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BOUNDARY BEHAVIOR OF SPACES OF ANALYTIC FUNCTIONS

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0. **Introduction.** We de fine for $p \geqq 1$, $b > 0$, the space $M_{p,b}$ of function $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$, analytic in the unit disc D, such that

$$\|f\|_{p,b} = \limsup_{r \to 1} (1 - r)^b \left[\int_0^{2\pi} |f(z)|^p d\theta/2\pi \right]^{1/p} < \infty,$$
$$z = r \exp i\theta.$$

Two functions f and g are identified in $M_{p,b}$ whenever $\|f - g\|_{p,b} = 0$. We also define for $a > 0$, the space $M_{\infty,a}$ of functions $f(z)$, analytic in D such that

$$\|f\|_{\infty,a} = \limsup_{r \to 1} (1 - r)^a \max_{|z|=r} |f(z)| < \infty;$$

two functions f and g in $M_{\infty,a}$ are identified whenever $\|f - g\|_{\infty,a} = 0$, that is $f(r \exp i\theta) - g(r \exp i\theta) = o(1 - r)^a$, uniformly in θ.

For $b = 0$ a space $M_{p,b}$ reduces to a Hardy space H^p; for a description of the Hardy space see [**1**, **6**]. If f is in a Hardy space H^p, then $\|f\|_{p,b} = 0$ for all $b > 0$.

In addition to the obvious relations $M_{p,b} \subseteqq M_{q,b}$ for $p \geqq q$ we also have

$$M_{p,a-1/p} \subseteqq M_{q,a-1/q} \tag{1}$$

for $1 \leqq p \leqq q < \infty$, $a > 1/p$; moreover there exist constants C, C' such that

$$\begin{aligned} \|f\|_{\infty,a} &\leqq C\|f\|_{p,a-1/p} \\ \|f\|_{q,a-1/q} &\leqq C'\|f\|_{p,a-1/p} \end{aligned} \tag{2}$$

(cf. [**1**, p. 84]).

The relations (2) shows that if a function f is in a space $M_{p,a-1/p}$, $p \geqq 1$, $a > 1/p$, then $(1 - |z|)^a f(z)$ must remain bounded as z approaches a boundary point of D. In this note we will obtain restrictions on the values which $(1 - |z|)^a f(z)$ approaches as z approaches the boundary of D for functions f in a space $M_{p,a-1/p}$. We will also study topological properties of the $M_{p,a}$ spaces.

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1. In this section we give estimates on the coefficients of a function in $M_{p,b}$ and also on the area of the region onto which a function in an $M_{p,b}$ space maps the disc $|z| \leqq r$. The results are essentially contained in [5]; we will give numerical estimates. We conjecture that for concave functionals on an $M_{p,b}$ space the largest possible value is taken for functions whose Taylor series contain huge gaps while the smallest possible value is taken at functions of the form $C(1-z)^{-a}$ for some constant C. We have been able to confirm our conjecture only in a few cases.

We let p' denote the quantity $p/(p-1)$ for $1 < p < \infty$; if $p = 1$, we let $p' = \infty$, while if $p = \infty$, we let $p' = 1$.

THEOREM 1. *If* $f \in M_{p,a-1/p}$, $1 \leqq p \leqq \infty$, $a > 1/p$, *then*

$$\limsup |\hat{f}(n)|/n|^{a-1/p} \leqq [e/(a-1/p)]^{a-1/p}\|f\|_{p,a-1/p}.$$

PROOF. We deal only with the case $1 < p < \infty$; the cases $p = 1$ and $p = \infty$ are somewhat simpler. We have

$$|\hat{f}(n)| = \left|\int_C f(\zeta)/\zeta^{n+1} d\zeta\right|/2\pi,$$

where C is the circle $|\zeta| = n/(n + a - 1/p)$. If we use Hölder's inequality to estimate $|\hat{f}(n)|$ we obtain the result.

In the opposite direction we have the following theorem.

