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A LIPSCHITZ INVARIANT OF NORMED LINEAR SPACES RELATED TO THE ENTROPY NUMBERS[1]

C. BESSAGA

ABSTRACT. An invariant of sigma-compact metric spaces under Lipschitz homeomorphisms is defined and used to construct an example of a separable infinite-dimensional normed linear space which is not Lipschitz homeomorphic to its subspace of codimension one.

0. A. Pelczynski [10] and independently A. N. Kolmogorov [6] defined linear-topological invariants of topological vector spaces based on estimates of the rank of growth of cardinalities $N_\varepsilon(K)$ of minimal ε-nets of compact subsets K of the space. The Kolmogorov's invariant is called the approximative dimension. The author [2, p. 282] has shown that the approximative dimension is in fact an invariant under uniform homeomorphisms of topological vector spaces and extends to an invariant of uniform spaces. The approximative dimension of a locally convex space is trivial, unless the space is of Schwartz type (cf. [4]). Therefore, in particular, it can not be used for distinguishing infinite-dimensional normed linear spaces.

Here we define an invariant of normed linear spaces under Lipschitz homeomorphisms which is based on a similar idea. (A Lipschitz homeomorphism between metric spaces is a homeomorphism h such that both h and h^{-1} satisfy the Lipschitz condition.) Instead of $N_\varepsilon(K)$ we use a more convenient measure of compactness, the entropy number $e_n(K)$ which is the infimum of positive numbers ε such that an ε-net for K of cardinality 2^n exists, introduced by A. Pietsch [9] in connection with the study of operator ideals. Our construction also extends some ideas of Rolewicz [11] and Dubinsky [5] and provides an example of an infinite-dimensional normed linear space which is not Lipschitz homeomorphic to its closed linear subspace of codimension one. (Examples of this kind, but related to linear homeomorphisms can be found in [5] and [11]).

1. Let K be a compact subset of a metric space. The nth entropy number $e_n(K)$ is the infimum of the positive ε's such that there exists an ε-net for K consisting of 2^n points of K. Let $e(K)$ be the class of all sequences (a_n) of positive numbers such that $\lim a_n/e_n(K) = 0$. For any sigma-compact metric space X we let

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$$\eta(X) = \bigcap \bigcup_{n=1}^{\infty} e(A_n)$$

with the intersection taken over all countable compact covers (A_n) for X.

LEMMA. *If L and K are compacta such that either $L \subset K$ or L is an image of K under a Lipschitz map, then $e(L) \subset e(K)$.*

PROOF. If $L \subset K$, we easily check that $e_n(L) \leqq 2e_n(K)$ for $n = 1, 2, \ldots$. If $L = g(K)$, where g satisfies Lipschitz condition with constant C, then obviously $e_n(L) \leqq Ce_n(K)$ for $n = 1, 2, \ldots$. Thus in both cases $e(L) \subset e(K)$.

The following is an immediate consequence of the Lemma.

PROPOSITION 1. *If X and Y are sigma-compact metric spaces and either X is a closed subspace of Y or X is an image of Y under a Lipschitz map, then $\eta(X) \subset \eta(Y)$. In particular, if X and Y are Lipschitz homeomorphic, then $\eta(X) = \eta(Y)$.*

Now we are going to prove our main result:

PROPOSITION 2. *If X is a normed linear space generated by a compact convex set K, i.e., $X = \bigcup_{n=1}^{\infty} nk$, then $\eta(X) = e(K)$.*

PROOF. Since $\|nx - ny\| = n\|x - y\|$ for $x, y \in x$, we have $e(nK) = e(K)$ for $n = 1, 2, \ldots$. Hence $\eta(X) \subset \bigcup_{n=1}^{\infty} e(nK) = e(K)$.

To obtain the other inclusion, assume $X = \bigcup_{n=1}^{\infty} A_n$, where all A_n are compact. Regarding K as a (complete) space and applying the Baire theorem we conclude that one of the sets $K \cap A_n$, say $K \cap A_{n_0}$, contains a relative open ball $B(x_0, \varepsilon) \cap K$ centred at a point $x_0 \in K$ of radius ε. The translate $K - x_0$ is compact, and hence bounded. Therefore there exists a $\delta > 0$ such that $\delta(K - x_0) \subset B(0, \varepsilon)$. Since K is convex, we conclude: $x_0 + \delta(K - x_0) \subset B(x_0, \varepsilon) \cap K \subset A_{n_0}$, and since the norm metric is translation-invariant and homogeneous, we get

$$e(K) = e(x_0 + \delta(K - x_0)) \subset e(A_{n_0}).$$

Hence $\bigcup_n e(A_n) \supset e(K)$ for each compact cover (A_n) of X. Therefore $\eta(X) = \bigcap \bigcup_{n=1}^{\infty} e(A_n) \supset e(X)$.

