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MODULAR FACE LATTICES: LOW DIMENSIONAL CASES

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ABSTRACT. Let K be a self dual cone with a modular lattice of faces. If dim $K = 4$, then K is strictly convex. If dim $K = 5$, then either K is strictly convex or every maximal face is of dimension 3. An example is given of a self dual cone K which has an orthomodular but not modular lattice of faces.

The notations and conventions are those of [2] and [3]. Recall that cone K in a real vector space V is a subset such that if $x, y \in K$, α, $\beta \geqq 0$, then $\alpha x + \beta y \in K$. The cones considered here will be topologically closed, pointed ($K \cap (-K) = \{0\}$), and full (non-empty interior). Also V is assumed to be finite dimensional. K defines an order in V by $x \geqq 0$ if and only if $x \in K$ (cf. [1]). We shall write: $x \geqq y$ if $x - y \in K$; $x > y$ if $x \geqq y$ and $x \neq y$; and $x \gg y$ if $x - y \in \operatorname{int} K$. A subset F of K is a *face* if and only if $0 \leqq x \leqq y$ and $y \in F$ implies $x \in F$. Let $\mathscr{F}(K)$ denote the set of all faces of K, and if $S \subset K$, put $\varphi(S) = \bigcap\{F\colon F \in \mathscr{F}(K)$ and $F \supset S\}$. Then $\mathscr{F}(K)$, ordered by inclusion, becomes a lattice if we define $F \vee G = \varphi(F \cup G)$, $F, G \in \mathscr{F}(K)$, and $F \wedge G = F \cap G$.

If $p \in K$ and $\varphi(p)$ is a ray, we call p an *extremal* and let Ext K denote the set of all extremals. If $F \in \mathscr{F}(K)$, we shall also write $F \lhd K$. More generally, since any face is full in its span, we may write $F \lhd G$ if F, G are faces of K and $F \subseteqq G$ (cf. [2]). Let $\langle F\rangle = F - F$ denote the linear span of F. We set $\dim F = \dim \langle F\rangle$. If $\mathscr{F}(K)$ is modular, then any two maximal chains linking $\{0\}$ and a face F will have the same number of elements. If there are $k + 1$ elements in a maximal chain between $\{0\}$ and F, we call k the *height* of F and write $h(F) = k$. (In the lattice theory this number is often called the dimension, but we wish to use the latter term for the algebraic dimension.) Note that if $F \in \mathscr{F}(K)$, then $h(F) \leqq \dim F$, and in general equality holds only when F is an atom or K is simplicial. If $h(K) = 2$, then either K is strictly convex or a two dimensional simplicial cone. More generally as theorem 2 of [3] and the following lemma show, it is enough to consider only indecomposable cones K. Recall that a cone K is *decomposable* (cf. [6]) if there are proper subsets $K_1, K_2 \subset K$ such that

(1) $\forall x \in K$, $\exists x_i \in K_i$ such that $x = x_1 + x_2$,

(2) span $K_1 \cap$ span $K_2 = \{0\}$

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If this is true we write $K = K_1 \oplus K_2$ and say that K is the *direct sum* of K_1 and K_2.

LEMMA. *Suppose $\mathscr{F}(K)$ is modular. Then $\mathscr{F}(K)$ is subdirectly irreducible if and only if K is indecomposabele.*

PROOF. Suppose $\mathscr{F}(K)$ is subdirectly irreducible but decomposable. Let $K = K_1 \oplus K_2$ and $p_i \in \operatorname{Ext} K_i$. Then there is a $p_3 \in \operatorname{Ext} K$ such that $\varphi(p_3) \neq \varphi(p_1)$, $\varphi(p_3) \neq \varphi(p_2)$, and $\varphi(p_3) \lhd \varphi(p_1) \vee \varphi(p_2)$ ([5] theorem 13.2). But by modulartity we have

$$\begin{aligned}\varphi(p_3) \lhd K_1 \wedge (\varphi(p_2) \vee \varphi(p_1)) &= \varphi(p_1) \vee (K_2 \wedge \varphi(p_2)) \\ &= \varphi(p_1) \vee \{0\} = \varphi(p_1).\end{aligned}$$

Hence $\varphi(p_3) \subseteqq \varphi(p_1)$ a contradiction.

Conversely suppose $\mathscr{F}(K)$ is reducible. Since a complemented modular lattice is relatively complemented, we have ([5] p. 94) that $\mathscr{F}(K)$ is a direct product of finitely many simple (hence subdirectly irreducible) lattices. As $\mathscr{F}(K)$ is reducible, this product contains at least two terms, say $\mathscr{F}(K) = \mathscr{F}_1 \oplus \mathscr{F}_2$ where $\mathscr{F}_1$ is simple and $\mathscr{F}_2$ is the product of the remaining terms. By theorem 2 of [3], there are $K_i \lhd K$ such that $\mathscr{F}_i = \mathscr{F}(K_i)$ and $K = K_1 \oplus K_2$. Thus K is decomposable and the result is established.

