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ORDER CONVERGENCE IN LATTICES

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0. **Introduction.** There are two essentially different possibilities to define convergence of a real sequence: the topological one, using open neighbourhoods, and the order-theoretical one, involving the notions of limit inferior and limit superior. The latter can be generalized to sequences or—since sequences are often inadequate—to nets or filters in arbitrary partially ordered sets. The net-theoretical generalization was developed by G. Birkhoff [**1**], O. Frink [**3**], B.C. Rennie [**11**], and others. Some years later, about 1954, the study of order convergence in terms of filters was started by A.J. Ward [**13**]. Ward's method was continued and generalized by D.C. Kent (cf. [**7, 8, 9**]) whose general filter-convergence theory provides an elegant and powerful method to describe and develop the theory of order convergence. However, until now, nearly all deeper results have been formulated and derived in the language of nets, often requiring rather complicated proof techniques (see, for example, [**1**], [**5, 6**], [**3**], [**11**]). Our purpose is to unify (and in some cases to correct) various results from the literature, and to complete them by giving several generalizations and new results, all formulated in the language of filters.

In §1, we compose the most important definitions and abbreviations concerning partially ordered sets and convergence theory.

In §2, we give several alternative characterizations of order convergence and show that it is always a localized convergence relation. Since it is well known that, in general, order convergence fails to be a limitierung, the question arises under which circumstances it is a limitierung, a pretopological or a topological convergence relation. For lattices, we answer this in §4.

A useful help for these investigations are the so-called conoids (cf. [**6**]) and ray filters (being in one-to-one correspondence to conoids): to any filter $\mathfrak{F}$, we can assign a ray filter such that $\mathfrak{F}$ order-converges to a point x if and only if the corresponding ray filter does. Hence, order convergence is completely described by the behaviour of ray filters (or conoids, respectively). A special class of ray filters is formed by the so-called interval filters, which possess a base of (closed bounded) intervals: the interval filters are just the bounded ray filters. Since every order-

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convergent filter is bounded, the investigations may be confined to the behaviour of interval filters.

Using the concept of conoids, we can prove that order convergence of a lattice is a limitierung if and only if the intersection of any two ideals with join x has also join x, and dually. An analogous condition is given for order convergence to be pretopological. Furthermore, the theory of conoids yields a description of the order topology in terms of ideals, without involving the convergence structure at all.

In general, it is not true that order convergence agrees with convergence in the order topology for all filters of a lattice. However, this coincidence holds for all interval filters. This is surprising enough in view of the fact that order convergence is completely determined by the convergence behaviour of interval filters.

Denoting by $\mathfrak{B}(x)$ the intersection of all filters order-converging to x and by $\mathfrak{U}(x)$ the neighbourhood filter in the order topology, we shall derive the following characterizations of pretopological and topological order convergence, respectively: The order convergence of a lattice is pretopological if and only if each $\mathfrak{B}(x)$ is an interval filter, and it is topological if and only if the same holds for each $\mathfrak{U}(x)$. Thereby, we generalize several results and correct some errors in two papers of A. Gingras [5, 6]. Moreover, most statements on interval filters are generalized to the much larger class of so-called pre-interval filters possessing a base of sets each of which has a greatest and a least element. These generalizations are finally applied to extend a result of D.P. Strauss on topological lattices.

1. **Notations and basic definitions.** The lattice of all filters on a set X is denoted $\mathbf{F}_0(X)$. Any filter $\mathfrak{F}$ with $\emptyset \notin \mathfrak{F}$ (i.e., $\mathfrak{F} \neq 2^X$) is referred to as a *proper filter*. By $\mathbf{F}(X)$, we denote the collection of all proper filters on X. For any set system $\mathfrak{Y} \subset 2^X$, $[\mathfrak{Y}]$ denotes the filter generated by $\mathfrak{Y}$. (Note that $[\mathfrak{Y}]$ is a proper filter only if any finite subsystem of $\mathfrak{Y}$ has a nonempty intersection.) For $x \in X$,

$$[x] := [\{\{x\}\}]$$

is the *principal ultrafilter generated by x*.

Partially ordered sets ("posets") will be indicated by the letter P, lattices by the letter L. For the corresponding partial orderings, we always use the symbol $\leqq$. If a subset $Y \subset P$ has a join (i.e., a least upper bound), then this is denoted $\bigvee Y$. Dually, a meet is indicated by the symbol $\bigwedge Y$. In a lattice, we write $y \vee z$ for $\bigvee \{y, z\}$ and $y \wedge z$ for $\bigwedge \{y, z\}$. Furthermore, we put

$$Y \vee Z := \{y \vee z \mid y \in Y, z \in Z\},$$
$$y \vee Z := \{y\} \vee Z = \{y \vee z \mid z \in Z\}$$

for $y \in L$ and subsets $Y, Z \subset L$. For any two elements y, z of a poset P, $y^* := \{x \in P \mid y \leqq x\}$ is a *closed upper ray*, $z^+ := \{x \in P \mid x \leqq z\}$ is a *closed lower ray*, and $[y, z] := y^* \cap z^+$ is a (*closed bounded*) *interval.* Notice that $[y, z]$ is nonempty if and only if $y \leqq z$.

In the following, all filters will be assumed to be proper unless we agree to admit the non-proper filter 2^X. (For studies on order convergence, this filter is not of interest.) A relation $\rightarrow$ between filters and points of a set X is called a *convergence relation* on X if it satisfies

($C1$) $[x] \rightarrow x$ for all $x \in X$, and

($C2$) $\mathfrak{F} \rightarrow x$ and $\mathfrak{F} \subset \mathfrak{G} \in \mathbf{F}(X)$ imply $\mathfrak{G} \rightarrow x$.

If, in addition,

($C3$) $\mathfrak{F} \rightarrow x$ implies $\mathfrak{F} \cap [x] \rightarrow x$ for all $x \in X$,

then we speak of a *localized convergence relation.* A *limitierung* is a convergence relation satisfying

($C4$) $\mathfrak{F} \rightarrow x$ and $\mathfrak{G} \rightarrow x$ imply $\mathfrak{F} \cap \mathfrak{G} \rightarrow x$.