COROLLARY. *There exists an infinite-dimensional normed linear space X which is not Lipschitz homeomorphic to its closed linear subspace of codimension one.*

PROOF. Let X be the space of real sequences $x = (x_n)_{n=0}$ for which $\sup_{n\geqq 0} |2^{2^n} x_n| < \infty$, equipped with the norm

$$\|x\| = \sup_{n\geqq 0} |x_n|, \tag{1}$$

and let $X_0 = \{x \in X\colon x_0 = 0\}$. Obviously X is generated by the compact convex set

$$A = \{x \in X\colon |x_n| \leqq 2^{-2^n} \text{ for } n = 0, 1, 2, \ldots\}$$

and X_0 is generated by $A_0 = A \cap X_0$.

Given a $k \geqq 1$, let $Y_k = \{x \in X\colon x_i = 0 \text{ for } i \geqq k\}$, $\varepsilon_k = 1/2^{2k}$. Let $V_1, \ldots, V_{N_k}$ be the relative closed balls (=cubes) of radius ε_k in $A_0 \cap Y_k$ which constitute the cubical division of $A_0 \cap Y_k$, i.e., their interiors are pair-wise disjoint and $\bigcup_i V_j = A_0 \cap Y_k$. We compute

$$\begin{aligned} N_k &= (2^{2^k}/2^{2^1})(2^{2^k}/2^{2^2}) \ldots (2^{2^k}/2^{2^k}) = (2^{2^k})^k/2^{(2^1+\cdots+2^k)} \\ &= 2^{k2^k}/2^{2^{k+1}-2} = 2^{(k-2)2^k+2}. \end{aligned}$$

Since for each $j \geqq k$, the jth coordinate of any point of A_0 has absolute value $\leqq 2^{-2^k}$, we conclude that for every $\varepsilon > \varepsilon_k$ the centers of the balls $V_1, \ldots, V_{N_k}$ constitute an ε-net for A_0. Hence

$$e_{(k-2)2^k+2}(A_0) \leqq 2^{-2^k}.$$

Now suppose that $\{z_1, \ldots, z_n\} \subset A$ $(N = 2^{(k-2)2^k+2})$ is an ε-net for A. Then the closed cubes $W_n = \{x \in X\colon x - z_n \leqq \varepsilon\} \cap Y_k$ $(n = 1, 2, \ldots, N)$ cover the set $A \cap Y_k$. Therefore the sum of their k-dimensional volumes is greater than or equal to the volume of $A \cap Y_k$, i.e.,

$$2^{(k-2)2^k+2}(2\varepsilon)^k \geqq (2\cdot 2^{-2^0})(2\cdot 2^{-2^1}) \ldots (2\cdot 2^{-2^{k-1}})$$

or

$$2^{(k-2)2^k+2}\,\varepsilon^k \geqq 2^{-(2^0+\cdots+2^{k-1})},$$

whence

$$\varepsilon \geqq 2^{(2^k-1)/k}2^{-2^k}.$$

This gives

$$e_{(k-1)2^k+2}(A) \geqq 2^{(2^k-1)/k}\cdot 2^{-2^k}.$$

Hence

$$\limsup_{n\to\infty} e_n(A)/e_n(A_0) \geqq \lim_k 2^{(2^k-1)/k} = \infty.$$

Therefore $\eta(X) = e(A) \neq e(A_0) = \eta(X_0)$; X and X_0 are not Lipschitz homeomorphic.

Remark. It can easily be shown that there exists a family of cardinality continuum of compact cubes in the space l_∞ which are differentiated by the invariant $e(\cdot)$. Hence the normed linear spaces generated by those cubes (in the l_∞ norm) constitute a family of cardinality continuum of separable normed linear spaces which are distinct with respect to Lipschitz homeomorphisms. However all these spaces are homeomorphic to each other (see [**2**, p. 274]).

2. **Open problems.** The questions below concern normed linear spaces generated by compact convex sets.

A. Give an example of a pair of spaces X, Y such that $\eta(X) = \eta(Y)$ but X and Y are not Lipschitz homeomorphic.

B. Assume that $\eta(X) = \eta(Y)$. Do there exist compact convex sets K, L generating X and Y, respectively, such that K is Lipschitz homeomorphic to L ?

C. Assume that X and Y are Lipschitz homeomorphic. Does this imply that X and Y are linearly homeomorphic?

In connection with the last problem one should try to find whether the classical Rademacher theorem on almost everywhere differentiability of Lipschitz maps extends to sigma-compact normed linear spaces. Let us mention that some extensions of the Rademacher theorem to infinite dimensions, but assuming the completeness of the spaces, have been obtained by Mankiewicz [**7**], [**8**] and Aronszajn [**1**].

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INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, WARSAW, POLAND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48104