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ASYMPTOTIC STABILITY OF A COUPLED DIFFUSION SYSTEM ARISING FROM GAS-LIQUID REACTIONS

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ABSTRACT. This paper is concerned with the asymptotic behavior of the time dependent solution in relation to the corresponding steady-state solution for a nonlinear coupled reaction-diffusion system arising from gas-liquid absorption. Existence and uniqueness of both time-dependent and steady-state solutions are discussed, and various boundary conditions are included in the discussion. It is shown in the case of a homogeneous system that for any non-negative initial function the time dependent solution converges exponentially to zero as $t \to \infty$ when the boundary condition is of either Dirichlet or mixed type. However, for Neumann type boundary condition, multiple constant steady-state solutions exist and the time-dependent solution may converge to any one of these steady-states. Depending on the relative magnitude between the initial functions, convergence of the time-dependent solution to one of these constant states is explicitly given. For a nonhomogeneous system with nonzero boundary or internal data the convergence of the time-dependent solutions also depends on the relative magnitude between the components of the steady-state solution. A characterization of the stability and instability of a steady-state solution is established, and in the case of stability an estimate of the stability region is given.

1. Introduction. In the theory of a gas-liquid diffusion reaction system in a p-dimensional medium Ω the concentration of the dissolved gas $u = u(t, x)$ and the reactant $v = v(t, x)$ are governed by the coupled reaction-diffusion equations (cf. [**2–4**, **6**, **12**])

$$\begin{aligned} u_t - D_1 \Delta u &= -k_1 uv \\ v_t - D_2 \Delta v &= -k_2 uv \end{aligned} \qquad (t > 0,\ x \in \Omega) \tag{1.0}$$

where Δ is the Laplacian operator, D_1, D_2 are the diffusion coefficients, k_1, k_2 are the reaction rate constants and $r_i = -k_i uv$ represent the rate of reactions. A more general reaction rate is given by

$$r_1(u, v) = -k_1 u^m v^{m'}, \quad r_2(u, v) = -k_2 u^n v^{n'}$$

and is called the (m, n)th order reaction (cf. [**4**]). Motivated by the above

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gas-liquid models we consider a more general coupled reaction-diffusion system in the form

$$\text{(1.1)} \qquad \begin{aligned} u_t - L_1 u &= -f_1(t, x, u, v) \\ v_t - L_2 v &= -f_2(t, x, u, v) \end{aligned} \qquad (t > 0,\ x \in \Omega),$$

where Ω is a bounded domain in the Euclidean space $R^p\,(p = 1, 2, \ldots)$, f_1, f_2 are continuous functions of $\mathbf{R}^+ \times \bar{\Omega} \times \mathbf{R}^+ \times \mathbf{R}^+$ ($\mathbf{R}^+ = [0, \infty)$), and for each $i = 1, 2$, L_i is a uniformly elliptic operator in the form

$$L_i = \sum_{k,\ell=1}^{p} a^i_{k\ell}(x)\partial^2/\partial x_k \partial x_\ell + \sum_{\ell=1}^{p} b^i_\ell(x)\partial/\partial x_\ell.$$

On the boundary surface $\partial\Omega$ and at time $t = 0$ we consider the following mixed boundary condition

$$\text{(1.2)} \qquad \begin{aligned} B_1[u] &\equiv \begin{cases} \beta_1(x)\partial u/\partial \nu + u = h_1(x) & (t > 0,\ x \in \Gamma_1) \\ \partial u/\partial \nu = 0 & (t > 0,\ x \in \partial\Omega - \Gamma_1) \end{cases} \\ B_2[v] &\equiv \begin{cases} \beta_2(x)\partial v/\partial \nu + v = h_2(x) & (t > 0,\ x \in \Gamma_2) \\ \partial v/\partial \nu = 0 & (t > 0,\ x \in \partial\Omega - \Gamma_2) \end{cases} \end{aligned}$$

and the initial condition

$$\text{(1.3)} \qquad u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \quad (x \in \Omega),$$

where Γ_1, Γ_2 are portions of the boundary $\partial\Omega$, ν is the outward normal (or conormal) derivative on Γ_1 or Γ_2 and β_1, β_2 are non-negative functions on Γ_1, Γ_2, respectively. The surface Γ_1 or Γ_2 (or both) is allowed to be the whole boundary $\partial\Omega$ but we assume that neither of them is empty. The pure Neumann-type boundary condition (i.e., both Γ_1, Γ_2 are empty)

$$\text{(1.4)} \qquad \partial u/\partial \nu = \partial v/\partial \nu = 0 \quad (t > 0,\ x \in \partial\Omega)$$

will be given special attention. The consideration of the boundary condition (1.2) is motivated by the physical situation which often involves various mixed type boundary conditions, including the present gas-liquid problem. An important example is the one-dimensional gas-liquid absorption problem considered in **[4]** where the boundary condition is given by

$$\text{(1.5)} \qquad u(t, 0) = u(t, \ell) = 0,\ v_x(t, 0) = 0,\ v(\ell, 0) = b_0 \quad (b_0 > 0).$$

A different nonhomogeneous boundary condition was considered in **[2, 6]**. In the present boundary condition (1.2) the values of β_1, β_2 are allowed to be zero on part or the whole of Γ_i so that (1.2) includes the Dirichlet or a combination of Dirichlet, Neumann and third type boundary conditions.

The model given by (1.0) was investigated by a number of researches in recent years (cf. **[2–4, 6, 9, 12]**). The work in **[3, 6]** gives some results on the asymptotic behavior of the solution under a Dirichlet type boundary condition. The more general (m, n)-*th* order reaction model was discussed in **[4]** in relation to the film model in gas-liquid absorption. The main purpose of this paper is to investigate the asymptotic behavior of the time-dependent solution of the gas-liquid problem in relation to the corresponding steady-state solution. Special attention is given to the model (1.0) under the Neumann boundary condition (1.4). A comparison between the two types of boundary conditions (1.2) and (1.4) exhibits some rather different asymptotic behavior of the solution. Existence and uniqueness of both time-dependent and steady-state solutions are included in the discussion. Our investigation of the stability problem also includes some explicit estimates for the stability and the instability regions of a steady-state solution. Here the definition of stability and instability is in the usual sense of Lyapunov in the space of continuous functions in $\bar{\Omega}$. It turns out that the estimated stability region is also an invariant set of the system.

In §2 we investigate the asymptotic behavior of the solution for the gas-liquid model (1.0) under the homogeneous boundary condition (1.2) and (1.4). It is shown that for the Neumann boundary condition (1.4), the time-dependent solution converges to one of the steady-state solutions in the form $(0, c_1)$ or $(c_2, 0)$, and the value of c_1 or c_2 depends explicitly on the relative magnitude of the spatial averages of the initial functions u_0, v_0. The convergence of the solution is usually at an exponential rate, but in certain special situations it converges at the rate $O(1/t)$. However, for a Dirichlet or mixed boundary condition the time-dependent solution always converges to zero exponentially, and independently of the initial function. In §3, we establish an existence-comparison theorem, in terms of upper and lower solutions, for the general system (1.1)–(1.3) using monotone arguments. This theorem not only leads to some information concerning the existence and the rate of convergence of the time-dependent solution, but more importantly it is the main tool in establishing the stability and instability criteria for the steady-state solutions of certain nonhomogeneous systems. §4 is devoted to such a nonhomogeneous system in relation to the gas-liquid problem. Here we prove the existence of a non-negative steady-state solution and establish conditions for ensuring the stability and instability of this solution. These conditions characterize the asymptotic behavior of the time-dependent solution under initial perturbations from a given steady-state solution.

2. The homogeneous gas-liquid problem. In this section we investigate the asymptotic behavior of the solution for the gas-liquid equation (1.0) under either the homogeneous boundary condition (1.2) (i.e., $h_1 = h_2 = 0$) or the Neumann boundary condition (1.4). Due to the distinct behavior of the solution under these two types of boundary conditions it becomes necessary to treat these systems by different arguments. We first discuss the Neumann boundary problem.