THEOREM 2A. *If* $f \in M_{\infty,a}$, *then*

$$\limsup |\hat{f}(n)|/n^{a-1} \geqq \|f\|_{\infty,a}/\Gamma(a);$$

if $f \in M_{p,a-1/p}$ *for some* p, $2 \leqq p < \infty$, $a \geqq 1$, *then*

$$\limsup |\hat{f}(n)|/n^{a-1} \geqq \|f\|_{p,a-1/p}[p'/[\Gamma(ap-1)/(p-1)]]^{1/p'}.$$

PROOF. We treat only the case $2 \leqq p < \infty$. Let $\lambda = \limsup |\hat{f}(n)|/n^{a-1}$ where $f(z) = \sum \hat{f}(n)z^n$ is a function in $M_{p,a-1/p}$. By the Hausdorff Young theorem (cf, [2: p. 145]).

$$\begin{aligned}\|f\|_{p,a-1/p} &= \limsup (1-r)^{a-1/p}\left(\int_0^{2\pi} |f(z)|^p d\theta/2\pi\right)^{1/p} \\ &\leqq \limsup (1-r)^{a-1/p}\left(\sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} r^{np'}\right)^{1/p'} \\ &= \lambda\Gamma[ap-1)/(p-1)]^{1/p'}/p'.\end{aligned}$$

Hence

$$\lambda \geqq p'\|f\|_{p,a-1/p}/\Gamma[(ap-1)/(p-1)]^{1/p'}.$$

If $1 \leqq p < 2$, we use the Hardy Littlewood theorem [1, p. 95] in place of the Hausdorff Young Theorem to obtain the following theorem.

THEOREM 2B. *If* $f \in M_{p,a-1/p}$ *for some* p, $1 \leqq p < 2$, *and* $a > 1/p$, *then*

$$\limsup (\hat{f}(n))/n^{a-1} \geqq \|f\|_{p,a-1/p}(p/\Gamma(ap-1))^{1/p}.$$

Theorem 1 is the best possible in that equality is achieved for the function $f(z) = \sum n_k^{a-1/p} z^n$, where the numbers n_k are chosen to increase sufficiently rapidly (for examples, the numbers must be chosen in such a way that n_{k+1}/n_k tends to infinity).

The first part of Theorem 2A is also the best possible; here equality is achieved for the function $f(z) = (1-z)^{-a}$; equality is also achieved in the second part of this theorem for these functions in the case $a = 1$, $p = 2$.

We let $A(r)$ denote the area of the region onto which the function $f(z)$ maps the disc $|z| \leqq r$.

LEMMA 1. *If* $f \in M_{\infty,a}$, *then* $\|f'\|_{\infty,a+1} \leqq (a+1)^{a+1}\|f\|_{\infty,a}/a^a$; *if* $f \in M_{p,b}$, $1 \leqq p < \infty$, $b > 0$, *then*

$$\|f'\|_{\infty,b+1+1/p} \leqq (pb+p+1)^{b+1+1/p}\|(1-z)^{-1}\|^2_{2p',(p+1)/2p}\|f\|_{p,b}/(pb)^b(p+1)^{1+1/p}.$$

Duren [1, pp. 65–66] showed that $\|(1-z)^{-1}\|_{2p',(p+1)/2p}$ is finite.

PROOF. By the Cauchy integral formula

$$|f'(z)| \leqq \int_C (|f(\zeta)|/(\zeta-z)^2|d\zeta|/2\pi.$$

For the first part of the theorem we take C as the circle $|\zeta - z| = (1-|z|)/(a+1)$; for the second part we take C as the circle $|\zeta| = |z| + (p+1)(1-|z|)/(pb+p+1)$ and apply Hölder's inequality.

We also have, following [5, p. 430], the next lemma.

LEMMA 2. *If* $f \in M_{2,b}$, *then* $\|f'\|_{2,b+1} \leqq (b+1)^{b+1}\|f\|_{2,b}/2b^b$.

THEOREM 3. *If* $f \in M_{\infty,a}$, $a > 0$, *then*

$$\limsup_{r\to 1}(1-r)^{2a+1}A(r) \leqq \pi(a+1)^{a+1}\|f\|^2_{\infty,a}/a^a;$$

if $f \in M_{p,b}$, $1 \leqq p < \infty$, $b > 0$, *then*

$$\limsup (1-r)^{2b+1}A(r) \leqq \pi[(p+1+pb)^{b+1+1/p}/(p+1)^{1+1/p}(pb)^b] \times \|(1-z)^{-1}\|_{2p',(p+1)/2p}\|f\|^2_{p,b}.$$

