} power Dirichlet convolution of $a(m)$ with itself. Here we take $r = 1$ and $A = gn/2$.

In [8, Lemma 6, p. 228] it is shown that there is a set of primes P satisfying (1.5) and such that $a(p) \geqq 1$ for all $p \in P$. Thus we may take

$a = 0$ in the Corollary. Thus if $E(x)$ is the error term associated with $f(s)$ by (1.4), we have, as $x \to \infty$,

$$E(x) = \Omega_{\pm}\left\{x^{(1/2)-(1/2gn)}\exp\left(c_{33}\frac{(\log\log x)^{(1/2)-(1/2gn)}}{(\log\log\log x)^{(1/2)+(1/2gn)}}\right)\right\}.$$

This betters the result of Berndt [1, Example 1, p. 201]. If we take $n = 1$ and $g = 2$ we have a result of Corradi and Katai [10, (1.8), p. 90]. For $n \geqq 2$ and $g = 1$ we have a result of Joris [8, Satz 1, p. 220].

EXAMPLE 2. Let $Q(x_1, \cdots, x_k)$ be a positive definite quadratic form in $k \geqq 2$ variables. Let $a(Q, n)$ denote the number of representations of n by the form Q. For $\mathrm{Re}(s) > k/2$, the Epstein zeta function is defined by

$$\zeta(Q, s) = \sum_{n=1}^{\infty} a(Q, n)n^{-s}.$$

Then $\zeta(Q, s)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(Q, s) = |Q|^{-1/2}\pi^{-((k/2)-s)} \cdot \Gamma((k/2) - s)\zeta(Q^{-1}, (k/2) - s),$$

where $|Q|$ is the determinant of Q and Q^{-1} is the inverse form to Q. Here we take $r = k/2$ and $A = 1$.

For even k Hecke in [7] and for odd k Petersson in [12] have shown that, for $k \geqq 5$,

$$a(Q, n) = A(Q, n)n^{(k/2)-1} + O(n^{k/4})$$

as $n \to \infty$, where $A(Q, n)$ is the singular series associated with Q. By a result of Tartakowsky [14] we know that $A(Q, n) \neq 0$ if $k \geqq 5$ and n belongs to certain residue classes determined by the form Q. Thus, for these n, we have

$$a(Q, n) \geqq c_{34}n^{(k/2)-1}.$$

If we let $P_k(x)$ be the error term associated with $\zeta(Q, s)$ then Theorem 1 gives, for $k \geqq 5$,

$$P_k(x) = \Omega_{\pm}\{(x \log\log x \log\log\log x)^{(k-1)/4}\}.$$

This extends the results of Szegö [13].

In the case that $Q(x_1, \cdots, x_k) = x_1^2 + \cdots + x_k^2$ we can obtain better results for the values $k = 2$, 4, and 8. For these values it is known [6] that $a(Q, n)/2k$ is a multiplicative function. Also in [6] it is shown that if $k = 2$ and $p \equiv 1 \pmod 4$, then $a(Q, n) = 1$ and for $k = 4$ and 8, that $A(Q, n) \neq 0$ for all n. Thus we may take $a = (k/2) - 1$ in the Corollary for these values of k. This gives

$$P_k(x) = \Omega_\pm \left\{ x^{(k-1)/4} \exp \left(c_{35} \frac{(\log \log x)^{(k-1)/4}}{(\log \log \log x)^{(5-k)/4}} \right) \right\}.$$

For $k = 2$ we have a result of Corradi and Katai [**10**, (1.7),p. 90].

However, for such Q, Walfisz in [**15**] has obtained better results for $k \geqq 4$. He shows that

$$P_4(x) = \Omega_\pm(x \log \log x)$$

and for $k \geqq 5$

$$P_k(x) = \Omega_\pm(x^{(k/2)-1}).$$

By dividing the values of k up into various residue classes modulo 8 he shows that this result is best possible for $k \geqq 7$, in the sense that there exist positive absolute constants C and D, depending only on k, such that

$$Cx^{(k/2)-1} < P_k(x) < Dx^{(k/2)-1}.$$

Example 3. Let $\{a(m)\}$ be a sequence of real numbers which are the coefficients of a cusp form of weight k with an Euler product. Here we have $r = k$ and $A = 1$.

As in [8, § 7], we can show that for all $x \geqq 1$ we have

$$\sum_{q \leqq x} |a(q)| q^{-((k/2)-(1/4))} \geqq c_{36} \log x.$$

Thus, by Theorem 1 with $\alpha = 0$ and $\beta = 1$, we have

$$\sum_{m \leqq x} a(m) = \Omega_\pm(x^{(k/2)-(1/4)} \log \log \log x).$$

This result was obtained by Joris in [9].

Example 4. Let $\sigma_v(n)$ denote the sum of the v^{th} powers of the divisors of n. Since $\sigma_0(n) = d(n)$, which we dealt with in Example 1, and $\sigma_{-v}(n) = n^{-v}\sigma_v(n)$, we may assume $v > 0$. For $\operatorname{Re}(s) > v + 1$, we have

$$\sum_{n=1}^{\infty} \sigma_v(n) n^{-s} = \zeta(s)\zeta(s - v).$$

Here we have $r = v + 1$ and $A = 1$.

We have $\sigma_v(p) > p^v$ for all p. Thus, in the Corollary we may take $a = v$. This gives

$$S_v(x) = \Omega_\pm\{x^{(v/2)+(1/4)} \cdot \exp(c_{38}(\log\log x)^{(v/2)+(1/4)}(\log\log\log x)^{(v/2)-(3/4)})\},$$

where $S_v(x)$ is the error term associated with the coefficients $\sigma_v(n)$. This result improves a result of Berndt [1, Example 3, p. 202].

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Department of Mathematics, Southern Illinois University,
Carbondale, Illinois 62901