For the Neumann problem (1.0) (1.3) (1.4) there are infinitely many constant steady-state solutions in the form $(c_1, 0)$, $(0, c_2)$, where c_1, c_2 are arbitrary constants. An interesting and delicate question about this system is whether the time-dependent solution converges to one of these constant solutions, and if it does to which one it converges. We shall answer this question in the affirmative and show that the limit of the solution depends solely on the relative magnitude of the spatial "averages" $\hat{u}_0$, $\hat{v}_0$, where

$$\hat{u}_0 = \int_\Omega u_0(x)dx, \ \hat{v}_0 = \int_\Omega v_0(x)dx.$$

In the following theorem we assume that the system (1.0)(1.3)(1.4) has a unique non-negative solution. The existence of such a solution will be justified in the next section.

THEOREM 2.1. *Let* $k_1 > 0$, $k_2 > 0$, $u_0 \geqq 0$, $v_0 \geqq 0$, *and let* (u, v) *be the non-negative solution of* (1.0), (1.3), (1.4). *Then for* $k_1\hat{v}_0 \geqq k_2\hat{u}_0$, *the solution* (u, v) *satisfies the relation*

$$\lim_{t\to\infty} u(t, x) = 0, \ \lim_{t\to\infty} v(t, x) = \hat{v}_0 - (k_2/k_1)\hat{u}_0; \ (x \in \Omega) \tag{2.1}$$

while for $k_1\hat{v}_0 \leqq k_2\hat{u}_0$,

$$\lim_{t\to\infty} u(t, x) = \hat{u}_0 - (k_1/k_2)\hat{v}_0, \ \lim_{t\to\infty} v(t, x) = 0. \ (x \in \Omega) \tag{2.2}$$

PROOF. By integrating (1.0) over Ω, applying the divergence theorem and using the boundary condition (1.4) we obtain

$$\hat{u}' = -k_1\widehat{uv}, \ \hat{v}' = -k_2\widehat{uv} \ (\hat{u}' \equiv d\hat{u}/dt, \ etc) \tag{2.3}$$

where $\widehat{uv} = m(\Omega)^{-1} \int_\Omega uvdx$. These equations imply that $\hat{u}$, $\hat{v}$ are monotone nonincreasing and

$$k_1\hat{v} - k_2\hat{u} = k_1\hat{v}_0 - k_2\hat{u}_0 = \text{const.} \quad (t \geqq 0) \tag{2.4}$$

It follows from the non-negative property of the solution that $\hat{u}$, $\hat{v}$ must approach some limits u_∞, v_∞ as $t \to \infty$. We first show that $u(t, x) \to u_\infty$, $v(t, x) \to v_\infty$ as $t \to \infty$.

Let $q(t, x) = -k_2(uv - \widehat{uv})$, $V_0(x) = v_0(x) - \hat{v}_0$, and consider the linear system

$$
\begin{aligned}
&V_t - D_2\Delta^2 V = q(t, x) && (t > 0,\ x \in \Omega)\\
(2.5)\qquad &\partial V/\partial\nu = 0 && (t > 0,\ x \in \partial\Omega)\\
&V(0, x) = V_0(x) && (x \in \Omega)
\end{aligned}
$$

Then by the principle of superposition we may write $V = V_1 + V_2$, where V_1, V_2 are solutions of (2.5) corresponding to V_0 and q, respectively. Notice that $\int_\Omega V_0 dx = \int_\Omega q(t, x)dx = 0$ and $\partial q/\partial\nu = 0$ for all $t \geqq 0$. By eigenfunction expansion, the solutions V_i may be written as

$$(2.6)\qquad V_i = \sum_{j=1}^{\infty} \alpha_j^{(i)}(t)\phi_j(x), \quad i = 1, 2,$$

where ϕ_j's are the (normalized) eigenfunctions of the operator $D_2\nabla^2$ under the Neumann boundary condition. For $i = 1$ ($q \equiv 0$), the Fourier coefficients $\alpha_j^{(1)}$ are given by $\alpha_j^{(1)} = c_j\exp(-\lambda_j t)$ where λ_j are the eigenvalues corresponding to ϕ_j and $c_j = \int_\Omega V_0\phi_j\, dx$. Since $c_0 = \int_\Omega V_0 dx = 0$, $\lambda_0 = 0$, $\lambda_n > 0$ for $n = 1, 2, \ldots$, it follows from (2.6) that $V_1(t, x) \to 0$ as $t \to \infty$. For $i = 2$ ($V_0 \equiv 0$) the functions $\alpha_j^{(2)}$ are determined from the Cauchy problem

$$(2.7)\qquad (\alpha_j^{(2)})' + \lambda_j\alpha_j^{(2)} = \gamma_j(t), \ \alpha_j(0) = 0, \quad j = 1, 2, \ldots,$$

where γ_j's are the Fourier coefficients of q given by

$$\gamma_j(t) = \int_\Omega q(t, x)\phi_j(x)dx, \quad j = 0, 1, 2, \ldots.$$

Since $\gamma_0 = 0$ and $\gamma_j(t) \to 0$ as $t \to \infty$ the solution $\alpha_j^{(2)}$ of (2.7) satisfies the relation $\alpha_0 = 0$ and $\alpha_j(t) \to 0$ as $t \to \infty$. By (2.6), $V_2(t, x) \to 0$ and therefore $V(t, x) \to 0$ as $t \to \infty$. But by uniqueness, the solution V of (2.5) coincides with $v - \hat{v}$; we conclude that $\lim v(t, x) = v_\infty$ as $t \to \infty$. A similar argument gives $u(t, x) \to u_\infty$ as $t \to \infty$. The above result implies that $\widehat{uv} \to m(\Omega)u_\infty v_\infty$ as $t \to \infty$. It follows from

$$\lim_{t\to\infty} \int_0^t uv(\tau)d\tau = k_1^{-1} \lim_{t\to\infty} (\hat{u}_0 - \hat{u}(t)) = k_1^{-1}(\hat{u}_0 - u_\infty)$$

that $u_\infty v_\infty = 0$, that is, either $u_\infty = 0$ or $v_\infty = 0$ (or both). Since by (2.4), $k_1 v_\infty - k_2 u_\infty = k_1\hat{v}_0 - k_2\hat{u}_0$, we conclude that $u_\infty = 0$, $v_\infty = \hat{v}_0 - (k_2/k_1)\hat{u}_0$ when $k_1\hat{v}_0 - k_2\hat{u}_0 \geqq 0$, and $v_\infty = 0$, $u_\infty = u_0 - (k_1/k_2)v_0$ when $k_1\hat{v}_0 - k_2\hat{u}_0 \leqq 0$. The results of (2.1) and (2.2) follow immediately from $u(t, x) \to u_\infty$, $v(t, x) \to v_\infty$ as $t \to \infty$.

We next consider (1.0) under the Dirichlet or the mixed boundary condition (1.2). It is well-known that for any non-negative initial function there exists a constant $\rho_i \geqq 0$ such that the solution u_i of the linear scalar system

(2.8) $$(u_i)_t - L_i u_i = 0, \quad B_i[u_i] = 0 \quad (i = 1, 2)$$

satisfies the relation $0 \leqq u_i(t, x) \leqq \rho_i \exp(-\lambda_i t)$, where λ_i is the least eigenvalue of the eigenvalue problem

(2.9) $$L_i\phi_i + \lambda_i\phi_i = 0, \quad B_i[\phi_i] = 0 \quad (i = 1, 2)$$

(*e.g.*, see **[11]**). The value of λ_i is real, positive and its corresponding eigenfunction ϕ_i is also positive in Ω (cf. **[13]**). Now if (u, v) is a nonnegative solution of (1.1)–(1.3), then

(2.10) $$u_t - L_1 u \leqq 0,\ B_1[u] = 0 \text{ and } v_t - L_2 v \leqq 0,\ B_2[v] = 0.$$

By the maximum principle the above relations imply that $u \leqq u_1$, $v \leqq u_2$. For comparison purposes we state this as the following theorem.