It is *pretopological* if, moreover,

($C5$) $\mathfrak{F} \rightarrow x$ for all $\mathfrak{F} \in \mathbf{F}$ implies $\bigcap \mathbf{F} \rightarrow x$ ($\emptyset \neq \mathbf{F} \subset \mathbf{F}(X)$).

Finally, a convergence relation $\rightarrow$ on X is *topological* if it coincides with convergence in a suitable topology $\mathfrak{T}$ on X, in other words, if there exists a topology $\mathfrak{T}$ on X such that

$$\mathfrak{F} \rightarrow x \Leftrightarrow \mathfrak{F} \xrightarrow[\mathfrak{T}]{} x \Leftrightarrow \mathfrak{U}_{\mathfrak{T}}(x) \subset \mathfrak{F}$$

where $\mathfrak{U}_{\mathfrak{T}}(x)$ is the neighbourhood filter of x in $\mathfrak{T}$. For any convergence relation $\rightarrow$ on X and every $x \in X$, $\mathfrak{V}(x) := \bigcap\{\mathfrak{F} \mid \mathfrak{F} \rightarrow x\}$ is a filter on X, called the *convergence-neighbourhood filter* of x. Obviously, $\rightarrow$ is pretopological if and only if $\mathfrak{V}(x) \rightarrow x$ for all $x \in X$ The set system $\mathfrak{T} := \{Y \subset X \mid x \in Y \text{ and } \mathfrak{F} \rightarrow x \text{ imply } Y \in \mathfrak{F}\}$ is a topology on X, and the convergence relation $\xrightarrow[\mathfrak{T}]{}$ defined by

$$\mathfrak{F} \xrightarrow[\mathfrak{T}]{} x \Leftrightarrow \mathfrak{U}_{\mathfrak{T}}(x) \subset \mathfrak{F}$$

is the least topological convergence relation containing $\rightarrow$. In other words, $\mathfrak{T}$ is the finest topology such that $\mathfrak{F} \rightarrow x$ implies $\mathfrak{F} \xrightarrow[\mathfrak{T}]{} x$. From this observation it follows easily that $\mathfrak{U}_{\mathfrak{T}}(x)$ is always contained in $\mathfrak{V}(x)$. Furthermore, the convergence relation $\rightarrow$ is topological if and only if $\mathfrak{U}_{\mathfrak{T}}(x) \rightarrow x$ for all $x \in X$. For these and further basic results on general filter convergence, the reader may consult D. Kent's paper [7]. (We note that Kent's theory bases on the concept of convergence functions instead of convergence relations; see also H.R. Fischer [2], H.-J. Kowalsky [10], and others. However, it is easy to check that both concepts are equivalent.)

2. **Order convergence.** Let P be a poset. For $Y \subset P$ and $\mathfrak{Y} \subset 2^P$, we define $Y^+ := \{x \mid x \leqq y \text{ for all } y \in Y\}$, $\mathfrak{Y}^+ := \bigcap \{Y^+ \mid Y \in \mathfrak{Y}\}$. In accordance with the previous definition, we have $\{y\}^+ = y^+$. Y^* and $\mathfrak{Y}^*$ are defined dually. Note that Y^+ and Y^* are the sets of all lower and upper bounds of Y, respectively. Now we say a filter $\mathfrak{F}$ on P *order-converges* to a point $x \in P$, in symbols $\mathfrak{F} \underset{0}{\to} x$, if $\bigvee \mathfrak{F}^+ = x = \bigwedge \mathfrak{F}^*$. We compose some simple rules concerning the operators $^+$ and *.

LEMMA 2.1. *Let* $y \in P$, $Y, Z \subset P$, $\mathfrak{F}, \mathfrak{G} \in \mathbf{F}(P)$, *and* $\mathbf{F} \subset \mathbf{F}(P)$. *Then*
(1) $Y \subset Z$ *implies* $Y^+ \supset Z^+$ *and* $Y^* \subset Z^*$;
(2) $Y^+ = \{x \mid Y \subset x^*\}$, $Y^* = \{x \mid Y \subset x^+\}$;
(3) $Y \subset Y^{*+}$, $Y \subset Y^{+*}$;
(4) $\bigwedge Y = y$ *if and only if* $Y^+ = y^+$, $\bigvee Y = y$ *if and only if* $Y^* = y^*$;
(5) $y^+ = y^{*+}$, $y^* = y^{+*}$;
(6) $\mathfrak{F} \subset \mathfrak{G}$ *implies* $\mathfrak{F}^+ \subset \mathfrak{G}^+$ *and* $\mathfrak{F}^* \subset \mathfrak{G}^*$;
(7) $\mathfrak{F}^+ = \{x \mid x^* \in \mathfrak{F}\}$, $\mathfrak{F}^* = \{x \mid x^+ \in \mathfrak{F}\}$;
(8) $\mathfrak{F}^{*+} = \bigcap \{x^+ \mid x^+ \in \mathfrak{F}\}$, $\mathfrak{F}^{+*} = \bigcap \{x^* \mid x^* \in \mathfrak{F}\}$;
(9) $\mathfrak{F}^* \subset \mathfrak{F}^{*+*} \subset \mathfrak{F}^{+*}$, $\mathfrak{F}^+ \subset \mathfrak{F}^{+*+} \subset \mathfrak{F}^{*+}$;
(10) $\mathfrak{F}^+ = \mathfrak{B}^+$ *and* $\mathfrak{F}^* = \mathfrak{B}^*$ *for every base* $\mathfrak{B}$ *of* $\mathfrak{F}$. *In particular,* $[y]^+ = y^+$, $[y]^* = y^*$;
(11) $(\bigcap \mathbf{F})^+ = \bigcap \{\mathfrak{F}^+ \mid \mathfrak{F} \in \mathbf{F}\}$, $(\bigcap \mathbf{F})^* = \bigcap \{\mathfrak{F}^* \mid \mathfrak{F} \in \mathbf{F}\}$;
(12) $y \leqq z$ *for all* $y \in \mathfrak{F}^+$ *and all* $z \in \mathfrak{F}^*$.

