ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 4, Fall 1982

HEREDITARY $\lambda(n, k)$-FAMILIES AND GENERALIZED CONVEXITY OF FUNCTIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. In this paper some results on the generalized convexity of real valued functions of a real variable are extended to the case where the members of the dominating family are determined by n conditions at $k(< n)$ points. In addition a partial answer to an open question related to the definition of generalized convexity of functions is given.

Throughout this paper $\lambda(n, k)$ will denote an *ordered k-partition of n* which is an ordered k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ where n, k and each λ_i are positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. For fixed n the set of all such $\lambda(n, k)$, $1 \leqq k \leqq n$, is denoted by $P(n)$. For $\lambda(n, k) \in P(n)$, $\|\lambda(n, k)\|$ denotes the positive integer $\max\{\lambda_i : 1 \leqq i \leqq k\}$, and $r = \|\lambda(n, k)\| - 1$.

Let $F \subset C^r(I)$ where I is an open interval of real numbers. Then F is called a $\lambda(n, k)$-*parameter family* on I (or for brevity a $\lambda(n, k)$-family) in case for any k points (nodes) $x_1 < x_2 < \cdots < x_k$ from I and any n real numbers α_{ij} there is a unique function $f \in F$ satisfying

$$(1) \qquad f^{(j)}(x_i) = \alpha_{ij}, = j = 0, 1, \ldots, \lambda_i - 1, i = 1, 2, \ldots, k.$$

A function $g \in C^r(I)$ is said to be a $\lambda(n, k)$-*convex function* with respect to the $\lambda(n, k)$-family F on I if for any k nodes $x_1 < x_2 < \cdots < x_k$ from I and any $f \in F$ satisfying

$$(2) \qquad f^{(j)}(x_i) = g^{(j)}(x_i), j = 0, 1, \ldots, \lambda_i - 1, i = 1, 2, \ldots, k,$$

it follows that

$$(3) \qquad (g(x) - f(x))(-1)^{M(i)} \geqq 0 \text{ whenever } x_{i-1} < x < x_i$$

for $i = 2, 3, \ldots, k$ where $M(i) = n + \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1}$.

Research supported by NRC (Canada) grants A3035, A4823, A7179, A7673, and A8130 while on leave at University of Alberta.
Mathematics subject classification (1980): 26A51.
Received by the editors on May 9, 1979, and in revised form on January 10, 1980.

In the case that $k = n$, so that each $\lambda_i = 1$, it can easily be shown that if for some fixed integer q, $1 \leqq q \leqq n + 1$, (3) always holds for any choice of the n nodes for $l = q$ whenever $f \in F$ satisfies (2), then indeed (3) holds for all i, $1 \leqq i \leqq n + 1$. (Throughout the paper x_0 will denote the left end point of I and x_{k+1} will denote its right endpoint. Also $M(1) = n$.) In fact Kemperman in **[4]** only requires that (3) holds for $i = n + 1$ in his definition of $\lambda(n, n)$-convexity, while Hartman in **[1]** requires that (3) holds for each i, $1 \leqq i \leqq n + 1$. The question of the equivalence of (3) holding for one or for all i in case $k < n$ is shown in section 2 for $\|\lambda(n, k)\| \leqq 3$ provided the condition that F be "hereditary", which holds vacuously when $k = n$, is imposed. The validity of this implication without these assumptions on $\lambda(n, k)$ and F remains an open question. Section 3 contains some results concerning relationships among the $\lambda(n, k)$-convex function for various choices of the partition $\lambda(n, k)$. The obvious analogues for *concave* functions ("$\leqq$" in place of "$\geqq$" in (3)) are valid but will not be stated.

$\lambda(n, k)$-families arise as natural generalizations of the dominating family of functions for classical convex or n-convex functions. From the point of view of differential equations, for an nth order differential equation in which the boundary value problems of the type described in (1) always have unique solutions which extend throughout the interval I, the family of solutions will form a $\lambda(n, k)$-family on I for that particular choice of the partition $\lambda(n, k)$ of n. From this point of view Theorems 2 and 3 can be interpreted as a restricted type of uniqueness theorems for solutions of certain related boundary value problems.

Lloyd K. Jackson, to whom this paper is dedicated, and some of his students have explored the use of "subfunctions" (For a subfunction g the restriction on F is relaxed so that (3) still must hold *if* there is an $f \in F$ satisfying (2), but the conditions described in (1) are not required to have a solution $f \in F$.) to prove existence theorems for two point boundary value problems for second order equations. See **[2]** and the references therein. Some progress in this direction was made by Jackson and Schrader for $n = 3$ in **[3]**, but essentially no results in this area have been obtained for $n \geqq 4$ at least in part due to the complicated nature of the subfunctions. We hope that results in this paper may shed light on the connection between generalized convexity of functions and existence theorems for various boundary value problems for ordinary differential equations.

2. Hereditary families. For $\lambda(n, k) \in P(n)$ the partition $\mu(n, k+1) = (\mu_1, \mu_2, \ldots, \mu_{k+1})$ is said to be obtained from $\lambda(n, k)$ by a *replacement* in case there is an m, $1 \leqq m \leqq k$, such that $\lambda_m > 1$, $\mu_i = \lambda_i$ for $i < m$, $\mu_{i+1} = \lambda_i$ for $i > m$, and $\mu_m + \mu_{m+1} = \lambda_m$. A replacement is said to be of *type* 1 if either $\mu_{m+1} = 1$ or $\mu_m = 1$. If $\mu_m = 1$, $\mu(n, k + 1)$

is denoted by $\lambda(n, k; m-)$, while if $\mu_{m+1} = 1$, $\mu(n, k + 1)$ is then denoted by $\lambda(n, k; m+)$. Also $\mu(n, j) < \lambda(n, k)$ in case $\mu(n, h)$ can be obtained by beginning with $\lambda(n, k)$ and applying a finite sequence of replacements. $R_1(\lambda(n, k))$ denotes the set of all $\mu(n, j) < \lambda(n, k)$ for which the sequence consists only of replacements of type 1. The $\lambda(n, k)$-family F is said to be *hereditary* in case F is a $\mu(n, j)$-family for each $\mu(n, j) \in R_1(\lambda(n, k))$. Observe that if $\|\lambda(n, k)\| \leqq 3$, then replacements on $\lambda(n, k)$ must all be type 1. Also if F is a $\mu(n, j)$-family on I for every $\mu(n, j) \in P(n)$, F is called an *unrestricted n-parameter family.*