THEOREM 2.2. *Let* $k_1 > 0$, $k_2 > 0$, $u_0 \geqq 0$, $v_0 \geqq 0$ *and let* (u, v) *be the non-negative solution of* (1.0)(1.2)(1.3). *Then there exist positive constants* ρ_1, ρ_2 *such that*

(2.11) $$0 \leqq u(t, x) \leqq \rho_1 e^{-\lambda_1 t},\ 0 \leqq v(t, x) \leqq \rho_2 e^{-\lambda_2 t} \quad (t > 0,\ x \in \bar{\Omega})$$

where λ_1, λ_2 *are the respective least eigenvalues of* (2.9).

REMARK 2.1. It is easily seen from the above argument that the result of Theorem 2.2 also holds for the general system (1.1)–(1.3) when $f_i(t, x, u, v) \geqq 0$ for $u \geqq 0$, $v \geqq 0$ $(i = 1, 2)$. In particular, this is the case for the (m, n)-th order reaction model where $f_1 = k_1 u^m v^{m'}$, $f_2 = k_2 u^m v^{n'}$.

It is seen from Theorems 2.1 and 2.2 that a change of boundary condition from Neumann type to Dirichlet or mixed type leads to rather different asymptotic behavior of the solution. In the latter situation, whether Dirichlet or mixed type, the zero steady-state solution is always globally asymptotically stable; while in the former case it may not even be locally asymptotically stable. It is interesting to note that in the special case of constant initial condition $u_0 = A_0$, $v_0 = B_0$ the solution (u, v) does converge to zero when $k_1 B_0 = k_2 A_0$ but its rate of convergence is of order $o(1/t)$. This fact together with the existence of a non-negative solution to the system (1.0)(1.2)(1.3) (and (1.0)(1.3)(1.4)) will be justified in the next section.

3. An existence-comparison theorem. In order to study the qualitative property of the solution of the gas-liquid problem with nonhomogeneous input data we first establish an existence-comparison theorem for the general system (1.1)–(1.3) using monotone arguments and the notion of upper and lower solutions. This approach is constructive and can be used to obtain similar existence-comparison theorems for the corresponding steady-state problem without any complication. In developing these theorems we make the usual Hölder continuity assumption on the coefficients of L_i, the boundary and initial data β_i, h_i, u_0, v_0, and the functions f_i in their respective domains. It is also assumed that β_i, h_i, u_0, v_0 are non-negative and the boundary of Ω is smooth.

Motivated by the gas-liquid reaction model we make the following quasi-monotone assumption on f_1, f_2:

(H) For each $u \geqq 0$, $f_1(\cdot, u, v)$ is monotone nondecreasing in $v \geqq 0$; and for each $v \geqq 0$, $f_2(\cdot, u, v)$ is monotone nondecreasing in $u \geqq 0$.

The above hypothesis implies that the functions $(-f_1)$ and $(-f_2)$ on the right-side of (1.1) are monotone nonincreasing in v and u, respectively. In order to use the monotone method to establish an existence theorem for the system (1.1)–(1.3), however, it is essential that $(-f_1)$ and $(-f_2)$ are quasi-monotone nondecreasing rather than nonincreasing. To overcome this difficulty, we make a transformation for u by letting $w = M_0 - u$. Then the system (1.1)–(1.3) is transformed into the form

$$\begin{aligned} w_t - L_1 w &= F_1(t, x, w, v) \\ v_t - L_2 v &= -F_2(t, x, w, v) \end{aligned} \qquad (t > 0,\ x \in \Omega) \tag{3.1}$$

$$B_1[w] = M_0 - h_1,\ B_2[v] = h_2 \qquad (t > 0,\ x \in \partial\Omega) \tag{3.2}$$

$$w(0, x) = M_0 - u_0(x),\ v(0, x) = v_0(x) \quad (x \in \Omega) \tag{3.3}$$

where

$$F_i(t, x, w, v) = f_i(t, x, M_0 - w, v) \qquad i = 1, 2. \tag{3.4}$$

The expression $B_2[v] = h_2$ represents the boundary condition in (1.2) for v while $B_1[w] = M_0 - h_1$ is in the sense that

$$\begin{aligned} \beta_1(x)\partial w/\partial\nu + w = M_0 - h_1 \text{ for } x \in \Gamma_1 \text{ and } \partial w/\partial\nu \\ = 0 \text{ for } x \in \partial\Omega - \Gamma_1 \end{aligned} \tag{3.5}$$

For the Neumann boundary condition (1.4), (3.2) should be replaced by

$$\partial w/\partial\nu = \partial v/\partial\nu = 0 \qquad (t > 0,\ x \in \partial\Omega). \tag{3.6}$$

With this transformation the functions F_1, $(-F_2)$ are quasi-monotone nondecreasing for $0 \leqq w \leqq M_0$, $v \geqq 0$.

Following the same idea as in **[1, 10, 11, 14]** for scalar systems we can obtain an existence-comparison theorem for the transformed problem (3.1)–(3.3) in terms of upper and lower solutions. Here by an upper solution of (3.1)–(3.3) we mean a pair of smooth functions (w, v) satisfying the inequalities

$$
(3.7)\qquad
\begin{aligned}
&\tilde{w}_t - L_1\tilde{w} \geqq F_1(t, x, \tilde{w}, \tilde{v}) && (t > 0,\ x \in \Omega)\\
&\tilde{v}_t - L_2\tilde{v} \geqq - F_2(t, x, \tilde{w}, \tilde{v}) && \\
&B[_1\tilde{w}] \geqq M_0 - h_1,\ B_2[\tilde{v}] \geqq h_2 && (t > 0,\ x \in \partial\Omega)\\
&\tilde{w}(0, x) \geqq M_0 - u_0,\ \tilde{v}(0, x) \geqq v_0 && (x \in \Omega).
\end{aligned}
$$

Similarly, the pair $(\underset{\sim}{u}, \underset{\sim}{v})$ is a lower solution if it satisfies the reversed inequalities in (3.7). Suppose that $\tilde{U} = (\tilde{w}, \tilde{v})$, $\underset{\sim}{U} = (\underset{\sim}{w}, \underset{\sim}{v})$ are upper and lower solutions of (3.1)–(3.3) such that $0 \leqq \underset{\sim}{w} \leqq \tilde{w} \leqq M_0, 0 \leqq \underset{\sim}{v} \leqq \tilde{v} \leqq \rho_0$, where M_0, ρ_0 are some given positive constants. Assume that there exist constants $M_i > 0$ such that

$$
(3.8)\qquad |f_i(t, x, u_2, v_2) - f_i(t, x, u_1, v_1)| \leqq M_i(|u_2 - u_1| + |v_2 - v_1|)
$$

for $0 \leqq u_i \leqq M_0$, $0 \leqq v_i \leqq \rho_0$ $i = 1, 2$. Then starting from the initial iteration $U^{(0)} = \tilde{U}$ we can construct a sequence $\{\bar{U}^{(k)}\} \equiv \{\bar{w}^{(k)}, \bar{v}^{(k)}\}$ from the uncoupled linear system for $k = 1, 2, \ldots$

$$
(3.9)\qquad
\begin{aligned}
&w_t^{(k)} - L_1 w^{(k)} + M_1 w^{(k)} = M_1 w^{(k-1)} + F_1(t, x, w^{(k-1)}, v^{(k-1)}),\\
&v_t^{(k)} - L_2 v^{(k)} + M_2 v^{(k)} = M_2 v^{(k-1)} - F_2(t, x, w^{(k-1)}, v^{(k-1)}),\\
&B[w^{(k)}] = M_0 - h_1, \quad B[v^{(k)}] = h_2,\\
&w^{(k)}(0, x) = M_0 - u_0, \quad v^{(k)}(0, x) = v_0.
\end{aligned}
$$

The existence of such a sequence follows from the existence theorem for linear scalar parabolic system (cf. [5]). Similarly, we can obtain a sequence from (3.9) with $U^{(0)} = \underset{\sim}{U}$, and denote this sequence by $\{\underline{U}^{(k)}\} \equiv \{\underline{w}^{(k)}, \underline{v}^{(k)}\}$. It is easily seen by a standard argument that these two sequences possess the following monotone properties: (i) $\{\bar{U}^{(k)}\}$ is monotone nonincreasing, (ii) $\{\underline{U}^{(k)}\}$ is monotone nondecreasing, and (iii) $\underline{U}^{(k)} \leqq \bar{U}^{(k)}$ for every $k = 1, 2, \ldots$. These monotone properties lead immediately to the following theorem which is our basis for the investigation of the asymptotic behavior of the solution.