PROOF. We consider only the second part of the theorem; the first part can be dealt with in a similar fashion. For $1 \leqq p < \infty$

$$\limsup(1 - r)^{2b+1}A(r)$$
$$= \pi \limsup(1 - r)^{2b+1}\sum_{n=1}^{\infty} n|\hat{f}(n)|^2 r^{2n}$$
$$= \limsup(1 - r)^{2b+1}\int_0^{2\pi} |f(z)f'(z)|d\theta/2, \quad (z = r \exp i\theta)$$
$$\leqq (2\pi)^{1/p'} \limsup(1 - r)^{b+1} \max|f'(z)| \cdot \limsup(1 - r)^b\left(\int_0^{2\pi} |f(z)|^p d\theta\right)^{1/p}/2,$$

where the maximum is taken over the circle $|z| = r$. Hence

$$\limsup(1 - r)^{2b+1}A(r)$$
$$\leqq \pi\|f\|_{p,b}\|f'\|_{\infty,b+1}$$
$$\leqq \pi[(p + 1 + pb)^{b+1+1/p}/(p + 1)^{1+1/p}(pb)^b]\cdot\|(1 - z)^{-1}\|^2_{2p',p+1/2p}\|f\|^2_{p,b}.$$

We can also conclude from Lemma 2 the following theorem.

THEOREM 3B. *If* $f \in M_{2,b}$, $b > 0$, *then*

$$\limsup_{r\to 1}(1 - r)^{2b+1}A(r) \leqq \pi(b + 1)^{b+1}\|f\|^2_{2,b}/2b^b.$$

In the opposite direction we have the following theorem.

THEOREM 4. *If* $f \in M_{2,a-1/2}$, $a > 1/2$. *tjen*

$$\limsup(1 - r)^{2a}A(r) \geqq (2a - 1)\pi\|f\|^2_{2,a-1/2}/2.$$

PROOF. We have

$$A(r) = \pi\sum_{n=1}^{\infty} n|\hat{f}(n)|^2 r^{2n}$$

so that if $\limsup (1 - r)^{2a}A(r) \leqq \lambda$, then for each $\varepsilon > 0$, there is a number r_0 in $(0, 1)$ such that $A(r) \leqq (\lambda + \varepsilon)/(1 - r)^{2a}$ for $r \geqq r_0$. We consider r_0 fixed; we take r in $(r_0, 1)$ and let r tend to one. We have

$$\|f\|^2_{2,a-1/2} = \limsup (1 - r)^{2a-1}\sum_{n=0}^{\infty}|\hat{f}(n)|^2 r^{2n}$$
$$= 2 \limsup (1 - r)^{2a-1}\sum_{n=1}^{\infty} n|\hat{f}(n)|^2(r^{2n+1} - r_0^{2n+1})/(2n + 1)$$
$$= 2 \limsup(1 - r)^{2a-1}\int_{r_0}^{r} A(r')dr'/\pi$$
$$\leqq 2(\lambda + \varepsilon)/(2a - 1)\pi.$$

Hence $\lambda \geqq (2a - 1)\pi\|f\|^2_{2,a-1/2}/2$. Since ε is arbitrary the result follows.

In the case $a = 1$, the theorem is the best possible; for $f(z) = (1 - z)^{-1}$, $\|f\|_{2,1/2} = 2^{-1/2}$ and $(1 - r)^2A(r)$ tends to $\pi/4$.

COROLLARY. *If* $f \in M_{2,a-1/2}$, $a > 1/2$, *then*

$$\limsup\, (1 - r)^{2a} A(r) \geqq (2a - 1)\pi \|f\|^2_{\infty,a}/2^{2a}.$$

PROOF. We first note that $\|f\|_{\infty,a} < \infty$. We have

$$\begin{aligned}\|f\|_{\infty,a} &\leqq \limsup\, (1 - r)^a \sum_{n=0}^{\infty} |\hat{f}(n)| r^n \\ &\leqq \limsup\, (1 - r)^a \Big(\sum_{n=0}^{\infty} |\hat{f}(n)|^2 r^n\Big)^{1/2} \Big(\sum r^n\Big)^{1/2}\end{aligned}$$

by the Schwarz inequality, and the above quantity is bounded by $2^{a-1/2}\|f\|_{2,a-1/2}$. The result follows from the preceding theorem.

Again equality is achieved in the case $a = 1$ for the function $(1 - z)^{-1}$.