A $\lambda(n, k)$-convex function is said to be $\lambda(n, k)$-**convex* with respect to F in case (3) holds for $i = 1$ and $i = k + 1$ as well as for $2, 3, \ldots, k$, and g is called *strictly* $\lambda(n, k)$-convex in case the strictly inequality holds in (3).

THEOREM 1. *Let g be in* $C^r(I)$ *and suppose that* $F \subset C^r(I)$ *is a hereditary* $\lambda(n, k)$*-family on I where* $\|\lambda(n, k)\| = r + 1 \leqq 3$. *Suppose also that there is a* q, $1 \leqq q \leqq k + 1$, *so that whenever* $x_1 < x_2 < \cdots < x_k$ *are in the open interval I and* $f \in F$ *satisfies* (2), *then* (3) *holds for* $i = q$. *It follows then that g is* $\lambda(n, k)$-**convex with respect to F on I.*

The validity of this result remains an open question for $\|\lambda(n, k)\| > 3$. The essential part of the proof of Theorem 1 is contained in the lemma in [2] and the following result.

LEMMA. *Suppose that* $F \subset C^2(I)$ *is an unrestricted* 3*-parameter family on I. Suppose also that the function* $g \in C^2(I)$ *has the property that for* $f \in F$, *if* $g - f$ *has a zero of order at least* 3 *at some* $z \in I$, *then*

$$g(x) < f(x) \text{ for all } x \in I \cap (-\infty, z). \tag{4}$$

It follows then that g is strictly μ-**convex with respect to F on I for all* $\mu \in P(3)$. *In particular* $g(x) > f(x)$ *for all* $x \in I \cap (z, \infty)$.

PROOF. We first show that g is strictly (1, 2)-convex. Let $x_1 < x_2$ be points in I and let $f \in F$ satisfy $f(x_1) = g(x_1)$, $f(x_2) = g(x_2)$, and $f'(x_2) = g'(x_2)$. We must show that $g(x) > f(x)$ for every $x \in x_1, x_2)$.

Suppose there were a point $u \in (x_1, x_2)$ such that $f(x) > g(x)$ for all $x \in (u, x_2)$. Pick $h \in F$ satisfying $h^{(j)}(x_2) = g^{(j)}(x_2)$ for $j = 0, 1, 2$. Since $f - g$ has a double zero at x_2 and is positive on (u, x_2), $f''(x_2) > g''(x_2) = h''(x_2)$. Also $g - h$ has a zero of order at least 3 at x_2, so by (4) $g(x) < h(x)$ for all $x < x_2$. f and h are distinct members of F, so $f(x) \neq h(x)$ for all $x < x_2$, and $h''(x_2) < f''(x_2)$ thus implies that $h(x) < f(x)$ for all $x < x_2$. Hence $g(x_1) < h(x_1) < f(x_1) = g(x_1)$, and we have a contradiction. We conclude that no such point u can exist. That fact will be used at several places in the remainder of the proof.

Since $f(x_1) = g(x_1)$ it follows that $g''(x_2) > f''(x_2)$ and there is some point $w \in [x_1, x_2)$ so that $g(x) > f(x)$ for all $x \in (w, x_2)$ and $g(w) = f(w)$. We shall now show that $w = x_1$, and that will show the (1, 2)-convexity of g. Suppose that $w > x_1$. Let $z_1 = (x_2 + w)/2$ and pick $h_1 \in F$ so that $h_1(x) = g(x)$ at x_1, z_1, and x_2. How h_1 and f agree at x_1 and x_2, so $h_1(z_1) = f(z_1)$ implies that $h_1(x) > f(x)$ for all $x \in [w, x_2)$. Thus $h_1'(x_2) < f'(x_2) = g'(x_2)$ and $h_1(x) > g(x)$ on $(x_2 - \delta, x_2)$ for some small $\delta > 0$. Next we claim that there are points u_1 and v_1 with $w < u_1 < v_1 < x_2$ such that $h_1(x) < g(x)$ for all $u_1 < x < v_1$. To see this observe that $h_1(w) > g(w) = f(w)$, so there is a least point u_1 in $(w, z_1]$ at which $h_1(x)$ and $g(x)$ are equal. If $h_1 - g$ had a zero of order 2 at u_1, then we would have $h_1(u_1) = g(h_1)$, $h_1'(u_1) = g'(u_1)$, $h(x_1) = g(x_1)$, and $g(x) < h_1(x)$ for all $x \in (w, u_1)$. That is exactly the situation (with f in place of g and x_2 in place of u_1) that was ruled out at the beginning of the proof. Consequently $h_1 - g$ changes sign at u_1. Since $h_1(x) - g(x) < 0$ for $x > u_1$ but alose to u_1 and $h_1(x) - g(x) > 0$ for $x < x_2$ but close to x_2, $h_1(x) = g(x)$ for some $x \in (u_1, x_2)$. Since $h_1(z_1) = g(z_1)$, either $u_1 = z_1$ or $v_1 \leqq z_1$. In either case, $2(v_1 - u_1) \leqq x_2 - w$. Next let $z_2 = (u_1 + v_1)/2$, and pick $h_2 \in F$ such that $h_2(x) = g(x)$ at x_1, z_2, and x_2. Then $h_2(x) > h_1(x)$ on (x_1, x_2), and by the argument just concluded there are points $u_2 < v_2$ in (u_1, v_1) with $u_2 = z_2$ or $v_2 \leqq z_2$, i.e., $2(v_2 - u_2) \leqq v_1 - u_1$, so that $h_2(x) < g(x)$ for all $x \in (u_2, v_2)$, $h_2(x) = g(x)$ at u_2 and v_2, and $h_2(x) > g(x)$ for all $x \in [w, u_2)$. We may continue in this