On account of these rules, order convergence may be described without using joins and meets (cf. [9]):

COROLLARY 2.2. *For* $\mathfrak{F} \in \mathbf{F}(P)$, *the following five statements are equivalent.*
(1) $\mathfrak{F} \underset{0}{\to} x$.
(2) $\mathfrak{F}^{+*} = x^*$ *and* $\mathfrak{F}^{*+} = x^+$.
(3) $\mathfrak{F}^{+*} \subset x^*$ *and* $\mathfrak{F}^{*+} \subset x^+$.
(4) $x \in \mathfrak{F}^{+*+} \cap \mathfrak{F}^{*+*}$.
(5) $\{x\} = \mathfrak{F}^{+*+} \cap \mathfrak{F}^{*+*}$.

For lattices, this result can be sharpened.

THEOREM 2.3. *In a lattice* L, *a filter* $\mathfrak{F} \in \mathbf{F}(L)$ *order-converges to* $x \in L$ *if and only if* $\{x\} = \mathfrak{F}^{+*} \cap \mathfrak{F}^{*+}$.

PROOF. By 2.2., $\mathfrak{F} \underset{0}{\to} x$ implies $\{x\} = x^* \cap x^+ = \mathfrak{F}^{+*} \cap \mathfrak{F}^{*+}$. Conversely, assume $\{x\} = \mathfrak{F}^{+*} \cap \mathfrak{F}^{*+}$. Then x is an upper bound of $\mathfrak{F}^+$ since $x \in \mathfrak{F}^{+*}$. If there would be another upper bound z of $\mathfrak{F}^+$ with $z < x$, then $x \in \mathfrak{F}^{*+}$ would imply $z \in \mathfrak{F}^{*+}$, and $z \in \mathfrak{F}^{+*} \cap \mathfrak{F}^{*+} = \{x\}$, a contradiction. Since L is a lattice, it follows that x is the least upper bound of $\mathfrak{F}^+$, i.e., $x = \bigvee \mathfrak{F}^+$, and by duality, $x = \bigwedge \mathfrak{F}^*$.

The *interval topology* $\mathfrak{J}$ on a poset P has as a subbase the set-complements of closed rays. Kent has shown in [9] that an ultrafilter $\mathfrak{U}$ converges to x in the interval topology if and only if $x \in \mathfrak{U}^{+*} \cap \mathfrak{U}^{*+}$. From 2.3., we conclude the following corollary.

COROLLARY 2.4. *In a lattice L, an ultrafilter* $\mathfrak{U}$ *order-converges to x if and only if x is the unique* $\mathfrak{J}$*-limit of* $\mathfrak{U}$.

The *order topology* of a poset P is defined by $\mathfrak{O} := \{Y \subset P \mid x \in Y$ and $\mathfrak{F} \xrightarrow[0]{} x$ imply $Y \in \mathfrak{F}\}$. Hence, convergence in $\mathfrak{O}$ is the least topological convergence relation containing order convergence, and, in particular, order convergence is topological if and only if it coincides with convergence in $\mathfrak{O}$ (cf. §1). A deviating definition of order topology in terms of nets has been given by G. Birkhoff [1] and others. One can show that at least in lattices (but not in all posets) both definitions coincide. It is well known that $\mathfrak{O}$ is always finer than the interval topology $\mathfrak{J}$ (cf. [9]). Hence *every closed ray is also closed in the order topology.* $\mathfrak{U}(x) := [\mathfrak{O} \cap [x]]$ will denote the neighbourhood filter of x with respect to $\mathfrak{O}$. Note that

$$\mathfrak{U}(x) \subset \mathfrak{V}(x) = \bigcap\{\mathfrak{F} \mid \mathfrak{F} \xrightarrow[0]{} x\}.$$

A topological space is T_2 if and only if every ultrafilter has at most one limit; on the other hand, a space is compact if and only if every ultrafilter has at least one limit. Thus, from 2.4, we infer the following two results due to A.J. Ward [13].

COROLLARY 2.5. *The interval topology of a lattice is* T_2 *if and only if for ultrafilters, order convergence coincides with interval convergence. (In this case,* $\mathfrak{J} = \mathfrak{O}$.)

COROLLARY 2.6. *The following three conditions are equivalent for a lattice L.*

(1) *L is complete, and the interval topology is* T_2.

(2) *The interval topology of L is compact and* T_2.

(3) *Every ultrafilter order-converges.*

If one of these conditions holds, then the interval topology $\mathfrak{J}$ *coincides with the order topology* $\mathfrak{O}$.

(The proof of (1) $\Leftrightarrow$ (2) involves a theorem of O. Frink [3] saying that the interval topology of a lattice L is compact if and only if L is complete.)

Kent has shown in [9] that order convergence always satisfies ($C1$) and ($C2$) (which is also clear from Lemma 2.1.). Moreover, one can prove the following theorem.

THEOREM 2.7. *The order convergence of any poset is a localized convergence relation.*

PROOF. In order to verify ($C3$), suppose $\mathfrak{F} \underset{0}{\rightarrow} x$. Then by 2.1. (9) and 2.2., $\mathfrak{F}^+ \subset \mathfrak{F}^{*+} \subset x^+$, whence $\mathfrak{F}^+ \cap x^+ = \mathfrak{F}^+$. Now an application of 2.1. (11) and (10) gives $(\mathfrak{F} \cap [x])^+ = \mathfrak{F}^+ \cap [x]^+ = \mathfrak{F}^+ \cap x^+ = \mathfrak{F}^+$. Hence, $(\mathfrak{F} \cap [x])^{+*} = \mathfrak{F}^{+*} \subset x^*$, and by duality, $(\mathfrak{F} \cap [x])^{*+} = \mathfrak{F}^{*+} \subset x^+$. From 2.2., we conclude that $\mathfrak{F} \cap [x] \underset{0}{\rightarrow} x$.

However, in general, order convergence is not a limitierung (cf. [**8**]). On lattices, necessary and sufficient conditions for order convergence to be a limitierung will be given in 4.1. and 4.2.