THEOREM 3.1. *Let* $(\tilde{w}, \tilde{v})$, $(\underset{\sim}{w}, \underset{\sim}{v})$ *be upper and lower solutions of* (3.1)–(3.3) *such that* $0 \leqq \underset{\sim}{w} \leqq \tilde{w} \leqq M_0$. $0 \leqq \underset{\sim}{v} \leqq \tilde{v} \leqq \rho_0$, *and let* (H) *and* (3.8) *hold. Then the sequence* $\{\bar{U}^{(k)}\}$ *obtained from* (3.9) *with* $\bar{U}^{(0)} = \tilde{U}$ *converges from above to a unique solution* $U = (w, v)$ of (3.1)–(3.3), *while the sequence* $\{\underline{U}^{(k)}\}$ *with* $\underline{U}^{(0)} = \underset{\sim}{U}$ *converges from below to* U. *Moreover, the solution* (w, v) *satisfies the relation*

$$
(3.10)\qquad 0 \leqq \underset{\sim}{w} \leqq w \leqq \tilde{w} \leqq M_0, 0 \leqq \underset{\sim}{v} \leqq v \leqq \tilde{v} \leqq \rho_0 \quad (t > 0, x \in \bar{\Omega})
$$

PROOF. The proof for the existence of a solution follows from the same argument as for scalar system using the quasi-monotone property of F_i (cf [**10**, **11**]). The uniqueness of the solution is a direct consequence of the Lipschitz condition (3.8). Details are omitted.

REMARK 3.1. A number of existence-comparison theorems for weakly coupled parabolic systems have been known and can be established by various methods. Some examples are the work by Kuiper [**8**] using a functional analytic approach, and by Bebernes, Chueh and Fulks [**3**] in connection with invariance sets. However, the present monotone argument is more constructive and provides a simpler and straightforward proof. Furthermore, the same approach can be used to obtain a similar existence-comparison theorem for the corresponding elliptic boundary-value problems simply by dropping the terms (w_t, v_t) and the initial conditions. This kind of boundary-value problem will be discussed in a later section when dealing with the existence and stability of a steady-state solution.

It is seen from Theorem 3.1 that if there exist upper and lower solutions $(\tilde{w}, \tilde{v})$, $(\underset{\sim}{w}, \underset{\sim}{v})$ of the transformed system (3.1)–(3.3) such that

$$0 \leqq \underset{\sim}{w} \leqq \tilde{w} \leqq M_0, \quad 0 \leqq \underset{\sim}{v} \leqq \tilde{v} \leqq \rho_0, \tag{3.11}$$

then by the transformation $u = M_0 - w$ the original system (1.1)–(1.3) has a unique solution (u, v) whcih satisfies the relation

$$0 \leqq M_0 - \tilde{w} \leqq u \leqq M_0 - \underset{\sim}{w} \leqq M_0, 0 \leqq \underset{\sim}{v} \leqq v \leqq \tilde{v} \leqq \rho_0. \tag{3.12}$$

Hence the existence and the asymptotic behavior of a solution for the system (1.1)–(1.3) can be determined through the construction of suitable upper and lower solutions for the transformed system (3.1)–(3.3). Since the only requirements on upper and lower solutions are the inequalities (and the reversed inequalities) in (3.7), it gives considerable flexibility in the construction of these functions. Consider, for example, the case $f_i = k_i uv$ and $h_i = 0$, $i = 1, 2$. Then for any constants $\rho_1 \geqq u_0$, $\rho_0 \geqq v_0$ and $M_0 \leqq \rho_1$ the constant functions $(\tilde{w}, \tilde{v}) = (M_0, \rho_0)$ and $(\underset{\sim}{w}, \underset{\sim}{v}) = (M_0 - \rho_1, 0)$ are upper and lower solutions and satisfy the relation (3.11). This can be verified by direct substitution into the respective inequalities in (3.7). Since f_1 and f_2 satisfy the conditions in (H) and (3.8), Theorem 3.1 ensures the existence of a unique solution (u, v) to the original system (1.0) (1.2) (1.3) such that

$$0 \leqq u(t, x) \leqq \rho_1, 0 \leqq v(t, x) \leqq \rho_0 \ (t > 0, x \in \bar{\Omega}). \tag{3.13}$$

The same conclusion holds for the system (1.0) (1.3) (1.4). This existence result has been used in Theorems 2.1 and 2.2. It is to be noted that the relation (3.13) implies that the zero steady-state solution of the Neumann problem (1.0) (1.3) (1.4) is stable (but it is not asymptotically stable).

As another application of Theorem 3.1 we consider the general equation

(1.1) under the Neumann condition (1.4). In this situation, upper and lower solutions of (1.1) (1.3) (1.4) can be obtained from the solution of the corresponding ordinary differential system

$$P_1' = -\bar{f}_1(t, P_1, P_2),\ P_2' = -\underline{f}_2(t, P_1, P_2) \tag{3.14}$$

$$P_1' = -\underline{f}_1(t, P_1, P_2),\ P_2' = -\bar{f}_2(t, P_1, P_2) \tag{3.15}$$

where $\bar{f}_i$, $\underline{f}_i$ denote, respectively, the supremum and infimum of f_i over Ω. It turns out that the behavior of the solution of (1.1) (1.3) (1.4) is directly related to the property of the solutions of (3.14), (3.15). Specifically, we have the following theorem.

THEOREM 3.2. *Let* f_1, f_2 *satisfy* (H_1) *and* (3.7) *and let* (P_1, P_2), $(\hat{P}_1, \hat{P}_2)$ *by any solutions of* (3.14), (3.15), *respectively, such that* $P_1(t) \leqq \hat{P}_1(t) \leqq M_0$, $\hat{P}_2(t) \leqq P_2(t)$, *where* $P_1(0) \leqq u_0(x) \leqq \hat{P}_1(0)$, $\hat{P}_2(0) \leqq v_0(x) \leqq P_2(0)$ *for* $x \in \bar{\Omega}$. *Then the system* (1.1) (1.3) (1.4) *has a unique solution* (u, v) *such that*

$$P_1(t) \leqq u(t, x) \leqq \hat{P}_1(t),\ \hat{P}_2(t) \leqq v(t, x) \leqq P_2(t)\ (t > 0,\ x \in \bar{\Omega}) \tag{3.16}$$

PROOF. To prove the theorem, it suffices to show that the transformed system (3.1) (3.3) (3.6) has a unique solution (w, v) such that

$$M_0 - \hat{P}_1(t) \leqq w(t, x) \leqq M_0 - P_1(t),\ \hat{P}_2(t) \leqq v(t, x) \leqq P_2(t)$$

where $M_0 \geqq \hat{P}_1(t)$. This will be done if we can show that the pairs $(\tilde{w}, \tilde{v}) = (M_0 - P_1, P_2)$ and $(\underset{\sim}{w}, \underset{\sim}{v}) = (M_0 - \hat{P}_1, \hat{P}_2)$ are upper and lower solutions of the transformed system, respectively. Since $\partial\tilde{w}/\partial\nu = \partial\tilde{v}/\partial\nu = 0$, we see from the hypothesis on u_0, v_0 that the boundary and initial requirements are fulfilled. Hence $(\tilde{w}, \tilde{v})$ is an upper solution if P_1, P_2 satisfy the relation

$$-P_1' \geqq F_1(t, x, M_0 - P_1, P_2),\ P_2' \geqq -F_2(t, x, M_0 - P_1, P_2). \tag{3.17}$$

In view of (3.4) the above relation is equivalent to

$$P_1' \leqq -f_1(t, x, P_1, P_2),\ P_2' \geqq -f_2(t, x, P_1, P_2)$$

which is clearly satisfied by the solution of (3.14). The proof of the lower solution $(\underset{\sim}{w}, \underset{\sim}{v})$ is similar.