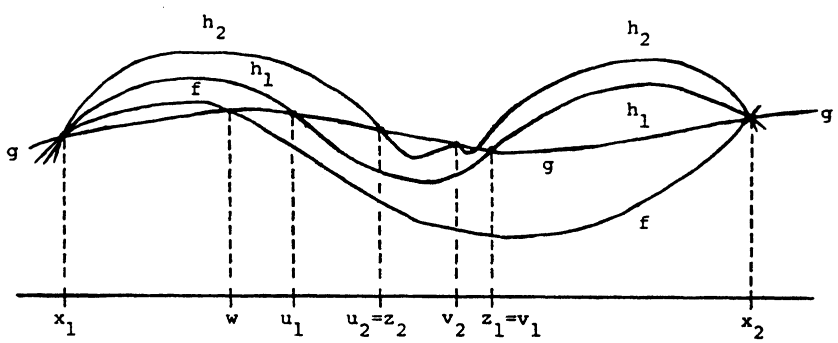

Fig. 1

fashion picking $z_n = (u_{n-1} + u_{n-1})/2$, $h_n \in F$ with $h_n - g$ zero at x_1, z_n, and x_2, $h_n < g$ on (u_n, v_n), $h_n(x) = g(x)$ at u_n and v_n, $h_n > g$ on $[w, u_n)$, and $2(v_n - u_n) \leqq v_{n-1} - u_{n-1}$. The strictly increasing sequence of numbers u_n has a limit $u_0 \in (w, x_2)$ with $u_n < u_0 < v_n$ for each n and also $v_n \to u_0$ as $n \to \infty$. Now let $h_0 \in F$ with $h_0(x) = g(x)$ at x_1, u_0, and x_2.

Since $u_n < u_0 < v_n$ for each n, $h_0(u_0) = g(u_0) > h_n(u_0)$. Therefore $h_0 > g$ on $[w, u_0)$ and $h'(u_0) \leqq g'(u_0)$. $h_0'(u_0) < g'(u_0)$ cannot hold because $h_0(v^n) > h_n(v_n) = g(v_n)$. Consequently $h_0 - g$ has a double zero at u_0 and is positive for all $w < x < u_0$. That again is the situation that was ruled out at the beginning of the proof. This contradiction shows that $w > x_1$ cannot hold, and the strict (1, 2)-convexity of g is demonstrated.

That $g(x) < f(x)$ for all x in $I \cap (-\infty, x_1)$ now follows from Theorem 2 in [8]. Also since $g(x_1) = f(x_1)$ and $g''(x_2) > f''(x_2)$, then $g(x) > f(x)$ for all $x > x_2$ but close to x_2. Suppose that $g > f$ on (x_2, z) and $g(z) = f(z)$. Let $h \in F$ satisfy $h(x_1) = g(x_1) = f(x_1)$, $h(z) = g(z) = f(z)$, and $h'(z) = g'(z) < f'(z)$. f and h are distinct members of F, so $h > f$ on (x_1, z). $h < g$ on (x_1, z) since g is strictly (1, 2)-convex. This gives the impossible situation that $f(x_2) < h(x_2) < g(x_2) = f(x_2)$, and we conclude that indeed $g > f$ on $I \cap (x_2, \infty)$ and that g is strictly (1, 2)-*convex with respect to F on I. It is now easy to show (or appeal to Theorems 3.1 and 3.2 in [6] to conclude) that g is strictly (1, 1, 1) and (2, 1)-*convex with respect to F on I. It remains to show the strict (3)-*convexity of g. Suppose that $f \in F$ and $f^{(j)}(z) = g^{(j)}(z)$, $j = 0, 1, 2$, at some $z \in I$. If $f(w) = g(w)$ for some $w > z$, then $g < f$ on (z, w) since g is strictly (2, 1)-convex. Pick u between z and w and let $h \in F$ satisfy $h(z) = f(z) = g(z)$, $h'(z) = f'(z) = g'(z)$, and $h(u) = g(u) < f(u)$. Then $h > g$ on (z, u), so $h''(z) \geqq g''(z)$. h and f are distinct, so it follows that $h''(z) > f''(z)$ and consequently that $f - h$ has a zero at some point in (z, u). That is impossible by the (2, 1) uniqueness of elements of F. The same contradiction would be reached if $f > g$ were to hold on $I \cap (z, \infty)$. It must be the case that $f < g$ on $I \cap (z, \infty)$. We conclude that g is strictly (3)-*convex, and the lemma is proved.

REMARKS. (i) Clearly the techniques used in the preceding proof could be used to show the conclusion of the lemma still holds if (4) is replaced by

$$g(x) > f(x) \text{ for all } x \in I \cap (z, \infty). \tag{5}$$

(ii) Indeed, only minor modifications in the preceding proof are needed to show that if the strict inequality in (4) (or (5)) is replaced by $\leqq$ (respectively by $\geqq$), then the conclusion of the lemma is that g is μ-*convex. Moreover if $f(z) = g(z)$ at a point z other than the given nodes, then f and g are identical between $\min\{z, x_1\}$ and $\max\{z, x_k\}$.

(iii) Finally, we observe that if "$<$" in (4) were to be replaced by "$>$", the conclusion of the lemma would be that g is strictly μ-*concave, but still $f - g$ would change sign at any point where it has a triple zero.