3. **Connections between filters and ideals.** In the following, we shall characterize order convergence of a lattice in terms of ideals and dual ideals. An *ideal* of a poset P is a subset I with $Y^{*+} \subset I$ for all finite subsets Y of I (cf. [**4**]). Notice that, in contrast to the classical ideal definition for lattices, the empty set is an ideal if and only if P has no least element. The order-theoretical definition of ideals has several advantages. For example, the ideal lattice of a poset P is, up to isomorphism, the least algebraic lattice L containing P such that every element of P is compact in L. This statement neither holds if we forbid empty ideals on principle, nor if the empty set is always required to be an ideal. A further consequence of the previous ideal definition is the following theorem.

THEOREM 3.1. *A subset I of a poset P is an ideal if and only if there exsits some $\mathfrak{F} \in \mathbf{F}_0(P)$ with $I = \mathfrak{F}^+$. Hence, the function $\mathfrak{F} \mapsto \mathfrak{F}^+$ is a $\bigcap$-homomorphism from the filter lattice $\mathbf{F}_0(P)$ onto the ideal lattice P. Furthermore, if I is a proper ideal (i.e., $I \subsetneq P$), then there is a proper filter $\mathfrak{F}$ with $\mathfrak{F}^+ = I$.*

PROOF. Let $\mathfrak{F} \in \mathbf{F}_0(P)$. For a finite subset Y of $\mathfrak{F}^+$, we obtain $y^* \in \mathfrak{F}$ for all $y \in Y$. Hence, $Y^* = \bigcap\{y^* \mid y \in Y\} \in \mathfrak{F}$, and $Y^{*+} \subset \mathfrak{F}^+$, so that $\mathfrak{F}^+$ is an ideal. Conversely, if I is any ideal in P, then $\mathfrak{F} := [\{y^* \mid y \in I\}]$ has the property $\mathfrak{F}^+ = I$. If $\mathfrak{F} = 2^P$ is the only member of $\mathbf{F}_0(P)$ with $\mathfrak{F}^+ = I$, then $I = [\{\varnothing\}]^+ = \varnothing^+ = P$ is not proper.

The following obvious observation will often apply in connection with order-convergent filters.

LEMMA 3.2. *An ideal which has a join is nonempty. In particular, $\bigvee \mathfrak{F}^+ = x$ implies $\mathfrak{F}^+ \neq \varnothing$ for any filter $\mathfrak{F}$.*

From now on, we always are considering a given lattice L.

Generalizing a definition introduced by A. Gingras [**6**], we say a *conoid* is a pair (I, D) where I is an ideal, D a dual ideal, and $y \leqq z$ for all $y \in I$,

$z \in D$. The collection of all conoids of L is denoted $\mathbf{C}(L)$. On account of 3.1. and 2.1.(12), *for any filter* $\mathfrak{F}$, $(\mathfrak{F}^+, \mathfrak{F}^*)$ *is a conoid.*

For any conoid (I, D), let $\mathbf{F}(I, D)$ denote the collection of all filters $\mathfrak{F}$ with $\mathfrak{F}^+ = I$ and $\mathfrak{F}^* = D$. That at least one such filter exists is a consequence of the following theorem.

THEOREM 3.3. *The system* $\{\mathbf{F}(I, D) \mid (I, D) \in \mathbf{C}(L)\}$ *forms a partition of* $\mathbf{F}(L)$. *Each class* $\mathbf{F}(I, D)$ *has a least element, the filter* $\mathfrak{F}(I, D)$ *generated by the system* $\mathfrak{S}(I, D) = \{y^* \mid y \in I\} \cup \{z^+ \mid z \in D\}$.

PROOF. For any conoid (I, D), every finite intersection formed by members of $\mathfrak{S}(I, D)$ is nonempty, so that $\mathfrak{S}(I, D)$ generates a (proper) filter. For $y \in I$, we get $y^- \in \mathfrak{S}(I, D) \subset \mathfrak{F}(I, D)$, $y \in \mathfrak{F}(I, D)^+$. Thus $I \subset \mathfrak{F}(I, D)^+$. Conversely, suppose $x \in \mathfrak{F}(I, D)^+$. Then $x^* \in \mathfrak{F}(I, D)$. If I is nonempty, then there exists an element $y \in I$ with $y^* \subset x^*$, and it follows that $x \in y^{*+} \subset I$ (since I is an ideal). On the other hand, if I is empty, then $x \in \mathfrak{F}(I, D)^+$ implies that x is a lower bound either of L or of some lower ray z^+ with $z \in D$. But then x is the least element of L, contradicting $I = \emptyset$. Summarizing, we obtain $I = \mathfrak{F}(I, D)^+$, and by duality, $D = \mathfrak{F}(I, D)^*$; thus $\mathfrak{F}(I, D) \in \mathbf{F}(I, D)$. Now let $\mathfrak{F}$ be an arbitrary filter in $\mathbf{F}(I, D)$. Then for $y \in I = \mathfrak{F}^+$ it follows that $y^* \in \mathfrak{F}$, and dually, $z \in D$ implies $z^+ \in \mathfrak{F}$. Hence, $\mathfrak{S}(I, D) \subset \mathfrak{F}$, and since $\mathfrak{F}(I, D)$ is generated by $\mathfrak{S}(I, D)$, $\mathfrak{F}(I, D) \subset \mathfrak{F}$.

On account of 3.3., $\mathfrak{F}(I, D)$ *is the least filter* $\mathfrak{F}$ *with* $\mathfrak{F}^* = I$ *and* $\mathfrak{F}^+ = D$. Furthermore, we have shown that (I, D) is a conoid if and only if there exists a filter $\mathfrak{F}$ with $(I, D) = (\mathfrak{F}^+, \mathfrak{F}^*)$.