The result of Theorem 3.2 can be used to establish the rate of convergence of the solution of the system (1.0) (1.3) (1.4). For convenience, we denote by $\bar{u}_0$, $\bar{v}_0$ and $\underline{u}_0$, $\underline{v}_0$ the respective least upper bound and greatest lower bound of u_0, v_0.

THEOREM 3.3. *Let* (u, v) *be the solution of* (1.0) (1.3) (1.4). *Then* (i) (u, v) *coneverges exponentially to* $(0, v_\infty)$ *with* $v_\infty = \hat{v}_0 - (k_2/k_1)\hat{u}_0$ *when* $k_1\underline{v}_0 >$

$k_2\bar{u}_0$, and in the order no less than $o(1/t)$ when $k_1\underline{v}_0 = k_2\bar{u}_0$; (ii) (u, v) *converges exponentially to (u_∞, o) with $u_\infty = \hat{u}_0 - (k_1/k_2)\hat{v}_0$ when $k_2\underline{u}_0 > k_1\bar{v}_0$ and in the order no less than $o(1/t)$ when $k_2\underline{u}_0 = k_1\bar{v}_0$; and* (iii) (u, v) *converges to (o, o) in the order $o(1/t)$ when u_0, v_0 are constants and $k_2u_0 = k_1v_0$.*

PROOF. (i) When $f_1 = k_1uv$, $f_2 = k_2uv$, the ordinary differential system (3.14) and (3.15) are both reduced to the form

$$P_1' = -k_1P_1P_2 \quad P_2' = -k_2P_1P_2. \tag{3.18}$$

The solution of this system is given by $(t > 0)$

$$\begin{aligned} P_1(t) &= CP_1(0)[k_1P_2(0)e^{Ct} - k_2P_1(0)]^{-1} \\ P_2(t) &= CP_2(0)[k_1P_2(0) - k_2P_1(0)e^{-Ct}]^{-1} \end{aligned} \tag{3.19}$$

when $C \equiv k_1P_2(0) - k_2P_1(0) \neq 0$. In the case of $C = 0$ this solution becomes

$$\begin{aligned} P_1(t) &= P_1(0)[1 + k_2P_1(0)t]^{-1}, \\ P_2(t) &= P_2(0)[1 + k_1P_2(0)t]^{-1}. \end{aligned} \tag{3.20}$$

Let (P_1, P_2) and $(\hat{P}_1, \hat{P}_2)$ be the solution of (3.18) with $P_1(0) = 0$, $P_2(0) = \bar{v}_0$ and $\hat{P}_1(0) = \bar{u}_0$, $\hat{P}_2(0) = \underline{v}_0$. Then Theorem 3.2 ensures that

$$0 \leqq u(t, x) \leqq \hat{P}_1(t), \ \hat{P}_2(t) \leqq v(t, x) \leqq \bar{v}_0. \tag{3.21}$$

In view of (3.19) and (3.20), u converges to zero exponentially as $t \to \infty$ when $k_1\underline{v}_0 - k_2\bar{u}_0 > 0$, and in the order no less than $o(1/t)$ when $k_1\underline{v}_0 - k_2\bar{u}_0 = 0$. The same is true for the function $q \equiv uv - \hat{u}\hat{v}$ since v is uniformly bounded. From the proof of Theorem 2.1 it follows that the solution V of the linear problem (2.5) converges to zero in the same fashion as u and so does $v \to v_\infty$ as $t \to \infty$. This proves (i). The proof of (ii) is similar. When u_0, v_0 are constants and $k_2u_0 = k_1v_0$, then (u, v) coincides with the functions in (3.20). This leads to the conclusion in (iii).

4. A non-homogeneous system. The equations considered in (1.0) and the boundary condition (1.2) are homogeneous so that the zero function is a steady-state solution. In many concrete physical systems, however, there is either an internal source or a boundary input, (e.g., see [**2–4**]). In such a system nontrivial steady-state solutions exist and the asymptotic behavior of the time-dependent solution depends not only on the strength of the reaction function and the initial function but also on the steady-state solution itself. The aim of this section is to study the existence and the stability or instability property of a non-zero steady-state solution. To avoid a list of general hypotheses for the system (1.1)–(1.3) we limit our discussion to the following simple gas-liquid absorption problem:

$$(4.1)\qquad \begin{aligned} u_t - L_1 u &= -k_1 uv + q_1(x) \\ v_t - L_2 v &= -k_2 uv + q_2(x) \end{aligned} \qquad (t > 0,\ x \in \Omega)$$

$$(4.2)\qquad B_1[u] = h_1(x),\ B_2[v] = h_2(x) \qquad (t > 0, x \in \partial\Omega)$$

$$(4.3)\qquad u(0, x) = u_0(x),\ v(0, x) = v_0(x) \quad (x \in \Omega)$$

where for each $i = 1, 2$, k_i is a positive constant and q_i, h_i are external sources which are assumed non-negative continuous in Ω and $\partial\Omega$ respectively. In (4.2) the boundary condition $B_i[w] = h_i$ means that $\beta_i \partial w/\partial\nu + w = h_i$ on Γ_i and $\delta w/\delta\nu = 0$ on $\partial\Omega - \Gamma_i$. The consideration of this type of boundary conditions is to make our results directly applicable to the systems discussed in [**2**–**4**, **6**, **12**]. Our first objective is to establish the existence (and uniqueness) of a nonnegative steady-state solution for (4.1) (4.2). This question of existence-uniqueness has been discussed in [**6**] for a Dirichlet type boundary condition without the source term q_i. In this paper, however, we shall use the notion of upper and lower solutions for elliptic systems in proving the existence theorem (e.g., see [**7**, **14**, **15**]). As is to be expected, this approach also leads to an upper and a lower bound of the steady-state solution which is useful in the study of its stability property. To achieve this goal for the steady-state problem (4.1) (4.2), we again make the transformation $w = M_0 - u$ as in §3. Then by dropping the terms u_t, v_t in (4.1) the transformed steady-state problem becomes

$$(4.4.)\qquad \begin{aligned} -L_1 w &= k_1(M_0 - w)v - q_1 \\ -L_2 v &= -k_2(M_0 - w)v + q_2 \end{aligned} \qquad (x \in \Omega)$$

$$(4.5)\qquad B_1[w] = M_0 - h_1, \quad B_2[v] = h_2 \qquad (x \in \partial\Omega).$$