If K is an indecomposable cone for which $\mathscr{F}(K)$ is modular and of height 4 or greater, then by the coordinatization theorem ([5], theorems 13.4 and 13.5) $\mathscr{F}(K)$ is isomorphic to the lattice of all subspaces of a vector space over some division ring. The example of the $n \times n$ positive semidefinite matrices over the quaternions shows that the division ring need not be a field. The classification of modular face lattics of height 3 remains open. However, if K is self dual, then something more can be said at least in the low dimensional cases. Let (v, w) denote an inner product in V. The (positive) dual of K is $K^* = \{w \in V\colon (w, v) \geqq 0, \forall v \in K\}$. Then K is self dual if and only if $K = K^*$. If K is an indecomposable self dual cone and $\mathscr{F}(K)$ is modular, then $\mathscr{F}(K)$ is a modular ortholattice, and hence orthomodular. In [3] such cones were called *perfect* and a geometric version of the orthomodularity of $\mathscr{F}(K)$ was studied.

THEOREM. *Let K be an indecomposable self dual cone for which $\mathscr{F}(K)$ is modular. If* $\dim K = 4$, *then K is strictly convex. If* $\dim K = 5$, *then either* $h(K) = 2$ *or* $h(K) = 3$. *In the latter case every maximal face is of dimension* 3 *and strictly convex.*

REMARK. We already know that if $\dim K = 2$, then K is simplicial. If $\dim K = 3$ and $\mathscr{F}(K)$ is modular, then either K is simplicial or it is strictly convex even without the assumption that K is self dual (cf. [2]).

PROOF. First note that in general an indecomposable perfect cone can have no face of codimension 1. For if H is such a face and $\varphi(x) = H^D = \{y \in K: (y, h) = 0, \forall h \in H\}$, then $H \oplus \varphi(x)$ is a full self dual cone contained in K. But then $H \oplus \varphi(x) = K$ which contradicts the indecomposability of K.

Let $\dim K = 4$. We have $2 \leqq h(K) \leqq 4$. But if $4 = h(K) = \dim K$ and if F is a maximal face of K, then $3 = h(F) \leqq \dim F \leqq 3$. Thus F is of codimension 1, which is impossible. So $h(K) \leqq 3$. If $h(K) = 2$, then K is strictly convex. So suppose $h(K) = 3$. Let $H \in \mathscr{F}(K)$ be a maximal face. Then $2 = h(F) = \dim F$ by the general note above. Then $H = \varphi(g) \oplus \varphi(h)$ where $g, h \in H$ and $(g, h) = 0$. Let $G = \varphi(g)$, and let f be an extremal not in H. Since $\mathscr{F}(K)$ is modular, then $f + g, f + h \in \partial K$ (the boundary of K). We have

(1) $\varphi(f) \vee (\varphi(f)^D \wedge [\varphi(f) \vee \varphi(g)]) = \varphi(f) \vee \varphi(g) = \varphi(f) \oplus \varphi(g)$,

(2) $\varphi(f) \vee \varphi((f)^D \wedge [\varphi(f) \vee \varphi(h)]) = \varphi(f) \vee \varphi(h) = \varphi(f) \oplus \varphi(h)$.

Equation (1) implies $\varphi(f)^D \rhd \varphi(g)$, and (2) implies $\varphi(f)^D \rhd \varphi(h)$. Thus $\varphi(f)^D = \varphi(g) \oplus \varphi(h)$, whence $\varphi(f) = H^D$. But f was arbitrary, so K has only three extremals, an impossibility. Thus $h(K) = 3$ cannot hold.

Now suppose $\dim K = 5$. Then $h(K) \in \{2, 3, 4, 5\}$. If $h(K) = 2$, then K is strictly convex. Further $h(K) = 5$ is impossible for then K would have a face of codimension 1. Suppose $h(K) = 4$ and F is a maximal face. Then $3 = h(F) \leqq \dim F \leqq 3$. So $\dim F = 3$ and $\mathscr{F}(F)$ is modular. Hence F is simplicial for every maximal face of K. However, K is not polyhedral. Let H_1 be a maximal face of K and H_2 a 2-dimensional face of H_1. Let $G_1 = \varphi(g_1) = H_1^D$, and let g_2 be an extremal of K not in the convex cone $H_1 \oplus \varphi(g_1) \subset K$. Let $G_2 = \varphi(g_1) \vee \varphi(g_2)$. Then $\dim G_2 = 2$ since $h(G_2) = 2$. We claim that $G_2 \wedge H_1 = \{0\}$. For suppose $z \in G_2 \wedge H_1$. Then $0 \leqq z = g_1 + g_2 = h$ (where we relable $\alpha_i g_i$ by g_i for $\alpha_i > 0$ if need be). But then $0 \leqq g_i \leqq h$, $i = 1, 2$. Therefore $g_i \in H_1$ a contradiction. Thus $G_2 \wedge H_1 = \{0\}$. Also $G_2 \vee H_1 \rhd \varphi(g_1) \vee H_1 = K$. Thus H_1 is a common complement for G_2 and G_1 contradicting the modularity of $\mathscr{F}(K)$. Thus $h(K) = 4$ is impossible.