2. In this section we investigate the values which $(1 - |z|)^a f(z)$ approaches as z approaches a boundary point non-tangentially, for functions f in $M_{\infty,a}$ or in some space $M_{p,a-1/p}$. We let the symbol $C(\alpha, \eta)$ denote the curve $\theta = \alpha + \eta(1 - r) + o(1 - r)$, $r \to 1 -$, where α is in $[0, 2\pi)$ and η is a real number, that is, $C(\alpha, \eta)$ is a stolz ray terminating at $\exp i\alpha$ and making an angle arc sin $\eta/(1 + \eta^2)^{1/2}$ with the radius to the point $\exp i\alpha$. We let $q(\eta)$ denote the limit of $[(1 - |z|)/(1 - z)]^a$ as z approaches the point 1 along $C(0, \eta)$, this quantity is also equal to the limit of $[(1 - |z|)/(\exp i\alpha - z)]^a$ as z approaches the point $\exp i\alpha$ along the Stolz ray $C(\alpha, \eta)$.

THEOREM 5. *If* $f \in M_{\infty,a}$, $a > 0$, *and* $(1 - |z|)^a f(z)$ *tends to* w *as* z *tends to* $\exp i\alpha$ *along* $C(\alpha, \eta)$, *then* $(1 - |z|)^a f(z)$ *tends to* $w\, q(\eta')/q(\eta)$ *as* z *tends to* $\exp i\alpha$ *along* $C(\alpha, \eta')$.

PROOF. The function $F(z) = (\exp i\alpha - a)^a f(z)$ is analytic in the domain bounded by the curves $C(\alpha, \pm(\max[|\eta|, |\eta'| + 1]))$ and the smaller arc of the circle $|z| = 1/2$. As z tends to the point $\exp i\alpha$ along $C(\alpha, \eta)$, $F(z)$ tends to $w/q(\eta)$. By a theorem of Lindelöf [7, p. 76] $F(z)$ tends to $w/q(\eta)$ as z tends to the point $\exp i\alpha$ along $C(\alpha, \eta')$, that is, $(1 - |z|)^a f(z)$ tends to $w\, q(\eta')/q(\eta)$.

THEOREM 6. *Let* $\{z_n\}$ *and* $\{z_n'\}$ *be two sequences from D, each approaching a point in* ∂D *in such a way that*

$$|z_n - z_n'|/(1 - |z_n|) \text{ and } |z_n - z_n'|/(1 - |z_n'|)$$

remain bounded by a constant M. If for some function f in $M_{\infty,a}$, $a > 0$,

$$\lim\, (1 - |z_n|)^a f(z_n) = w, \text{ and } \lim\, (1 - |z_n'|)^a f(z_n') = w',$$

then

$$|w - w'| \leqq M[(a + 1)^{a+1}/a^a + a]\|f\|_{\infty,a}.$$

PROOF. We treat only the case $a \geqq 1$; the case $a < 1$ is dealt with in a similar manner. Without loss in generality, we take $|z_n| \geqq |z_n'|$. We have

$$\begin{aligned} &|(1 - z_n)^a f(z_n) - (1 - |z_n'|)^a f(z_n')| \\ &\quad \leqq (1 - |z_n|)^a |f(z_n) - f(z_n')| + (1 - |z_n|)^a - (1 - |z_n'|)^a|\,|f(z_n')| \\ &\quad \leqq (1 - |z_n|)^a |z_n - z_n'| \max|f'(z)| + a|z_n - z_n'| \max(1 - |z|)^{a-1}|f(z_n')|, \end{aligned}$$

where the above maxima are taken over the line segment L joining z_n to z_n'. Thus

$$\begin{aligned} &|(q - |z_n|)^a f(z_n) - (1 - |z_n'|)^a f(z_n')| \\ &\quad \leqq M(1 - |z_n)|^{a+1} \max|f'(z)| + Ma(1 - |z_n'|)^a |f(z_n')|. \end{aligned}$$