Our existence proof is based on the quasi-monotone property of the reaction functions and the explicit construction of upper and lower solutions. Here by an upper solution, we mean a pair of functions $(\tilde{w}, \tilde{v})$ satisfying the inequality "$\geqq$" in (4.4)(4.5) when (w, v) is replaced by $(\tilde{w}, \tilde{v})$. Lower solution is defined analogously and is denoted by $(\underset{\sim}{w}, \underset{\sim}{v})$. Since the functions on the right-side of (4.4) are quasi-monotone whenever $0 \leqq w \leqq M_0$, $v \geqq 0$, the same argument as for Theorem 3.1 shows that the system (4.4), (4.5) has at least one solution if there exists an upper solution $(\tilde{w}, \tilde{v})$ and a lower solution $(\underset{\sim}{w}, \underset{\sim}{v})$ such that $0 \leqq \underset{\sim}{w} \leqq \tilde{w} \leqq M_0$, $0 \leqq \underset{\sim}{v} \leqq \tilde{v}$. In fact, by using upper and lower solutions as initial iterations one can construct two monotone sequences which converge from above and below, respectively, to a solution $(\bar{w}, \bar{v})$, $(\underline{w}, \underline{v})$ of (4.4), (4.5) (cf, [**1**, **7**, **14**, **15**]). This can also be seen from (3.9) by dropping the terms $w_t^{(k)}$, $v_t^{(k)}$ and the initial conditions. The above construction implies that the solutions $(\bar{w}, \bar{v})$, $(\underline{w}, \underline{v})$ satisfy the relation

$$(4.6)\qquad \underset{\sim}{w}(x) \leqq \underline{w}(x) \leqq \bar{w}(x) \leqq \tilde{w}(x),\quad \underset{\sim}{v}(x) \leqq \underline{v}(x) \leqq \bar{v}(x) \leqq \tilde{v}(x).$$

Hence the existence problem will be resolved if we can find such a pair of upper and lower solutions. In the following theorem we construct some explicit upper and lower solutions in terms of the eigenfunctions ϕ_i of (2.9). For convenience, we denote by $\bar{h}_i$ the least upper bound of h_i on $\partial\Omega$ and set

$$(4.7)\qquad \bar{q}_i = \max\{q_i(x)/\phi_i(x);\ x \in \bar{\Omega}\},\quad \bar{q}_m = \max\{\bar{q}_1,\ \bar{q}_2\}.$$

THEOREM 4.1. *The steady-state problem* (4.1) (4.2) *has at least one non-negative solution* (u^*, v^*) *such that*

$$(4.8)\qquad 0 \leqq u^*(x) \leqq (\bar{q}_m/\lambda_1)\phi_1(x) + \bar{h}_1, 0 \leqq v^*(x) \leqq (\bar{q}_m/\lambda_2) + \bar{h}_2,$$

If, in addition, there exist positive constants c_1, c_2 *such that* $L_2 = c_1L_1$, $B_2 = c_2B_1$, *then there exists exactly one non-negative solution satisfying* (4.8).

PROOF. To prove the existence and the relation (4.8) it suffices to show that for some $M_0 \geqq (\bar{q}_m/\lambda_1) + \bar{h}_1$ the transformed system (4.4) (4.5) has a solution (w^*, v^*) satisfying

$$(4.9)\qquad M_0 - (\bar{q}_m/\lambda_1)\,\phi_1 - \bar{h}_1 \leqq w^* \leqq M_0, 0 \leqq v^* \leqq (\bar{q}_m/\lambda_2)\phi_2 + \bar{h}_2.$$

This will be done if we can show that $\tilde{w} = M_0$, $\tilde{v} = (\bar{q}_m/\lambda_2)\,\phi_2 + \bar{h}_2$ and $\underset{\sim}{w} = M_0 - (\bar{q}_m/\lambda_1)\,\phi_1 - \bar{h}_1$, $v = 0$ are upper and lower solutions respectively. However, this follows immediately from

$$(4.10)\qquad \begin{aligned} -L_1\tilde{w} &= 0 \geqq -q_1, -L_2\tilde{v} = \bar{q}_m\phi_2 \geqq q_2 && (x \in \Omega) \\ B_1[\tilde{w}] &= M_0, B_2[\tilde{v}] = \bar{h}_2 \geqq h_2(x) && (x \in \partial\Omega) \end{aligned}$$

and a similar expression for $(\underset{\sim}{w}, \underset{\sim}{v})$. To show the uniqueness of the solution when $L_2 = c_1L_1$, $B_2 = c_2B_1$ we let $U = u^* - \check{u}$, $V = v^* - \check{v}$, where $(\check{u}, \check{v})$ is any non-negative steady-state solution of (4.1)(4.2). Then (U, V) satisfies the equations

$$(4.11)\qquad \left.\begin{aligned} L_1U &= k_1(u^*v^* - \check{u}\check{v}) = k_1(v^*U + \check{u}V) \\ L_2V &= k_2(u^*v^* - \check{u}\check{v}) = k_2(v^*U + \check{u}V) \end{aligned}\right. \quad (x \in \Omega)$$

and the boundary condition $B_1[U] = B_2[V] = 0$. Since

$$(4.12)\qquad k_1^{-1}L_1U = k_2^{-1}L_2V = c_1k_2^{-1}L_1V, B_1[V] = c_2^{-1}B_2[V] = 0,$$

we see that the function $W = k_1^{-1}U - c_1k_2^{-1}V$ satisfies $L_1[W] = 0$ in Ω, $B_1[W] = 0$ on $\partial\Omega$ and therefore $W = 0$, that is, $V = k_2(c_1k_1)^{-1}U$. Substituting V into (4.11) leads to

$$(4.13)\qquad L_1U = (k_1v^* + k_2c_1^{-1}\check{u})U.$$

In view of the non-negative property of $(k_1 v^* + k_2 c_1^{-1}\breve{u})$ the maximum principle implies that $U = 0$, and therefore $V = 0$. This proves the uniqueness of the solution and thus the theorem.

REMARK 4.1. When ϕ_1 or ϕ_2 vanishes on $\partial\Omega$ and q_i/ϕ_i is unbounded, we may replace ϕ_i by the positive function $\hat{\phi}_i$ satisfying $L_i\hat{\phi}_i + \lambda\hat{\phi}_i \leqq 0$ for some $\lambda \leqq \lambda_i$ (cf. [11]). In this case the corresponding functions $(\tilde{w}, \tilde{v})$, $(\underset{\sim}{w}, \underset{\sim}{v})$ with respect to $\hat{\phi}_i$ are upper and lower solutions of (4.4) (4.5), and thus the upper bounds for (u^*, v^*) in (4.8) should be with respect to $\hat{\phi}_1, \hat{\phi}_2$. Notice that if $q_1 = q_2 = 0$, then the bound for (u^*, v^*) becomes $0 \leqq u^*(x) \leqq \bar{h}_1$, $0 \leqq v^*(x) \leqq \bar{h}_2$.

We next investigate the stability and the instability of a steady-state solution for the system (4.1)–(4.3). This is done through suitable construction of upper and lower solutions for the transformed time-dependent system:

$$\text{(4.14)} \quad \begin{aligned} w_t - L_1 w &= k_1(M_0 - w)v - q_1 \\ v_t - L_2 v &= -k_2(M_0 - w)v + q_2 \end{aligned} \qquad (t > 0, x \in \Omega)$$

$$\text{(4.15)} \quad B_1[w] = M_0 - h_1, \; B_2[v] = h_2 \qquad (t > 0, x \in \partial\Omega)$$

$$\text{(4.16)} \quad w(0, x) = M_0 - u_0(x), \; v(0, x) = v_0(x) \quad (x \in \Omega)$$

where $w = M_0 - u$. The definition of upper and lower solutions for the above system follows from (4.14)–(4.16) by replacing the equality by the corresponding inequality relation. Notice that the present transformation is again to obtain the quasi-monotone property for the nonlinear functions on the right-side of (4.14). Our main result on the asymptotic stability and instability problem is contained in the following theorem.