The filters $\mathfrak{F}(I, D)$ (where (I, D) is a conoid) can be characterized intrinsically as so-called ray filters. By a *ray filter*, we mean a filter which has a subbase consisting of closed rays. $\mathfrak{F}$ is referred to as an *interval filter* if it has a base consisting of intervals. $\mathfrak{F}$ will be called a *pre-interval filter* if it has a base consisting of sets each of which has a greatest and a least element. Finally, a filter $\mathfrak{F}$ is said to be *bounded* if it contains at least one bounded member (and has therefore a base of bounded sets). Equivalently, a bounded filter $\mathfrak{F}$ may be characterized by the inequality $\mathfrak{F}^+ \neq \emptyset \neq \mathfrak{F}^*$. By 3.2, *every order-convergent filter is bounded.*

Let us describe the previously introduced types of filters by several alternative conditions.

LEMMA 3.4. *For a filter* $\mathfrak{F}$, *the following four conditions are equivalent.*

(1) $\mathfrak{F}$ *is a ray filter.*

(2) $\mathfrak{F}$ *has a base consisting of intervals and closed rays.*

(3) $\mathfrak{F} = \mathfrak{F}(I, D)$ *for some conoid* (I, D).

(4) $\mathfrak{F} = \mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$.

PROOF. The implications (4) ⇒ (3) ⇒ (1) ⇔ (2) are clear. Let us show (1) ⇒ (4). By 3.3, we have $\mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*) \subset \mathfrak{F}$. Now let F be any member of $\mathfrak{F}$. Then there are finite subsets $Y, Z \subset L$ with

$$B = \bigcap\{y^* \mid y \in Y\} \cap \bigcap\{z^+ \mid z \in Z\} \subset F$$

and $y^* \in \mathfrak{F}$ for all $y \in Y$, $z^+ \in \mathfrak{F}$ for all $z \in Z$. Thus, $Y \subset \mathfrak{F}^+$, $Z \subset \mathfrak{F}^*$, and we conclude $B \subset F \in \mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$.

LEMMA 3.5. *For a filter $\mathfrak{F}$, the following four conditions are equivalent.*

(1) *$\mathfrak{F}$ is a pre-interval filter.*

(2) *For all $F \in \mathfrak{F}$, there are elements $y, z \in F$ with $[y, z] \in \mathfrak{F}$.*

(3) *$F \cap \mathfrak{F}^+ \neq \emptyset$ and $F \cap \mathfrak{F}^* \neq \emptyset$ for all $F \in \mathfrak{F}$.*

(4) *There are (proper) filters $\mathfrak{F}_+$ and $\mathfrak{F}_*$ with $\mathfrak{F} \cup \{\mathfrak{F}^+\} \subset \mathfrak{F}_+$, $\mathfrak{F} \cup \{\mathfrak{F}^*\} \subset \mathfrak{F}_*$.*

PROOF. (1) ⇒ (2): For $F \in \mathfrak{F}$, there exists a $G \in \mathfrak{F}$ with least element y, greatest element z, and $G \subset F$. It follows that $y, z \in F$ and $G \subset [y, z] \in \mathfrak{F}$.

(2) ⇒ (3): Choose $y, z \in F$ with $[y, z] = y^* \cap z^+ \in \mathfrak{F}$. Then $y^* \in \mathfrak{F}$ and $z^+ \in \mathfrak{F}$, whence $y \in F \cap \mathfrak{F}^+ \neq \emptyset$, $z \in F \cap \mathfrak{F}^* \neq \emptyset$.

(3) ⇔ (4): Clear.

(3) ⇒ (1): For $F \in \mathfrak{F}$, choose $y \in F \cap \mathfrak{F}^+$ and $z \in F \cap \mathfrak{F}^*$. Then $y^* \in \mathfrak{F}$, $z^+ \in \mathfrak{F}$, and $G := F \cap y^* \cap z^+$ is a member of $\mathfrak{F}$ with least element y, greatest element z and $G \subset F$.

Now the main relationships between ray filters and (pre-) interval filters are listed in the following lemma.

LEMMA 3.6. *For a filter $\mathfrak{F}$, the following six conditions are equivalent.*

(1) *$\mathfrak{F}$ is an interval filter.*

(2) *$\mathfrak{F}$ is a pre-interval filter possessing a base of convex sets.*

(3) *$\mathfrak{F} = \mathfrak{F}(I, D)$ for some conoid (I, D) with $I \neq \emptyset \neq D$.*

(4) *$\mathfrak{F} = \mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$, and $\mathfrak{F}$ is bounded.*

(5) *$\mathfrak{F}$ is a bounded ray filter.*

(6) *$\mathfrak{F}$ is both a ray filter and a pre-interval filter.*

PROOF. (1) ⇒ (6) ⇒ (5): obvious.

(5) ⇒ (4) ⇒ (3): 3.4.

(3) ⇒ (2): If I and D are not empty, then obviously $\mathfrak{B} = \{[y, z] \mid y \in I, z \in D\}$ is a base for the filter $\mathfrak{F}$ consisting of convex subsets each of which has a greatest and a least element.

(2) ⇒ (1): Let $F \in \mathfrak{F}$. Then there is a convex $C \in \mathfrak{F}$ with $C \subset F$. By 3.5, we find elements $y, z \in C$ such that $[y, z] \in \mathfrak{F}$. But then, by convexity of C, it follows that $[y, z] \subset C \subset F$. Thus, $\mathfrak{F}$ has a base of intervals.

Now consider the equivalence relation $\sim$ on $\mathrm{F}(L)$, defined by

$$\mathfrak{F} \sim \mathfrak{G} \Leftrightarrow \mathfrak{F}^+ = \mathfrak{G}^+ \text{ and } \mathfrak{F}^* = \mathfrak{G}^*.$$

By 3.3, the corresponding partition $\mathbf{F}(L)/\sim$ is the system $\tilde{\mathbf{F}}(L) := \{\mathbf{F}(I, D) \mid (I, D) \in \mathbf{C}(L)\}$. Summarizing, we obtain the following theorem.

THEOREM 3.7. (1) *The mapping $(I, D) \mapsto \mathfrak{F}(I, D)$ is a bijection between the set $\mathbf{C}(L)$ of all conoids and the set $\mathbf{F}_R(L)$ of all ray filters of L. The inverse mapping is given by $\mathfrak{F} \mapsto (\mathfrak{F}^+, \mathfrak{F}^*)$.*

(2) *The mapping $(I, D) \mapsto \mathbf{F}(I, D)$ is a bijection between $\mathbf{C}(L)$ and the partition $\tilde{\mathbf{F}}(L)$.*

(3) *$\mathbf{F}_R(L)$ is a representative system for $\tilde{\mathbf{F}}(L)$, and $\mathfrak{F}(I, D)$ is the least element of the class $\mathbf{F}(I, D)$.*

(4) *For any conoid (I, D) the following conditions are equivalent.*

(a) $I \neq \emptyset \neq D$.