By Lemma 1 for each positive ε

$$(1 - |z_n|)^{a+1}|f'(z)| \leqq (1 - |z|)^{a+1}|f'(z)| \leqq [(a + 1)^{a+1}/a^a](\|f\|_{\infty,a} + \varepsilon)$$

for each point z on L provided z_n and z_n' are sufficiently close to one. (We note that the boundedness of $|z_n - z_n'|/(1 - |z_n|)$ and $|z_n - z_n'|/(1 - |z_n'|)$ insures that if $|z_n|$ and $|z_n'|$ are close to one, then each point z on L is close to one.) We now have for $|z_n|$ and $|z_n'|$ sufficiently close to one

$$\begin{aligned} &|(1 - |z_n|)^a f(z_n) - (1 - |z_n'|) f(z_n')| \\ &\quad \leqq M[a + 1)^{a+1}(\|f\|_{\infty,a} + \varepsilon)/a^a] + aM(\|f\|_{\infty,a} + \varepsilon). \end{aligned}$$

If we let n tend to infinity and thus let $|z_n|$ and $|z_n'|$ tend to one and ε tend to zero, we obtain the result.

If a function f is in a space $M_{p,a-1/p}$, $p \geqq 1$, $a > 1/p$, then $(1 - |z|)^a f(z)$ is bounded; moreover the next theorem shows that there are restrictions on the way in which $(1 - |z|^a|f(z)|$ may tend to a positive limit as z approaches a boundary point of the disc.

THEOREM 7A. *Let* $\{z_n^{(i)}\}$, $i = 1, 2, \ldots,$ *be a collection of sequences of points from D such that*

(3) $$|z_n^{(1)}| = |z_n^{(2)}| = \cdots = r_n,$$

(4) $$\lim_{n\to\infty} r_n = 1, \textit{ and}$$

(5) *there exists a positive constant* ζ *such that for* $i \neq j$,

$$|z_n^{(i)} - z_n^{(j)}| \geqq \zeta(1 - |z_n^{(i)}|),$$

then in order that there exist a function f *in some space* $M_{p,a-1/p}$, $p \geqq 1$, $a > 1/p$, *such that*

$$\lim_n (1 - r_n)^a f(z_n^{(i)}) = w_i,$$

$i = 1, 2, \ldots$, *uniformly in i, it is necessary that the numbers* w_i *satisfy the condition*

$$\sum_i |w_i|^{p+1} \leqq K\|f\|^p_{p,a-1/p}$$

for some constant K depending only on p and a.

PROOF. For sufficiently large n, $|f(z_n^{(i)})| \geqq |w_i|/2(1 - r_n)^a$ for all i. (We may assume that all w_i are different from zero.) There is a constant K_1 depending only on p and a such that

$$|f'(z)| \leqq K_1 \|f\|_{p,a-1/p}/(1 - |z|)^{a+1}$$

(cf. [5, pp. 430–431]). Let $I_{n,i}$ denote the arc with $|z| = r_n$ and

$$|\theta - \arg z_n^{(i)}| \leqq w_i(1 - r_n) \min(\zeta/3, 1/4K_1\|f\|_{p,a-1/p}).$$

On $I_{n,i}$, if r_n is sufficiently close to one,

$$\begin{aligned} |f(z)| & \\ &\geqq w_i/2(1 - r_n)^a \\ &\quad - [w_i(1 - r_n) \min(\zeta/3, 1/4K_1\|f\|_{p,a-1/p})] \max|f'(z)| \\ &\geqq w_i/4(1 - r_n)^a. \end{aligned}$$

Since the arcs $I_{n,i}$ are disjoint, if r_n is sufficiently close to one,

$$\begin{aligned} \int_{|z|=r_n} |f(z)|^p d\theta &\geqq \sum_i \int_{I_{n,i}} |f(z)|^p d\theta \\ &\geqq K_2 \sum_i |w_i|^{p+1}/(1 - r_n)^{ap-1}, \end{aligned}$$

where K_2 is a universal constant. The result follows.

We are actually able to prove slightly more.

$$\text{Let } \mathscr{F}_r(\theta) = \max_{0 \leqq |z| \leqq r} f(|z| \exp i\theta).$$

Then there is a constant K_3 such that

$$\int_{|z|=r} |\mathscr{F}_r(\theta)|^p d\theta \leqq K_3 \int_{|z|=r} |f(z)|^p d\theta,$$

$1 < p < \infty$ (cf. [4, p. 103–108]). Hence, we have the following theorem.