THEOREM 4.2. *Let* (u^*, v^*) *be a non-negative steady-state solution of* (4.1) (4.2). *If there exist constants* $\delta > 0$, $\varepsilon > 0$ *such that*

$$\text{(4.17)} \quad \begin{aligned} \lambda_1 + k_1 v^*(x) - \delta k_1(\phi_2(x)/\phi_1(x))u^*(x) &\geqq \varepsilon \\ \lambda_2 + k_2 u^*(x) - \delta^{-1}k_2(\phi_1(x)/\phi_2(x))v^*(x) &\geqq \varepsilon, \end{aligned} \qquad (x \in \bar{\Omega})$$

then the system (4.1)–(4.3) *has a unique solution* (u, v) *such that*

$$\text{(4.18)} \quad \begin{aligned} u^*(x) - P_1(t)\phi_1(x) &\leqq u(t, x) \leqq u^*(x) + P_2(t)\phi_1(x) \\ v^*(x) - \delta P_2(t)\phi_2(x) &\leqq v(t, x) \leqq v^*(x) + \delta P_1(t)\phi_2(x) \end{aligned} \qquad (t > 0, x \in \bar{\Omega})$$

whenever u_0, v_0 *satisfy the relations*

$$\text{(4.19)} \quad \begin{aligned} u^* - \rho_1\phi_1 &\leqq u_0 \leqq u^* + \rho_2\phi_1, \\ v^* - \delta\rho_2\phi_2 &\leqq v_0 \leqq v^* + \delta\rho_1\phi_2 \end{aligned} \qquad (x \in \Omega)$$

with $\rho_1 < \eta_1$, $\rho_2 < \eta_2$, *where* $\eta_1 = \varepsilon/k_2$, $\eta_2 = \varepsilon/(\delta k_1)$ *and*

(4.20) $$P_i(t) = [\eta_i^{-1} + (\rho_i^{-1} - \eta_i^{-1})e^{\varepsilon t}]^{-1}, i = 1, 2, \quad (t > 0).$$

PROOF. We first seek an upper solution for (4.14)–(4.16) in the form of $\tilde{w} = w^* + P(t)\phi_1(x)$, $\tilde{v} = v^* + \delta P(t)\phi_2(x)$, where $w^* = M_0 - u^*$ and $P(t)$ is a positive function to be determined. Since $B_1[\tilde{w}] = B_1[w^*] = M_0 - h_1$, $B_2[\tilde{v}] = B[v^*] = h_2$, and since $\tilde{w}(0, x) = M_0 - u^* + P(0)\phi_1$ and $\tilde{v}(0, x) = v^* + \delta P(0)\phi_2$, we see that $(\tilde{w}, \tilde{v})$ is an upper solution if $P(0) = \rho_1$ and

(4.21) $$\begin{aligned} &-L_1 w^* + (P'\phi_1 - PL_1\phi_1) \geqq k_1(M_0 - w^* - P\phi_1)(v^* + \delta P\phi_2) - q_1 \\ &-L_2 v^* + \delta(P'\phi_2 - PL_2\phi_2) \geqq -k_2(M_0 - w^* - P\phi_1)(v^* + \delta P\phi_2) \\ &+ q_2. \end{aligned}$$

But since (w^*, v^*) is a solution of (4.4) and $M_0 - w^* = u^*$, the above inequalities are equivalent to

(4.22) $$\begin{aligned} (P' + \lambda_1 P)\,\phi_1 &\geqq k_1(\delta u^* P\phi_2 - v^* P\phi_1 - \delta P^2\phi_1\phi_2) \\ \delta(P' + \lambda_2 P)\,\phi_2 &\geqq -k_2(\delta u^* P\phi_2 - v^* P\phi_1 - \delta P^2\phi_1\phi_2). \end{aligned}$$

In view of the hypothesis (4.17) it suffices to find P such that

(4.23) $$\begin{aligned} P' + \varepsilon P &\geqq -\delta k_1 P^2\phi_2 \\ P' + \varepsilon P &\geqq k_2 P^2\phi_1. \end{aligned}$$

However, this follows immediately by the choice of $P = P_1(t)$, where P_1 is given by (4.20). We next construct a lower solution of (4.14)-(4.16) in the form of $\underset{\sim}{w} = w^* - P(t)\phi_1(x)$, $\underset{\sim}{v} = v^* - \delta P(t)\phi_2(x)$ by a suitable choice of P. It is easily seen by the same argument that $(\underset{\sim}{w}, \underset{\sim}{v})$ is a lower solution if P satisfies $P(0) = \rho_2$ and the relations

(4.24) $$P' + \varepsilon P \geqq \delta k_1 P^2\phi_2, \quad P' + \varepsilon P \geqq -k_2 P^2\phi_1,$$

where we have used the hypothesis (4.17). The above inequalities are clearly satisfied by the function $P = P_2$ given by (4.20). Hence by Theorem 3.1 the transformed system (4.14)-(4.16) has a unique solution (w, v) such that

(4.25) $$w^* - P_2\phi_1 \leqq w \leqq w^* + P_1\phi_1, \quad v^* - \delta P_2\phi_2 \leqq v \leqq v^* + \delta P_1\phi_2.$$

The above relation is equivalent to (4.18).

The result of Theorem 4.2 implies that under the condition (4.17) the steady-state solution (u^*, v^*) is asymptoticly stable. If this condition is replaced by

(4.26) $$\begin{aligned} \lambda_1 + k_1 v^*(x) - \delta k_1(\phi_2(x)/\phi_1(x))u^*(x) &\leqq -\varepsilon \\ \lambda_2 + k_2 u^*(x) - \delta^{-1}k_2(\phi_1(x)/\phi_2(x))v^*(x) &\leqq -\varepsilon, \end{aligned} \qquad (x \in \bar{\Omega})$$

then we have the following instability conclusion.

THEOREM 4.3. *If the steady-state solution* (u^*, v^*) *satisfies the condition* (4.26) *for some positive constants* δ, ε, *then there exists a unique solution* (u, v) *to* (4.1)–(4.3) *such that*

$$\begin{aligned} u(t, x) &\geqq u^*(x) + P_3(t)\phi_1(x), \\ v(t, x) &\leqq v^*(x) - \delta P_3(t)\phi_2(x) \end{aligned} \quad (t > 0, x \in \bar{\Omega}) \tag{4.27}$$

whenever (u_0, v_0) *satisfies the relation*

$$u_0 \geqq u^* + \rho_3\phi_1, \; v_0 \leqq v^* - \delta\rho_3\phi_2, \quad (x \in \Omega), \tag{4.28}$$

where $\rho_3 > 0$ *is any constant satisfying* $v^* - \delta\rho_3\phi_2 \geqq 0$ *and*

$$P_3(t) = [\eta_1^{-1} - (\rho_3^{-1} - \eta_1^{-1})e^{-\varepsilon t}]^{-1} \textit{ with } \eta_1 = \varepsilon/k_2 \quad (t > 0). \tag{4.29}$$

PROOF. Let $\tilde{w} = w^* - P(t)\phi_1(x)$, $\tilde{v} = v^* - \delta P(t)\phi_2(x)$, where P is a positive function to be choosen. It is easily seen that $(\tilde{w}, \tilde{v})$ is an upper solution if $P(t)$ satisfies $P(0) \geqq \rho_3$ and the relation

$$\begin{aligned} (P' + \lambda_1 P)\phi_1 &\leqq k_1(\delta u^* P\phi_2 - v^* P\phi_1 + \delta P^2\phi_1\phi_2) \\ \delta(P' + \lambda_2 P)\phi_2 &\leqq -k_2(\delta u^* P\phi_2 - v^* P\phi_1 + \delta P^2\phi_1\phi_2). \end{aligned} \tag{4.30}$$

By the condition (4.26), the above inequalities hold if

$$P' - \varepsilon P \leqq \delta k_1 P^2\phi_2 \text{ and } P' - \varepsilon P \leqq -k_2 P^2\phi_1. \tag{4.31}$$

This follows by choosing $P = P_3$ given by (4.29). To complete the proof we need to find a lower solution $(\underset{\sim}{w}, \underset{\sim}{v})$ such that $\underset{\sim}{w} \leqq \tilde{w}$, $\underset{\sim}{v} \leqq \tilde{v}$. A suitable choice is given by $\underset{\sim}{w} = M_0 - \rho_4 e^{\gamma t}$, $\underset{\sim}{v} = 0$, where $\gamma > 0$ and ρ_4 is a positive constant satisfying $\rho_4 \geqq \max\{\bar{q}_1/\gamma, \bar{h}_1, \bar{u}_0\}$. Verification of this pair being a lower solution follows directly from definition. This proves the existence of a unique solution (w, v) to (4.14)–(4,16) and

$$M_0 - \rho_4 e^{\gamma t} \leqq w \leqq w^* - P_3\phi_1, \; 0 \leqq v \leqq v^* - \delta P_3\phi_2. \tag{4.32}$$

The above conclusion implies the existence of a unique solution (u, v) to (4.1)–(4.3) satisfying the relation (4.27).