(b) *$\mathfrak{F}(I, D)$ is bounded.*

(c) *$\mathfrak{F}(I, D)$ is an interval filter.*

(d) *Every filter in $\mathbf{F}(I, D)$ is bounded.*

Now we say a conoid (I, D) *order-converges* to a point x, written $(I, D) \xrightarrow[0]{} x$, if $\bigvee I = x = \bigwedge D$. From the preceding considerations, we obtain immediately the following theorem.

THEOREM 3.8. (1) *A conoid (I, D) order-converges to x if and only if the ray filter $\mathfrak{F}(I, D)$ does. In this case, $\mathfrak{F}(I, D)$ is already an interval filter.*

(2) *A filter $\mathfrak{F}$ order-converges to x if and only if the conoid $(\mathfrak{F}^+, \mathfrak{F}^*)$ does. (In this case, $\mathfrak{F}$ is bounded.)*

(3) *A filter $\mathfrak{F}$ order-converges to x if and only if the ray filter $\mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$ does. (In this case, $\mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$ is an interval filter.)*

Thus, order convergence in a lattice is completely determined by the behaviour of conoids or of ray filters, respectively, and even by that of interval filters.

4. **Pretopological and topological order convergence.** We are now able to give several necessary and sufficient conditions for order convergence to be a limitierung, a pretopological or a topological convergence relation.

THEOREM 4.1. *Order convergence is a limitierung if and only if for any two conoids (I_1, D_1) and (I_2, D_2) order-converging to x, the "intresection" conoid $(I_1 \cap I_2, D_1 \cap D_2)$ also order-converges to x.*

PROOF. 1) Suppose $(I_1, D_1) \xrightarrow[0]{} x$ and $(I_2, D_2) \xrightarrow[0]{} x$. Then, by 3.8.(1), $\mathfrak{F} = \mathfrak{F}(I_1, D_1) \xrightarrow[0]{} x$, $\mathfrak{G} = \mathfrak{F}(I_2, D_2) \xrightarrow[0]{} x$, and if order convergence is a limitierung, $\mathfrak{F} \cap \mathfrak{G} \xrightarrow[0]{} x$. Thus

$$x = \bigvee(\mathfrak{F} \cap \mathfrak{G})^+ = \bigvee(\mathfrak{F}^+ \cap \mathfrak{G}^+) = \bigvee(I_1 \cap I_2).$$

By duality, $x = \bigwedge (D_1 \cap D_2)$.

2) To show that the condition stated in 4.1 is sufficient for order convergence to be a limitierung, let $\mathfrak{F} \underset{0}{\to} x$ and $\mathfrak{G} \underset{0}{\to} x$. Then, by 3.8.(2), $(\mathfrak{F}^+, \mathfrak{F}^*) \underset{0}{\to} x$ and $(\mathfrak{G}^+, \mathfrak{G}^*) \underset{0}{\to} x$, whence

$$((\mathfrak{F} \cap \mathfrak{G})^+, (\mathfrak{F} \cap \mathfrak{G})^*) = (\mathfrak{F}^+ \cap \mathfrak{G}^+, \mathfrak{F}^* \cap \mathfrak{G}^*) \underset{0}{\to} x,$$

and, again by 3.8.(2), $\mathfrak{F} \cap \mathfrak{G} \underset{0}{\to} x$.

COROLLARY 4.2. *Order convergence is a limitierung if and only if for any pair of ideals I_1, I_2, $\bigvee I_1 = \bigvee I_2 = x$ implies $\bigvee (I_1 \cap I_2) = x$, and dually.*

We call a lattice L $\wedge$*-continuous* if for any ideal I possessing a join and any element $x \in L$,

$$(\wedge) \quad x \wedge \bigvee I = \bigvee (x \wedge I).$$

Note that we do not postulate completeness for L. (Obviously, L is $\wedge$-continuous if $(\wedge)$ holds for every directed set I, cf. **[1]**, p. 187. Rennie calls a lattice L $\wedge$-continuous if $(\wedge)$ holds for all chains I. Although this definition is equivalent with the previous one in complete lattices, it is not always equivalent in arbitrary lattices.) Furthermore, a straightforward computation shows that L is $\wedge$-continuous if and only if

$$\bigvee I_1 \wedge \bigvee I_2 = \bigvee (I_1 \wedge I_2),$$

or equivalently,

$$\bigvee I_1 \wedge \bigvee I_2 = \bigvee (I_1 \cap I_2)$$

for all ideals I_1, I_2. Applying 4.2., we obtain the following corollary.

COROLLARY 4.3. *In $\wedge$-and $\vee$-continuous lattices, order convergence is a limitierung.*

For each element x of a lattice L, define $I(x) := \bigcap\{I \mid I \text{ is an ideal with } \bigvee I = x\}$, $D(x) := \bigcap\{D \mid D \text{ is a dual ideal with } \bigwedge D = x\}$. Then $I(x)$ is an ideal, $D(x)$ a dual ideal, and $(I(x), D(x))$ a conoid. For any ideal I with $\bigvee I = x$, $\mathfrak{F} = \mathfrak{F}(I, x^*)$ is a filter such that $\mathfrak{F}^+ = I$ and $\mathfrak{F} \underset{0}{\to} x$ (see 3.8.). This together with 3.1 yields the following lemma.

LEMMA 4.4. $\mathfrak{V}(x)^+ = I(x)$, $\mathfrak{V}(x)^* = D(x)$.

Now, an argument similar to that in the proof of 4.1 shows the following theorem.