PROOF OF THEOREM 1. Suppose that (3) always holds for $i = q$. Let $x_1 < x_2 < \cdots < x_k$ be in I and let $f \in F$ satisfy (2). Suppose that $M(q)$ is odd. Then $g(x) \leqq f(x)$ for all $x_{q-1} < x < x_q$. Suppose that $\lambda_q = 3$. Then the subfamily H of F consisting of all $f \in F$ satisfying (2) except for $i = q$ is a

$\lambda(3, j)$-family on the interval (x_{q-1}, x_{q+1}) for $j = 1, 2, 3$. If $h^{(j)}(z) = g^{(j)}(z), j = 0, 1, 2$, for some $z \in (x_{q-1}, x_{q+1})$ and some $h \in H$, then $g(x) \leqq h(x)$ for all $x_{q-1} < x < z$, i.e., (4) holds with $I = (x_{q-1}, x_{q+1})$, with "$<$" replaced by "$\leqq$", and with F replaced by H. Then by the conclusion of the lemma (See remark (ii).), $g(x) \geqq f(x)$ for all $x_q < x < x_{q+1}$, and thus (3) holds for $i = q + 1$. If $M(q)$ is even, the analog of the lemma for concave functions (See remark (iii).) can be applied to show that $f - g$ changes sign at x_q. If $\lambda_q = 2, f - g$ does not change sign at x_q by the lemma in **[7]**. If $\lambda_1 = 1, f - g$ changes sign at x_q by Theorem 2 in **[8]**. We conclude that in all cases (3) holds for $i = q + 1$. By reapplying the above argument it follows that (3) holds for all $q \leqq i \leqq k + 1$. Similarly inequality (3) can be extended to the left for $i = q - 1, q - 2, \ldots, 1$, using the lemma with (4) replaced by (5) and the corresponding results in [7] and **[8]** for the cases that $f - g$ has zero of order 2 or 1 at some node. With these observations the proof is complete.

3. Convex Functions. In **[8]** Tornheim showed that if two members of a $\lambda(n, n)$-family F intersect at $n - 1$ points, then their difference must change sign at those $n - 1$ intersection points. Lazarević in **[5]** observed that if $f, g \in F$ and $f - g$ has p changes of sign and q zeros at which it does not change sign, then $p + 2q < n$. Also see [7] in this regard. In order to establish some of the relationships among the various types of convexity a generalization of this change of sign result is needed.

If a function h, which has a continuous derivative of order $m + 1$ at a point z on the real line, has a zero of order $m + 1$ at z, then $h^{(m+1)}(x)$ is of constant sign (either always positive or always negative) for x in a neighborhood of z. Consequently there is a number $d > 0$ so that $(x - z)^{m+1}h(x)$ is of constant sign for $0 < |x - z| < d$. It is this property we take to extend the definition of a zero of order $m + 1$ at a point where the function fails to have a derivative of order m. Suppose that h has a continuous derivative of order $m - 1$ in a neighborhood of a point z, $h^{(m)}(z)$ fails to exist, but $h^{(i)}(z) = 0$ for $i = 0, 1, \ldots, m - 1$. Then h is said to have a *zero of order* $m + 1$ at z in case there is a $d > 0$ so that $(x - z)^{m+1}h(x)$ is of constant sign for $0 < |x - z| < d$.

Suppose that $\lambda(n, k)$ is given and that $h \in C^r(I)$, $r + 1 = \| \lambda(n, k) \|$. Let the function h have the points $x_1 < x_2 < \cdots < x_k$ as zeros of orders $\mu_1, \mu_2, \ldots, \mu_k$ respectively. (The case that some of the numbers μ_i are 0 is not excluded.) For each i, $1 \leqq i \leqq k$, let z_i be the largest nonnegative integer such that $z_i + \mu_i$ is even, $z_i \leqq \mu_i$, and $z_i \leqq \lambda_i + 1$. For $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ define $Z(\mu, \lambda(n, k)) = (z_1, z_2, \ldots, z_k)$. For $k = n$ the number $z_1 + z_2 + \cdots + z_k$ is just the number $p + 2q$ considered by Lazarević. The theorem that follows is the analogue for $k < n$ of Theorem 3 in **[8]**.

THEOREM 2. *Let F be a hereditary $\lambda(n, k)$-family on I. Let* $\mu =$

$(\mu_1, \mu_2, \ldots, \mu_k)$, *and let there be two integers* $p \neq q$ *so that* $\mu_i = \lambda_i$ *for* $i \neq p$ *and* $i \neq q$, $\mu_p = \lambda_p - 1$, *and* $\mu_q = \lambda_q + 1$. *Then the difference of two functions from* F *may not have a* ν *distribution of zeros in* I *for which* $Z(\nu, \lambda(n, k)) = \mu$.