REMARK 4.2. (a) In the case of $\phi_1(x) = 0$, $\phi_2(x) \neq 0$ (or $\phi_2(x) = 0$ $\phi_1(x) \neq 0$) at some point $x \in \partial\Omega$, we may replace ϕ_1 by $\hat{\phi}_1$ (resp., ϕ_2 by $\hat{\phi}_2$) in the relation (4.17). However, this replacement should not be used for the instability condition (4.26). (b) When $u^* = v^* = 0$ is a steady-state solution the condition (4.17) is trivially satisfied for any $\varepsilon \leqq \min\{\lambda_1, \lambda_2\}$. Thus Theorem 4.2 includes the result of Theorem 2.2.

The implication of Theorem 4.2 is that if a steady-state solution (u^*, v^*) of (4.1) (4.2) satisfies the condition (4.17) then it is asymptotically stable since both $P_1(t)$ and $P_2(t)$ converge to zero as $t \to \infty$. A stability region is given by

$$\Lambda_1 = \{(u_0, v_0);\ -\varepsilon k_2^{-1}\phi_1 < u_0 - u^* < \varepsilon(\delta k_1)^{-1}\phi_1,\\ -\varepsilon k_1^{-1}\phi_2 < v_0 - v^* < \varepsilon\delta k_2^{-1}\phi_2\} \tag{4.33}$$

However, if (u^*, v^*) satisfies (4.26), then by (4.27), (4.29),

$$\liminf_{t\to\infty} u(t, x) \geqq u^*(x) + \eta_1\phi_1(x),\\ \limsup_{t\to\infty} v(t, x) \leqq v^*(x) - \delta\eta_1\phi_2(x) \tag{4.34}$$

Since $\eta_1 = \varepsilon k_2^{-1}$ is independent of initial perturbations, the relation (4.34) implies that (u^*, v^*) is unstable. In fact, an instability region is given by (4.28). Therefore the conditions in (4.17) and (4.26) characterize the asymptotic stability and the instability behavior of a steady-state solution. Since this characterization depends only on the magnitude (but not the derivatives) of (u^*, v^*), it has rather interesting practical implications in certain specific situations. Consider, for example, the one-dimensional system (4.1) with $q_1 = q_2 = 0$ and with the boundary condition

$$u(t, 0) = u(t, \ell) = 0,\ v_x(t, 0) = 0,\ v(t, \ell) = b_0, \tag{4.35}$$

where b_0 is a positive constant. This system has been treated in [**4**] for the simple case $L_1 = L_2 = \partial^2/\partial x^2$. Since $(0, b_0)$ is a steady-state solution, condition (4.17) is clearly satisfied by choosing a sufficiently large δ and a suitable $\varepsilon > 0$. This means that $(0, b_0)$ is a stable steady-state solution. In fact, we have the following conclusion for a more general system.

THEOREM 4.4. *Let $q_1 = h_1 = 0$ and let $v^*(x)$ be the solution of the linear system*

$$-L_2 v = q_2(x)\ \ (x \in \Omega),\ \ B_2[v] = h_2(x)\ \ (x \in \partial\Omega). \tag{4.36}$$

Then $(0, v^*)$ *is a steady-state solution of* (4.1)(4.2) *and is asymptotically stable. Similarly, if $q_2 = h_2 = 0$ and u^* is the solution of*

$$-L_1 u = q_1(x)\ \ (x \in \Omega),\ \ B_1[u] = h_1(x)\ \ (x \in \partial\Omega), \tag{4.37}$$

then $(u^*, 0)$ *is a steady-state solution which is asymptotically stable. A stability region for* $(0, v^*)$ (or $(u^*, 0)$) *is given by* (4.33).

PROOF. For the steady-state solution $(0, v^*)$ condition (4.17) is reduced to

$$\lambda_1 + k_1 v^*(x) \geqq \varepsilon,\ \ \lambda_2 - \delta^{-1}k_2(\phi_1(x)/\phi_2(x))v^* \geqq \varepsilon. \tag{4.38}$$

Since $v^*(x)$ is non-negative, the above conditions are fulfilled by choosing $\varepsilon < \min\{\lambda_1, \lambda_2\}$ and a sufficiently large δ. The asymptotic stability of $(0, v^*)$ follows from Theorem 4.2. Notice that if (ϕ_1/ϕ_2) is unbounded on $\partial\Omega$, we should replace ϕ_2 by $\hat{\phi}_2$. The proof for $(u^*, 0)$ is similar.

The result in Theorems 4.2 and 4.3 also holds for the Neumann boundary condition (1.4). Specifically, we have the following theorem.

THEOREM 4.5. Let (u^*, v^*) be a non-genative steady-state solution of (4.1)(1.4). If there exist positive constants δ, ε such that

$$k_1(v^* - \delta u^*) \geqq \varepsilon, \; k_2(u^* - \delta^{-1}v^*) \geqq \varepsilon \quad (x \in \Omega), \tag{4.39}$$

then (u^*, v^*) is asymptotically stable. In case (4.39) is replaced by

$$k_1(v^* - \delta u^*) \leqq -\varepsilon, \; k_2(u^* - \delta^{-1}v^*) \leqq -\varepsilon \quad (x \in \Omega), \tag{4.40}$$

then (u^*, v^*) is unstable.

PROOF. The proof follows from the same argument as for Theorems 4.2 and 4.3 with $\lambda_1 = \lambda_2 = 0$ and $\phi_1 = \phi_2 = 1$. Details are omitted.

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REFERENCES

1. H. Amman, *On the existence of positive solutions of nonlinear elliptic boundary value problems*, Indiana Univ. Math. J. **21** (1971), 125–146.

2. L. E. Bobisud, *A nonlinear perturbation problem arising in chemical engineering*, Nonlinear Anal.: Theory, Method and Appl. **3** (1979), 337–345.

3. J. W. Bebernes, K. N. Chueh and W. Fulks, *Some applications of invariance for parabolic systems*, Indiana Math. J. **28** (1979), 269–277.

4. P. V. Danckwerts, *Gas Liquid Reactions*, McGraw-Hill, New York, 1970.

5. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs, N. J., 1964.

6. C. S. Kahane, *On a system of nonlinear parabolic equations arising in chemical engineering*, J. Math. Anal. Appl. **53** (1976), 343–358.

7. H. B. Keller and D. S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech. **16** (1967), 1361–1376.

8. H. J. Kuiper, *Existence and comparison theorems for nonlinear diffusion systems*, J. Math. Anal. Appl. **60** (1977), 166–181.

9. R. H. Martin, *Asymptotic behavior for semilinear differential equations in Banach spaces* (to appear).

10. C. V. Pao, *Positive solution to a nonlinear boundary value problem of parabolic type*, J. Differential Equations **22** (1976), 145–163.

11. ———, *Asymptotic behavior and nonexistence of global solutions of a class of nonlinear boundary value problems of parabolic type*, J. Math. Anal. Appl. **65** (1978), 616–637.

12. J. R. A. Pearson, *Diffusion of one substance into a semi-infinite medium containing another with a second order reaction*, Appl. Sci. Res., Section A, **11** (1963), 321–340.

13. M. H. Protter and H. F. Weinberger, *On the spectrum of general second order operators*, Bull. Amer. Math. Soc. **72** (1966), 251–255.

14. D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **21** (1972), 979–1000.

15. R. B. Simpson and D. S. *Cohen, Positive solutions of nonlinear elliptic eigenvalue problems*, J. Math. Mech. **19** (1970), 895–910.

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