THEOREM 7B. *Let* A_1 *and* A_2 *be two positive constants and for each i, let* $\{z_n^{(i)}$ *denote a sequence of points such that*

(3′) $$A_1 \leqq (1 - |z_n^{(i)}|)/(1 - |z_n^{(j)}|) \leqq A_2,$$

(4′) $$\lim_{n\to\infty} |z_n^{(i)}| = 1, \; i = 1\; 2, \ldots, \textit{ and}$$

(5′) *there exists a positive constant* ζ *such that*

$$|z_n^{(i)} - z_n^{(j)}| \geqq \zeta/(1 - |z_n^{(i)}|)$$

for all n, i, j, such that $i \neq j$, *then in order that there exist a function* $f \in M_{p,a-1/p}$ *for some* $p \geqq 1$, $a > 1/p$, *such that*

$$\lim(1 - |z_n^{(i)}|)^a f(z_n^{(i)}) = w_i,$$

$i = 1, 2, \ldots$, *it is necessary that*

$$\sum |w_j|^{p+1} \leqq K\|f\|_{p,a-1/p}^p$$

for some constant K depending only on p, and a.

3. In this section we determine the duals of the $M_{p,b}$ spaces; we will also give some necessary conditions for weak convergence in the $M_{p,b}$ spaces.

If X is a locally compact space, then X can be densely imbedded in a compact space βX in such a way that every bounded continuous complex function has a continuous extension f^β to βX. The space βX is called the Stone-Cech compactification of X (for a description of the Stone-Cech compactification, cf. [**3**, pp. 82–93]). We will use the symbol βX to denote the Stone-Cech compactification of X; if f is a bounded continuous function on X, then f^β will always denote its continuous extension to βX; if ν is a point in βX the symbol f_ν^β will express the fact that the function f^β has been evaluated at ν.

If f is a function in $M_{\infty,a}$, $a > 0$, then the function $F(r, \theta) = (1 - r)^a f(r \exp i\theta)$ is bounded and continuous in D and consequently has a continuous extension F^β to βD. We now respresent $M_{\infty,a}$ as a space of continuous functions on a compact space Δ formed from $\beta D - D$ by identifying two points ν_1 and ν_2 in $\beta D - D$ whenever $F_{\nu_1}^\beta = F_{\nu_2}^\beta$ for all $f \in M_{\infty,a}$, that is

$$[(1 - r)^a f(z)]_{\nu_1}^\beta = [(1 - r)^a f(z)]_{\nu_2}^\beta;$$

we give Δ the weakest topology which makes all functions $[(1 - r)^a f(r \exp i\theta)]^\beta$ continuous. The space Δ admits the metric d given by

$$\begin{aligned} d(\nu_1, \nu_2) &= \mathrm{Lub}|F_{\nu_1}^\beta - F_{\nu_2}^\beta| \\ &= \mathrm{lub}|[(1 - r)^a f(r \exp i\theta]_{\nu_1}^\beta - (1 - r)^a f(r \exp i\theta)]_{\nu_2}^\beta| \end{aligned}$$

where the lub is taken over all functions f in $M_{\infty,a}$ such that $\|f\|_{\infty,a} = 1$.

It can be shown that Δ does not contain an analytic disc. To see this note that the function $(1 - z)^{-a}$ is in $M_{\infty,a}$ and that the corresponding function $\{[(1 - r)/(1 - z)]^a\}_\nu^\beta$ vanishes when ν is outside the closure in Δ of each Stolz angle with vertex at $z = 1$, while this function takes values on some curve in the complex plane when ν is in some Stolz angle with vertex at $z = 1$.

As in [**6**, pp. 166–168] we may form the fiber W_α above each point exp $i\alpha$ in Δ; W_α consists of all limit points in Δ of all nets $\{z_\mu\}$ which tend to exp $i\alpha$. No point in Δ which is in the closure of the Stolz angle with vertex at the point 1 can lie in the closure of any union of W_α, $\alpha \neq 0$.

We denote the half-open interval [0, 1) by I.