PROOF. The proof is by induction on the length $k \geqq 2$ of $\lambda(n, k)$. We begin with $k = 2$ and $\lambda(n, 2) = (\ell, m)$ where $\ell \geqq 1$ and $m \geqq 1$. A distribution of zeros of the form $(\ell - 1, m + 1)$ for differences of functions from F will be shown to be impossible. Let $\ell = 1$. Suppose that for some members f_1 and f_2 of F the difference $f_1 - f_2$ has a zero of order $m + 1$ at some point $b \in I$. Assume that m is even. Then without loss of generality it follows that there are points a and c in I with $a < b < c$ such that $f_1 < f_2$ on $[a, b)$ and $f_1 > f_2$ on $(b, c]$, i.e., $f_1 - f_2$ must have a sign change at b. Now pick $h \in F$ so that $h(a) = f_1(a)$, $h(c) = f_2(c)$, and $h^{(i)}(b) = f_1^{(i)}(b) = f_2^{(i)}(b)$ for $i = 0, 1, \ldots, m - 2$. Since F is a $(1, m - 1, 1)$-family such an h must exist and $h^{(m-1)}(b) \neq f_1^{(m-1)}(b)$. Since $h(c) = f_2(c) < f_1(c)$, the $(1, m - 1, 1)$ uniqueness implies that $h < f_1$ on (b, c). Because $f_1^{(m-1)}(b) = f_2^{(m-1)}(b)$, then $h(x) - f_1(x) < 0$ must hold on (a, b), for otherwise $f_2(x_1) = h(x_1)$ for some $x_1 \in (a, b)$ which is impossible. But this implies that $h - f_1$ does not change its sign at b which is also impossible. The case that m is odd is similar (no condition is placed on h at b if $m = 1$.), and the result follows for $\ell = 1$. For $\ell > 1$ the hereditary property of F implies that F is an $(\ell - 1, 1, m)$-family on I. Suppose that $f_1, f_2 \in F$ with $f_1 - f_2$ having an $(\ell - 1, m + 1)$ distribution of zeros at a and b where $a < b$ and both a and b are in I. Let G consist of all $f \in F$ which satisfy $f^{(j)}(a) = f_1^{(j)}(a), j = 0, 1, \ldots, \ell - 2$. Then G is $(1, m)$-family on $I \cap (a, \infty)$, and thus no difference of functions from G may have a zero of order $m + 1$ in $(1, \infty) \cap I$. But $f_1, f_2 \in G$ and $f_1 - f_2$ has a zero of order $m + 1$ at b. We conclude therefore no such f_1 and f_2 may exist in F. Consequently $Z(\nu, (\ell, m)) = (\ell + 1, m - 1)$ is impossible, and by symmetry $z(\nu, (\ell, m)) = (\ell + 1, m - 1)$ is also impossible. In fact, by the previous argument using G, it follows that it is sufficient to prove the theorem only for the case $\lambda_1 = 1$.

Suppose next that the theorem is valid for all $\lambda(n, j)$ with $j < k$. Let $\lambda(n, k) = (1, \lambda_2, \lambda_3, \ldots, \lambda_k)$. We claim that the only distribution of zeros not immediately ruled out with $p < q$ is the case where $p = 1$ and $q = k$, i.e., $(\lambda_2, \lambda_3, \ldots, \lambda_{k-1}, \lambda_k + 1)$. To see this observe that if $f_1 - f_2$ has distribution of zeros which is claimed impossible at $x_1 < x_2 < \cdots < x_k$ in I, then the functions $f \in F$ satisfying $f^{(j)}(x_i) = f_1^{(j)}(x_i)$, $i = 1, 2, \ldots, p - 1$, $j = 0, 1, \ldots, \lambda_i - 1$, when restricted to $J = I \cap (x_{p-1}, \infty)$ form a hereditary μ-family H on J for $\mu = (\lambda_p, \lambda_{p+1}, \ldots, \lambda_q, \ldots, \lambda_k)$. The number of entries in the ordered partition μ of $n - (\lambda_1 + \lambda_2 + \cdots + \lambda_{p-1})$ is $k - p + 1 < k$ since $p > 1$. Then by the induction assumption no difference of functions from H may have a $(\lambda_p - 1, \lambda_{p+1}, \cdots, \lambda_q + 1, \ldots, \lambda_k)$ distribution of zeros in J. But f_1 and f_2 restricted to J are in H and give such a distribution of zeros. Consequently we may assume $p = 1$. Similarly it suffices to consider $q = k$. Consider first the case $\lambda_k = 1$. Suppose that $f_1 - f_2$ has a $(\lambda_2, \lambda_3, \ldots, \lambda_{k-1}, 2)$ distribution of zeros at $x_2 < x_3 < \cdots < x_k$ in I. Pick $u, v \in I$ with $u < x_2$ and $v > x_k$, and pick $h \in F$ so that $h - f_1$ has a $(1, \lambda_2, \lambda_3, \ldots, \lambda_{k-1})$ distribution of zeros at $u < x_2 < x_3 < \cdots < x_{k-1}$ and $h(v) = f_2(v)$. $h - f_1$ cannot have a zero in (x_{k-1}, v) because of the uniqueness on I of $h \in F$ satisfying (1). By the induction assumption and the hereditary property of $F, f_1 - f_2$ cannot change sign in any of the intervals (x_i, x_{i+1}) for $2 \leqq i \leqq k - 1$. To see this suppose that $f_1 - f_2$ changed sign at $w \in (x_t, x_{t+1})$. The subfamily H of F consisting of all $f \in F$ for which $f - f_2$ has a zero of order at least λ_i at x_i for each $i \geqq t + 1$ is a $(1, \lambda_2, \ldots, \lambda_t)$-family and hence a hereditary $(1, \lambda_2, \lambda_3, \ldots, \lambda_{t-1}, \lambda_t - 1, 1)$-family on $(-\infty, x_{t+1}) \cap I$. Thus a $(\lambda_2, \lambda_3, \ldots, \lambda_t, 1)$ distribution of zeros for difference of functions from H is impossible. Hence, as claimed, $f_1 - f_2$ cannot change sign except at the given nodes. Now since $f_1 - f_2$ has an even order zero at x_k and $h(x_k) \neq f_1(x_k)$, the graph of f_2 lies between graphs of f_1 and h on (x_{k-1}, v). By the induction assumption neither $h - f_2$ nor $h - f_1$ can have a zero at x_i of order greater than λ_i for $i = 2, 3, \ldots, k - 1$, and therefore the graph of f_2 separates the graphs of h and f_1 on $(-\infty, v) \cap I$. But this contradicts the fact that $h(u) = f_1(u) \neq f_2(u)$, and consequently the result is established for $\lambda_k = 1$.