Finally consider the case of $h(K) = 3$. If F is a maximal face, then $2 \leqq \dim F \leqq 3$. Suppose F is a maximal face of dimension 2. Then $F = \varphi(f_1) \oplus \varphi(f_2)$ where $(f_1, f_2) = 0$, and $F^D = \varphi(f)$ where $(f, f_i) = 0$, $i = 1, 2$ If g is any extremal of K not in $\varphi(f_1)$, $\varphi(f_2)$ or $\varphi(f)$, $G = \varphi(f) \vee \varphi(g)$ is a maximal proper face. So by orthomodularity $G = \varphi(f) \vee (F \wedge G)$. Thus $F \wedge G \neq \{0\}$. Say $F \wedge G = \varphi(f_1)$. Then $G = \varphi(f) \vee \varphi(f_1)$. But g was arbitrary so every extremal of K is either in $G = \varphi(f) \vee \varphi(f_1)$ or in $H = \varphi(f) \vee \varphi(f_2)$. Then $G^D = \varphi(f_2)$ and $H^D = \varphi(f_1)$. Now pick extremals $g \in G$, $g \notin G \wedge H$ and $h \in H$, $h \notin G \wedge H$. Let $F_1 = \varphi(g) \vee \varphi(h)$. Then $F_1 \wedge G = \varphi(g)$ and $F_1 \wedge H = \varphi(h)$. If there is an extremal in F_1 other than g or h, then it is in either G or H and $F_1 = G$ or $F_1 = H$. Thus $F_1 = \varphi(g)$

$\oplus\ \varphi(h)$ and $F_1^D = \varphi(g_1) = \varphi(g)^D \wedge \varphi(h)^D$. Either $\varphi(g_1) \lhd G$ or $\varphi(g_1) \lhd H$. Let $G_1 = \varphi(g) \vee \varphi(g_1)$, $H_1 = \varphi(h) \vee \varphi(g_1)$. If $g_1 \in G$, then $G_1 = G$; while if $g_1 \in H$, then $H_1 = H$. As before we use modularity to show that every extremal is in G_1 or in H_1. But if $g_1 \in G$, we replace F_1 above by H_1 to conclude that $H_1 = \varphi(h) \oplus \varphi(g_1)$. In the case $g_1 \in H$, we conclude that $G_1 = \varphi(g) \oplus \varphi(g_1)$. In any event one of the G_1 and H_1 is of dimension 2 so that the other must be of dimension 3. Now we have $\dim(\langle H_1\rangle + \langle G\rangle) = 5$ since every extremal is in either G_1 or H_1 so that

$$\begin{aligned} 5 &= \dim(\langle H_1\rangle + \langle G_1\rangle) \\ &= \dim\langle H_1\rangle + \dim\langle G_1\rangle - \dim(\langle H_1\rangle \cap \langle G_1\rangle) \\ &= 5 - \dim(\langle H_1\rangle \cap \langle G_1\rangle), \end{aligned}$$

whence $\dim(\langle H_1\rangle \cap \langle G_1\rangle) = 0$. This contradicts $H_1 \wedge G_1 = \varphi(g_1)$. Thus K can have no two dimensional faces and the theorem is proven.

CONJECTURE. The case $h(K) = 3$ is not even possible if $\dim K = 5$.

EXAMPLES. There are cones of heights 3 and 4. If K_1 (respectively K_2) is the cone of 3×3 (respectively 4×4) positive semidefinite matrices in the space of all real symmetric matrices, then $h(K_1) = 3$ and $h(K_2) = 4$. However, $\dim K_1 = 6$ and $\dim K_2 = 10$.

There is also a perfect cone of dimension 4 which is not strictly convex, so that $\mathscr{F}(K)$ is not modular. To construct this example note first that by theorem 2 of **[4]** and by the process described in **[7]** it is enough to construct a convex surface over a triangular region in the plane which is strictly convex over the interior of the region and coincides with the triangle along the boundary. The following example was constructed by George Poole and myself. For $0 \leqq z \leqq 1$ we let $x \in [-\pi(1 - z)/2, \pi(1 - z)/2]$ and set

$$y = \begin{cases} .2 \sin \pi z \cos\left[\dfrac{x}{1 - z}\right] + \dfrac{\pi\sqrt{3}}{6}(1 - z), & \text{for } z \neq 1 \\ 0, \text{ for } z = 1. \end{cases}$$

Then the triple (x, y, z) describes one side of the desired surface. If the constructed surface is rotated around the origin by $\pm 2\pi/3$ radians, the three pieces together form the desired surface.

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