THEOREM 8. *The set of linear functionals on $M_{p,b}$ given by*

$$L(f) = [(1 - r)^b \int_0^{2\pi} f(r \exp i\theta)\phi(r, \theta)d\theta]_\rho^\beta, \tag{6}$$

$f \in M_{p,b}$, where $\phi(r, \theta)$ ranges over the space $\Lambda_{p'}$ of functions which are continuous in r on I, and such that $\int_0^{2\pi}|\phi(r, \theta)|^{p'}$ remains bounded for $0 \leqq r < 1$, and ρ ranges over $\beta I - I$ are woak $*$ dense in the dual of $M_{p,b}$. Conversely each functional of the form (6) *represents a bounded linear functional on $M_{p,b}$ such that*

$$\|L\| \leqq (2\pi)^{1/p} \limsup_{r\to 1}\left(\int_0^{2\pi} |\phi(r, \theta)|^{p'} d\theta)\right)^{1/p'}.$$

PROOF. It is easily seen from Hölder's inequality that each functional of the form (6) is a bounded functional whose norm satisfies the stated inequality. To see that functionals of the form (6) are weak $*$ dense in the dual of $M_{p,b}$ we let f be an element of $M_{p,b}$ such that $\|f_{p,b} > 0$. We will construct a functional L of the form (6) such that $L(f) \neq 0$. Let

$$\phi(r, \theta) = \begin{cases} |f(z)|^{p-2}\bar{f}(z) & \text{if } f(z) \neq 0 \\ 0 & \text{if } f(z) = 0. \end{cases}$$

We then have, for some $\rho \in \beta I - I$,

$$L(f) = \text{lub}[(1 - r)^b \int_0^{2\pi} |f(z)|^p \, d\theta]_\rho^\beta = 2\pi \|f\|_{p,b}^p > 0.$$

The result follows.

With a slight modification of the above argument we have

THEOREM 8B. *The set of functionals on $M_{1,b}$ given by*

$$L(f) = [(1 - r)^b \int_0^{2\pi} f(z)\phi(r, \theta)d\theta]_\rho^\beta, \tag{6'}$$

$f \in M_{1,b}$, where ϕ ranges over the space Λ_∞ of functions which remain bounded in D and ρ ranges over $\beta I - I$ are weak $*$ dense in the dual of $M_{1,b}$. Conversely each functional L of the form (6′) *represents a bounded functional on $M_{1,b}$ such that*

$$\|L\| \leqq 2\pi \limsup_{r\to 1} |\phi(r, \theta)|.$$

If we let

$$\phi(r, \theta) = \begin{cases} \bar{f}(z) & \text{when } f \neq 0 \\ 0 & \text{when } f = 0, \end{cases}$$

then ϕ need not be continuous in r; however, ϕ can be approximated by a continuous function.

THEOREM 9. *The set of functionals on $M_{\infty,a}$, given by*

$$L(f) = [(1 - r)^a \int_0^{2\pi} f(z) d\mu_r(\theta)]^\beta_\rho, \tag{7}$$

$f \in M_{\infty,a}$, where $\mu_r(\theta)$ ranges over the measures defined on each circle $|z| = r$, $0 \leqq r < 1$, which depend continuously on r and which are such that $\int_0^{2\pi} |d\mu_r(\theta)|$ is uniformly bounded, and ρ ranges over $\beta I - I$ are weak $*$ dense in the dual of $M_{\infty,a}$. Conversely each functional of the form (7) is in the dual of $M_{\infty,a}$ and

$$\|L\| \leqq \lim_r \sup \int_0^{2\pi} |d\mu_r(\theta)|.$$

PROOF. To see that the functionals of the form (7) are weak $*$ dense in the dual of $M_{\infty,a}$ we let f be an element of $M_{\infty,a}$ such that $\|f\|_{\infty,a} > 0$. Then there is a sequence of points $\{z_n\}$ approaching the boundary of D such that

$$\lim \sup (1 - |z_n|)^a |f(z_n)| > 0.$$

For $\mu_r(\theta)$ we take a measure which is the Dirac delta measure concentrated at z_n on each circle $|z| = r_n$ and which depends continuously on r. Then

$$\lim \sup (1 - r)^a \int_0^{2\pi} f(z) d\mu_r(\theta) > 0.$$

The result follows.

The $M_{p,a}$ spaces are not complete for $a > 0$. However, each space $M_{p,a}$ can be imbedded in a complete space $\mathscr{M}_{p,a}$ consisting of equivalence classes of Cauchy sequences $\{f_n\}$ of elements from $M_{p,a}$; two Cauchy sequences in $M_{p,a}\{f_n\}$ and $\{g_n\}$ are equivalent if $\|f_n - g_n\|_{p,a}$ tends to zero as n tends to infinity. As usual the norm of an element of $\mathscr{M}_{p,a}$ can be defined as $\lim_{n\to 0} \|f_n\|_{p,a}$ where $\{f_n\}$ is a Cauchy sequence of elements from $M_{p,a}$ which represents f; clearly this limit does not depend on the choice of Cauchy sequence. It should be noted that the elements of $\mathscr{M}_{p,a-1/p}$ are limits (in the norm topology) of Cauchy sequences in $M_{\infty,a}$ and hence each element $\{f_n\}$ of $\mathscr{M}_{p,a-1/p}$ induces the continuous function

$$\lim_{n\to\infty} [(1 - r)^a f_n(r \exp i\theta)]^\beta_{\nu}$$

on Δ.