For $\lambda_k > 1$ the above arguments needs to be modified only slightly. With the same assumptions about the zeros of $f_1 - f$, except that now x_k is a zero of order $\lambda_k + 1$, pick h as before and in addition require that $h - f_1$ and $h - f_2$ both have a zero of order $\lambda_k - 1$ at x_k. Again the graph of f_2 must separate the graphs of f_1 and h on $(-\infty, v) \cap I$, and the same contradiction is reached. The result then follows for $p < q$. The symmetric situation with $q < p$ clearly follows in an analogous fashion. This concludes the proof of the theorem.

Some technical notation is needed for the statement of the next theorem. The family F of functions on I satisfies $U(\lambda(n, k))$ [alternatively $E(\lambda(n, k))$] in case for any k nodes $x_1 < x_2 < \cdots < x_k$ in I and any n real numbers α_{ij} there is at most [alternatively at least] one $f \in F$ satisfying (1). The set $A(\lambda(n, k), m+)$ consists of all $\mu(n, j) \in P(n)$ with $j = k$ or $k + 1$ that can be obtained from $\lambda(n, k)$ by inserting *a* 1 between λ_i and λ_{i+1} for $i \geqq m$ or after λ_k and then replacing λ_m by $\lambda_m - 1$ deleting a resulting entry of 0. $A(\lambda(n, k), m-)$ is defined analogously with the 1 being placed in a position to the left of λ_m, and $A(\lambda(n, k), m)$ is just the union of $A(\lambda(n, k), m+)$ and $A(\lambda(n,k), m-)$. The subset $B(\lambda(n, k), m)$ consists of all $\mu(n, j)$ in $P(n)$ with

$j = k - 1$ or k that can be obtained from $\lambda(n, k)$ by replacing λ_i, $i \neq m$, by $\lambda_i + 1$ and then replacing λ_m by $\lambda_m - 1$ again deleting an entry of 0.

THEOREM 3. *Let $\lambda(n, k) \in P(n)$ and suppose that the $\lambda(n, k)$-family F satisfies $U(\mu)$ for all $\mu \in B(\lambda(n, k), m) \cup A(\lambda(n, k), m)$ and is a $\lambda(n, k; m+)$-family on I. Suppose that $g \in C^r(I)$ is $\lambda(n, k; m+)$-convex with respect to F on I. Then g is also $\lambda(n, k)$-convex with respect to F on I provided either* (i) *$\lambda_j \leqq 3$ for all $j < m$, or else* (ii) *F satisfies $E(\mu)$ for each $\mu \in A(\lambda(n, k), m-)$. The analogous result, with $\lambda(n, k; m+)$, $A(\lambda(n - k), m-)$, and $j < m$ replaced by $\lambda(n, k; m-)$, $A(\lambda(n, k), m+)$, and $j > m$ respectively, also holds.*

COROLLARY 1. *Let F be a hereditary $\lambda(n, k)$-family on I. If g is $\lambda(n, n)$-convex with respect to F on I, then g is $\lambda(n, k)$-*convex with respect to F on I.*

The corollary is easily established by observing that $\lambda(n, n)$ may be obtained from $\lambda(n, k)$ by a sequence of replacements of type 1 so that at each step condition (i) of Theorem 3 is satisfied. The necessary uniqueness conditions follow from the hereditary property and Theorem 2 as is pointed out in the proof that follows. An examination of the proof shows that Theorem 3 remains valid for *convexity in place of convexity. By placing the strong "hereditary" condition on the family F in Theorem 3, Corollary 1 follows without the even stronger assumption that F is an unrestricted n-parameter family which was used by Umamaheswaram in [**10**, page 764] to get the same conclusion. Also see Theorem 4.5 in [**9**].

PROOF OF THEOREM 3. There are four cases to consider depending on the parity of λ_m and $\lambda_{m+1} + \cdots + \lambda_k$. We will consider only the case that both these quantities are odd since the arguments in the other cases are similar. Let $x_1 < x_2 < \cdots < x_k$ be in I and let $f \in F$ satisfy (2). $M(m + 1) = 2n - (\lambda_{m+1} + \cdots + \lambda_k)$ is odd and $M(m)$ is even, so we must show that $g \geqq f$ on (x_{m-1}, x_m) and $g \leqq f$ on (x_m, x_{m+1}). Suppose there is a $u \in (x_m, x_{m+1})$ so that $g(u) > f(u)$. Pick $h \in F$ so that (2) is satisfied with h in place of f except for the one pair $(i, j) = (m, \lambda_m - 1)$, and let $h(u) = g(u)$. Since g is $\lambda(n, k; m+)$-convex, we know that $g \geqq h$ on (x_m, u). $f - h$ cannot have a zero in (x_m, u) because the fact that F is a hereditary $\lambda(n, k)$-family guarantees $f - h$ cannot have a $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, \lambda_m - 1, 1, \lambda_{m+1}, \ldots, \lambda_k)$ distribution of zeros, and hence $f < h \leqq g$ on (x_m, u). But that implies that $f - h$ has a zero of order λ_m at x_m which contradicts the uniqueness of solutions of (1) for member of F. It then follows that $g \leqq f$ on (x_m, x_{m+1}). If $f(u) > g(u)$ for some $x_m < u < x_{m+1}$, then the function h defined above has the property that its graph lies between the graphs of g and f on each of the intervals (x_i, x_{i+1}) for $i = m + 1, m + 2, \ldots, k - 1$ and consequently (3) holds for $i \geqq m + 2$. To see that h must be as claimed we observe $h - f$ must have at x_i for