THEOREM 4.5. *For order convergence of a lattice to be pretopological, each of the following conditions is necessary and sufficient.*

(1) *For all $x \in L$ and any nonempty set of conoids order-converging to x, the "intersection" of these conoids order-converges to x.*

(2) *For all $x \in L$, the conoid $(I(x), D(x))$ order-converges ot x (cf.* [6]).

(3) *If $\mathfrak{Y}$ is a nonempty set of ideals with $\bigvee I = x$ for all $I \in \mathfrak{Y}$, then $\bigvee \bigcap \mathfrak{Y} = x$, and dually.*

(4) *For all $x \in L$, $\bigvee I(x) = x = \bigwedge D(x)$ (cf.* [8]).

Before giving some criteria for order convergence to be topological, it is convenient to describe the order topology in terms of ideals and dual ideals, without using the convergence relation.

THEOREM 4.6. *A subset U of a lattice L is open with respect to the order topology $\mathfrak{O}$ if and only if for all ideals I and all dual ideals D with $\bigvee I = \bigwedge D \in U$, there are elements $y \in I$ and $z \in D$ with $[y, z] \subset U$.*

PROOF. First, suppose $U \in \mathfrak{O}$ and $\bigvee I = x = \bigwedge D \in U$ for some ideal I and some dual ideal D. Then the filter $\mathfrak{F}(I, D)$ order-converges to $x \in U$, and we have $U \in \mathfrak{F}(I, D)$. Thus $[y, z] \subset U$ for some $y \in I$, $z \in D$. Conversely, if the condition holds, then for every $x \in U$ and every filter $\mathfrak{F}$ with $\mathfrak{F} \xrightarrow[0]{} x$, it follows that $\bigvee \mathfrak{F}^+ = x = \bigwedge \mathfrak{F}^* \in U$, and there are elements $y \in \mathfrak{F}^+$, $z \in \mathfrak{F}^*$ with $[y, z] \subset U$. But $y^* \in \mathfrak{F}$ and $z^+ \in \mathfrak{F}$ imply $y^* \cap z^+ = [y, z] \subset U \in \mathfrak{F}$. Hence, $x \in U$ and $\mathfrak{F} \xrightarrow[0]{} x$ yields $U \in \mathfrak{F}$, and $U \in \mathfrak{O}$.

COROLLARY 4.7. *If a directed subset Y of a lattice L has a join, then this is contained in the $\mathfrak{O}$-closure of Y. In particular, if L is complete, every $\mathfrak{O}$-closed sublattice is subcomplete (i.e., arbitrary joins and meets are the same as in L) and has, therefore, a greatest and a least element. Furthermore, an ideal of a complete lattice is $\mathfrak{O}$-closed if and only if it is principal.*

PROOF. Let Y be directed and $x = \bigvee Y$. Then $I := \{y \in L \mid y \leqq y' \text{ for some } y' \in Y\}$ is an ideal in L with $\bigvee I = x$. If there existed an $\mathfrak{O}$-open set U disjoint from Y but containing x, then $x = \bigvee I = \bigwedge x^*$ would imply $[y, x] \subset U$ for some $y \in I$, and we would find a $y' \in Y$ with $y \leqq y' \leqq x$, a contradiction. Hence, no such U exists, and x is in the closure of Y.

Although order convergence does not agree with convergence in the order topology for arbitrary filters, one can show the following theorem.

THEOREM 4.8. *For a pre-interval filter $\mathfrak{F}$, the following three conditions are equivalent.*

(1) *$\mathfrak{F}$ order-converges to x.*

(2) *$\mathfrak{F}$ converges to x in the order topology $\mathfrak{O}$.*

(3) *$\mathfrak{F}$ converges to x in the interval topology $\mathfrak{J}$.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Clear since $\mathfrak{J} \subset \mathfrak{O}$.

(3) $\Rightarrow$ (1): Suppose $\mathfrak{F}$ converges to x in $\mathfrak{F}$. We have to show that $\mathfrak{F}^{+*}$

$\subset x^*$ and $\mathfrak{F}^{*+} \subset x^+$. $w \in \mathfrak{F}^{+*}$ means $\mathfrak{F}^+ \subset w^+$. Assuming $w \notin x^*$, we obtain $x \in L\backslash w^+$, and $L\backslash w^+$ is open in $\mathfrak{I}$. Thus, $L\backslash w^+ \in \mathfrak{U}_{\mathfrak{I}}(x) \subset \mathfrak{F}$, and by 3.5., $(L\backslash w^+) \cap \mathfrak{F}^+ \neq \emptyset$, contradicting the inclusion $\mathfrak{F}^+ \subset w^+$.

COROLLARY 4.9. *For any filter $\mathfrak{F}$, the following four conditions are equivalent.*

(1) *$\mathfrak{F}$ order-converges to x.*
(2) *$\mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$ is an interval filter order-converging to x.*
(3) *$\mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$ is an interval filter converging to x in the order topology.*
(4) *$\mathfrak{F}(\mathfrak{F}^+, \mathfrak{F}^*)$ is an interval filter converging to x in the interval topology.*

Another consequence of 4.8. is the following corollary.

COROLLARY 4.10. *A conoid (I, D) order-converges to x if and only if for all $U \in \mathfrak{U}(x)$, there are elements $y \in I$ and $z \in D$ with $[y, z] \subset U$ (in other words, if and only if the interval filter $\mathfrak{F}(I, D)$ converges to x in the order topology).*

This has been proved by Gingras in [6], under the hypothesis of topological order convergence (and only for complete lattices). He conjectured that this hypothesis would be necessary for the equivalence in 4.10; but on the contrary, it holds without any restriction in arbitrary lattices.

Now we can prove our main result.