THEOREM 10. *Let $\{z_n\}$ be an infinite sequence of points on the unit circle.*

Let $\{f_n\}$ *be a sequence of functions in a space* $M_{p,a-1/p}$, $1 \leqq p \leqq \infty$, $a > 1/p$, *such that for each* m $\limsup\,(1 - |z|)^a|f_m(z)|$ *is greater than some positive constant* ζ *as* z *tends to* z_m *while for* $n \neq m$ $(1 - |z|)^a|f_n(z)|$ *tends to zero as* z *tends to* z_m. *The set* $\{f_m\}$ *does not have compact closure in* $\mathscr{M}_{p,a-1/p}$.

PROOF. The result is rather trivial for $p = \infty$. If $p < \infty$, there is a sequence of points $\{z_n^{(j)}$ from D which tends to z_n as j tends to infinite such that

$$(1 - |z_n^{(j)}|)^a f_n(z_n^{(j)}) \geqq \zeta/2$$

provided j is sufficiently large. As in the proof of Theorem 7A we construct an arc I containing the point $z_n^{(j)}$ such that

$$(1 - |z|)^{ap-1}\int_I |f_n(z)|^p d\theta$$

exceeds a positive constant; on the other hand if $m \neq n$,

$$(1 - |z|)^{ap-1}\int_I |f_m(z)|^p d\theta$$

can be made arbitrarily small if j is sufficiently large. Thus the distance between each two distinct elements of $\{f_m\}$ exceeds some positive constant. Thus the set $\{f_m\}$ cannot have compact closure.

We give some necessary conditions for weak convergence in $M_{p,b}$.

THEOREM 11. *If* $\{f_n\}$ *is a sequence of functions in* $M_{p,a-1/p}$, $1 \leqq p \leqq \infty$, $a > 1/p$, *which is weakly convergent to zero, then*

$$\lim_{n\to\infty}\limsup_{r\to 1}\,(1 - r)^a|f_n(z)| = 0.$$

PROOF. This result follows immediately from the fact that for each point ρ in βI − I the functional L on $M_{p,a-1/p}$ given by

$$L(f) = [(1 - r)^a f(r \exp i\theta)]_\rho^\beta,$$

$f \in M_{p,a-1/p}$, is continuous.

THEOREM 12. *For each* r, $0 \leqq r < 1$, *let* $E(r)$ *denote a measurable subset the circle* $|z| = r$ *such that the measure of* $E(r)$, $|E(r)|$, *depends continuously on* r. *If* $\{f_m(z)\}$ *is a sequence from* $M_{p,a-1/p}$ *for some* $p > 1, a > 1/p$, *which converges weakly to zero, then*

$$\lim_{m\to\infty}\limsup_{r\to 1}\,(1 - r)^a\left|\int_{E(r)} f_m(z)d\theta\right| = 0.$$

The result follows from the fact that for each point $\rho \in \beta I - I$ the functional on $M_{p,a-1/p}$ given by

$$L(f) = [(1 - r)^a \int_{E(r)} f(r \exp i\theta) d\theta]_\rho^\beta,$$

$f \in M_{p, a-1/p}$, is bounded.

THEOREM 13. *If* $\{f_m\}$ *is a sequence of functions in* $M_{p, a-1/p}$, $1 \leqq 0 \leqq \infty$, $a > 1$, *which is weakly convergent to zero, then*

$$\lim_{m \to \infty} \limsup_{n} n^{1-a} |\hat{f}_m(n)| = 0.$$

Let $\mathbf{N}$ denote the discrete space of natural numbers. The result follows immediately from the fact that for each $\lambda \in \beta\mathbf{N} - \mathbf{N}$ the functional on $M_{p, a-1/p}$ given by $L(f) = [n^{1-a}\hat{f}(n)]_\lambda^\beta$ is continuous.

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