$i \geqq m + 2$ a zero of order λ_i by property $U(\mu)$ for $\mu \in B(\lambda(n, k), m)$. Actually all that is required here is that the zero of $h - f$ at x_i must be of the same parity as λ_i, and that follows from Theorem 1. This observation is needed in the proof of Corollary 1. Also $h - f$ cannot have a zero in $(u, \infty) \cap I$ except at some x_i, for if it did, then the property $U(\mu)$ for $\mu \in A(\lambda(n, k), m)$ would be violated. Actually we can get by with a weaker hypothesis here. If all λ_i for $i > m$ are 1, that the sign of $f - g$ is correct follows from Theorem 1. Otherwise the hereditary property implies that F is a hereditary μ-family for $\mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, 1, \lambda_{i+1}, \ldots, \lambda_k)$ where $i > m$ and $\lambda_i > 1$. Then Theorem 2 (with $p = m$ and $q = i$) can be applied to show that $\nu(n, k + 1)$ with $Z(\nu, \mu) = (\lambda_1, \ldots, \lambda_{m+1}, \lambda_m - 1, \ldots, \lambda_i, 1, \lambda_{i+1}, \ldots, \lambda_k)$ cannot be a distribution of zeros for $f - h$. This observation is also needed for the proof of the corollary. If $f(x) = g(x)$ for all $x \in (x_m, x_{m+1})$, then by the $\lambda(n, k; m+)$-convexity of g it follows that $f - g$ has the correct sign on (x_i, x_{i+1}) for $m + 1 \leqq i \leqq k$. It then remains only to show that (3) holds for all $i \leqq m$. Suppose there were a v in (x_{m-1}, x_m) for which $f(v) > g(v)$. Pick $f_1 \in F$ so that (2) holds with f_1 in place of f for all i, j except the pair $(i, j) = (m, \lambda_m - 1)$, and let $f_1(v) = (f(v) + g(v))/2$. Now property $U(\mu)$ for $\mu = \lambda(n, k)$, $\lambda(n, k; m+)$, and $\lambda(n, k; m-)$ implies that $f_1 - f$ cannot change sign in (x_{m-1}, x_{m+1}). Also $f_1 - g$ cannot have a zero in (x_m, x_{m+1}) by the $\lambda(n, k; m+)$ convexity of g. If $f_1^{(j)}(x_m) < g^{(j)}(x_m)$ for $j = \lambda_m - 1$, then $f_1(x) < g(x)$ if $0 < |x - x_m| < \delta$ for some $\delta > 0$. Let $u = x_m + \delta/2$, and pick h as before. Then $h \leqq g$ on (x_{m-1}, u), and consequently $h - f_1$ must change sign at some point in (v, u). But that would contradict the uniqueness of solutions to (1) in F for $\lambda(n, k)$, $\lambda(n, k; m+)$, or $\lambda(n, k; m-)$. Thus $f \leqq g$ on (x_{m-1}, x_m). Now if $\lambda_j \leqq 3$ for all $j < m$, the result follows by Theorem 1. If $\lambda_{m-1} > 3$, we must make use of condition (ii). For definiteness suppose that λ_{m-1} is odd and that $f(z) < g(z)$ at some $z \in (x_{m-2}, x_{m-1})$. With h as before pick $f_2 \in F$ so that (2) holds with f replaced by f_2 except for the pair $(m, \lambda_m - 1)$, and let $f_2(z)$ be chosen so that $f(z) < f_2(z) < g(z) \leqq h(z)$. Then the graph of f_2 separates the graphs of f and h on (x_{i-1}, x_i) for $i = m - 1$, m, $m + 1$. In addition $f^{(j)}(x_m) = g^{(j)}(x_m) \neq f_2^{(j)}(x_m)$ for $j = \lambda_m$, so $f - g$ must have a zero in (x_m, u). Then by the $\lambda(n, k; m+)$ convexity of g it follows that $f_2(z) \geqq g(z)$ which is impossible. Hence $f \geqq g$ on (x_{m-2}, x_{m-1}). If λ_m is even, the same argument shows that $f \leqq g$ on (x_{m-2}, x_{m-1}) and thus (3) holds for $i = m - 1$. Similar arguments work for $i < m - 1$, and the theorem is proved.

Umamaheswaram in **[10]** defined a function g which is $\lambda(n, k)$-*convex with respect to F on I to have property $P(\lambda(n, k))$ in case for any $x_1 < x_2 < \cdots < x_k$ in I and $f \in F$, the conditions in (2) together with $g(z) = f(z)$ for some $z \in I$ with $z \neq x_i$, $i = 1, 2, \ldots, k$, imply that f and g are identical on the closed interval $[\min\{x_1, z\}, \max\{x_k, z\}]$. If F is a hereditary $\lambda(n, k)$-family on I and g is $\lambda(n, n)$-*convex with respect to F on I, then g is also $\lambda(n, j)$-*convex for any $\lambda(n, j) \leqq \lambda(n, k)$. In fact the arguments in the proof of Theorem 3 work as well for $i = k + 1$ and $i = 1$ if one knows that g is $\lambda(n, k; m+)$-*convex. Indeed if $f \in F$ satisfies (2) and $f(z) = g(z)$ where $z \in (x_m, x_{m+1})$, then the $\lambda(n, k; m+)$-convexity of g (or the $\lambda(n, k; m-)$ convexity if $m = 0$) shows that f and g are identical on $[x_m, z]$. Then by using the $\lambda(n, k; m+)$-*convexity of g and picking k of the $k + 1$ nodes in $[x_m, z]$, it follows that f and g are identical on $[x_m, x_k]$. A similar argument using $\lambda(n, k; 1-)$ gives the property $P(\lambda(n, k))$. Therefore the conclusions of Theorem 4.5 in [9] follow for hereditary $\lambda(n, k)$-families as well as unrestricted n-parameter families.

In Theorem 3 conditions are given under which convexity for a given partition implies convexity for a partition larger in the sense of the partial order on $P(n)$. Theorem 4 deals with the converse problem.