THEOREM 4.11. *For any element x of a lattice L and any filter $\mathfrak{F}$ on L with $\mathfrak{U}(x) \subset \mathfrak{F} \subset \mathfrak{B}(x)$, the following conditions are equivalent.*

(1) $\mathfrak{F} \underset{0}{\to} x$.
(2) *$\mathfrak{F} = \mathfrak{F}(I(x), D(x))$, and $\mathfrak{F}$ is bounded.*
(3) *$\mathfrak{F}$ is a bounded ray filter.*
(4) *$\mathfrak{F}$ is an interval filter.*
(5) *$\mathfrak{F}$ is a pre-interval filter.*

PROOF. (1) $\Rightarrow$ (2): $\mathfrak{F} \underset{0}{\to} x$ and $\mathfrak{F} \subset \mathfrak{B}(x)$ imply $\mathfrak{F} = \mathfrak{B}(x)$ since $\mathfrak{B}(x)$ is the intersection of all filters order-converging to x. Furthermore, by 4.4. and 3.3., $\mathfrak{F}(I(x), D(x)) = \mathfrak{F}(\mathfrak{B}(x)^+, \mathfrak{B}(x)^*) \subset \mathfrak{B}(x)$. On the other hand, $\mathfrak{B}(x) = \mathfrak{F} \underset{0}{\to} x$ implies $\mathfrak{F}(I(x), D(x)) \underset{0}{\to} x$, and we have $\mathfrak{B}(x) \subset \mathfrak{F}(I(x), D(x))$. Thus, $\mathfrak{F} = \mathfrak{B}(x) = \mathfrak{F}(I(x), D(x))$. Finally,as remarked before, any order-convergent filter is bounded.

(2) $\Rightarrow$ (3): Clear.
(3) $\Rightarrow$ (4): See 3.6.
(4) $\Rightarrow$ (5): Clear.
(5) $\Rightarrow$ (1): $\mathfrak{U}(x) \subset \mathfrak{F}$ implies $\mathfrak{F} \underset{0}{\to} x$, by 4.8.

Choosing for $\mathfrak{F}$ the extremal cases $\mathfrak{F} = \mathfrak{B}(x)$ and $\mathfrak{F} = \mathfrak{U}(x)$, respectively, we obtain the following corollaries.

COROLLARY 4.12. *In a lattice, order convergence is pretopological if and only if one of the equivalent conditions in* 4.11 *holds for each of the filters* $\mathfrak{B}(x)$ (*instead of* $\mathfrak{F}$).

COROLLARY 4.13. *In a lattice, order convergence is topological if and only if one of the equivalent conditions in* 4.11 *holds for each of the filters* $\mathfrak{U}(x)$ (*instead of* $\mathfrak{F}$).

In particular, for order convergence of a lattice to be topological, it is necessary and sufficient that every topological neighbourhood filter $\mathfrak{U}(x)$ be an interval filter. This equivalence has been asserted by Gingras in **[5]** and **[6]**, but his proof was essentially based on a wrong hypothesis requiring the equivalence of the notions "pretopological" and "topological".

Applying 2.7, we can show a useful modification of 4.13.

THEOREM 4.14. *In a complete lattice, order convergence is topological if and only if there exists a* T_3*-topology* $\mathfrak{T}$ *with* $\mathfrak{J} \subset \mathfrak{T} \subset \mathfrak{O}$ *such that each neighbourhood filter* $\mathfrak{U}_{\mathfrak{T}}(x)$ *has a base of sublattices. If such a topology exists, then it must coincide with the order topology* $\mathfrak{O}$.

PROOF. Necessity is clear by 4.13 since every interval filter has a base of (closed) sublattices, namely of intervals. To show sufficiency, we prove that $\mathfrak{U}_{\mathfrak{T}}(x)$ is a pre-interval filter. By the T_3-axiom, $\mathfrak{U}_{\mathfrak{T}}(x)$ has a base of $\mathfrak{T}$-closed subsets. Let U be one of them and V a sublattice neighbourhood of x contained in U. Then $\bigvee V \in U$ and $\bigwedge V \in U$, by 4.7. Thus $F := V \cup \{\bigvee V, \bigwedge V\}$ is a neighbourhood of x possessing a greatest and a least element and contained in U. Now, as in the proof of 4.8, we conclude $\mathfrak{U}_{\mathfrak{T}}(x) \underset{0}{\to} x$, and in particular, $\mathfrak{U}_{\mathfrak{T}}(x) \supset \mathfrak{U}(x)$. By assumption, $\mathfrak{O}$ is finer than $\mathfrak{T}$, and consequently, $\mathfrak{U}(x) = \mathfrak{U}_{\mathfrak{T}}(x) \underset{0}{\to} x$, so that order convergence is topological and $\mathfrak{T} = \mathfrak{O}$.

Now we can prove the following generalization of a theorem on topological lattices due to D.P. Strauss (cf. **[12]**, Theorem 5).

THEOREM 4.15. *Let* $\mathfrak{T}$ *be a compact* T_2*-topology on a lattice L which is finer than the interval topology. If each neighbourhood filter* $\mathfrak{U}_{\mathfrak{T}}(x)$ *has a base of sublattices, then* $\mathfrak{J} = \mathfrak{T} = \mathfrak{O}$, *and order convergence is topological.*

PROOF. First, we observe that the interval topology $\mathfrak{J}$ is also compact (being contained in $\mathfrak{T}$), and consequently, L is complete [3]. Hence, we may apply 4.14 if we can show that $\mathfrak{T}$ is contained in the order topology $\mathfrak{O}$. Let $\mathfrak{U}$ be any ultrafilter order-converging to x. Then, by 2.3, $\{x\} = \mathfrak{U}^{+*} \cap \mathfrak{U}^{*+}$. Since $\mathfrak{T}$ is compact, $\mathfrak{U}$ has at least one limit y in $\mathfrak{T}$. But $\mathfrak{T}$ is finer than $\mathfrak{J}$, so $\mathfrak{U}$ $\mathfrak{J}$-converges to y, that is, $y \in \mathfrak{U}^{+*} \cap \mathfrak{U}^{*+} = \{x\}$. Hence, y must coincide with x, and $\mathfrak{U}$ $\mathfrak{T}$-converges to x. Observing that $\mathfrak{O}$ is the finest topology such that all order limits of any ultrafilter are also topological limits, we obtain $\mathfrak{T} \subset \mathfrak{O}$. By 4.14, order convergence coincides with $\mathfrak{T}$-convergence. Since $\mathfrak{T}$ is compact, every ultrafilter order-converges. Now from 2.6, we infer that $\mathfrak{J} = \mathfrak{O} = \mathfrak{T}$, as desired.

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