THEOREM 4. *Suppose that F is a hereditary $\lambda(n, k)$-family on I. Let g be $\lambda(n, k)$-convex with respect to F on I. Then g is also $\lambda(n, k; m+)$-convex with respect to F on I provided $\lambda_i \leqq 3$ for $i > m$, $\lambda_m \leqq 4$, and $1 < m < k$. If g is $\lambda(n, k)$-*convex, then the condition $1 < m < k$ can be deleted, and the conclusion is that g is $\lambda(n, k; m+)$-*convex. The analogous result for $\lambda(n, k; m-)$ and $\lambda_i \leqq 3$ for $i < m$ is also valid.*

PROOF. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{k+1}) = \lambda(n, k; m+)$, let $x_1 < x_2 < \cdots < x_{k+1}$ be points in I, and let $f \in F$ satisfy (2) with μ in place of λ and $k + 1$ in place of k. Observe that μ has length $j = k + 1$, so $x_{j+1} = x_{k+2}$ would now denote the right endpoint of I using the previously introduced convention. Pick $h \in F$ so that $h - g$ has a λ distribution of zeros at $x_1, x_2, \ldots, x_m, x_{m+2}, \ldots, x_{k+1}$. For definiteness assume that λ_m is even and $M(m + 1)$ is odd. (Since $\lambda(n, k; m+) = \lambda(n, k)$ if $\lambda_m = 1$, we can assume that $\lambda_m \geqq 2$.) Then $h \geqq g$ on (x_m, x_{m+2}). Suppose that $h(x_{m+1}) > g(x_{m+1})$. Since $f(x_{m+1}) = g(x_{m+1})$ and $f - h$ cannot change sign in (x_m, x_{m+1}), it follows that $f(x) < h(x)$ on (x_m, x_{m+1}) and $f^{(j)}(x_m) < h^{(j)}(x_m)$ for $j = \lambda_m - 1$. Since $\lambda_m - 1$ is odd, $f > h$ on a small interval with right endpoint x_m. $f - h$ cannot change sign in (x_{i-1}, x_i) for $i = 1, 2, \ldots, m$ by the hereditary property and Theorem 2, and $f - h$ has a zero of order exactly λ_i at x_i since f and g are distinct members of F. Therefore the graph of h separates the graphs of f and g on $I \cap (-\infty, x_m)$. Consequently $g - f$ and $g - h$ cannot have opposite signs in any interval (x_{i-1}, x_i) for $i \leqq m$ and (3) must hold for $i \leqq m$ since it holds for $i < m$ if f is replaced by h. Next since $\mu_i \leqq 3$ for $i \geqq m$, it follows that $f - g$ also has the correct sign on (x_{i-1}, x_i) for $i > m$. Clearly if $h(x_{m+1}) = g(x_{m+1})$ then h and f are identical and (3) holds for $i < m$ with μ in place of λ. Thus (3) holds also for $i > m$ and μ in place of λ since $\mu_i \leqq 3$ for $i \geqq m$. Observe in this case that f and g are identical on (x_m, x_{m+1}). Then it follows from the fact that g is both $\lambda(n, k; k+)$ and $\lambda(n, k; 1-)$-convex that f and g must be identical on $[x_1, x_{k+1}]$.

If g is known to be $\lambda(n, k)$-*convex, the argument above clearly extends to the intervals (x_0, x_1) and (x_k, x_{k+1}) to show that g is $\lambda(n, k; m+)$-*convex.

COROLLARY. *If g is $\lambda(n, k)$-*convex with respect to the hereditary $\lambda(n, k)$-family F and if $\|\lambda(n, k)\| \leqq 4$, then g is also $\lambda(n, n)$-convex with respect to F. The analogous result holds for strict convexity.*

In the case of an unrestricted n-parameter family F, Umamaheswaram has shown (see [**10**; Theorem 3.3]) that if g is $\mu(n, j)$-*convex for all $\mu(n, j) \in P(n)$ which have at most one entry equal to 1, then g is $\lambda(n, n)$-convex with respect to F. Observe that by the above corollary the $\lambda(n, n)$-convexity of g will follow if g is $\mu(n, j)$-convex for any $\mu(n, j)$ with $\|\mu(n, j)\| \leqq 4$. According to [**10**; page 764] in the case $n = 4$ if g is strictly μ-*convex with respect to the unrestricted 4-parameter family F for $\mu = (1, 3)$, $(3, 1)$ and $(2,2)$, then g is strictly $(1, 1, 1, 1)$-convex. By the corollary above the strict μ-*convexity of g for any of $(1, 3)$, $(3, 1)$, $(2, 2)$ or (4) implies the strict $(1, 1, 1, 1)$-convexity of g.

The final result gives a sufficient condition under which a $\lambda(n, k)$-*convex function g has property $P(\lambda(n, k))$ with respect to a hereditary $\lambda(n, k)$-family F.

THEOREM 5. *Let F be a hereditary $\lambda(n, k)$-family on I and let g be $\lambda(n, k)$-*convex with respect to F on I. Then g has property $P(\lambda(n, k))$ provided $\lambda_1 \leqq 4$, $\lambda_k \leqq 4$, and for each $i = 3, 4, \ldots, k-1$ either $\lambda_i \leqq 4$ or $\lambda_{i-1} \leqq 4$.*

PROOF. Suppose that $x_1 < x_2 < \cdots < x_k$ are in I and that (2) holds for some $f = f_0 \in F$. Suppose also that $f_0(z) = g(z)$ for some $z \in (x_{q-1}, x_q)$ and $1 \leqq q \leqq k+1$. We first consider $1 < q < k+1$. Let us assume that $\lambda_{q-1} \leqq 4$ since the argument in the case $\lambda_q \leqq 4$ involves the same ideas. Let H consist of all $f \in F$ for which (2) holds for all pairs (i, j) with $i \neq q-1$. Then H is an unrestricted λ_{q-1}-parameter family on (x_{q-2}, x_q), and g is either (λ_{q-1})-*convex or *concave with respect to H on (x_{q-2}, x_q) depending on the parity of $M(q+1)$. Also by Theorem 4 or its analogue for concave functions g is $(\lambda_{q-1} - 1, 1)$-*concave with respect to H on (x_{q-2}, x_q). Consequently f_0 and g are identical on $[x_{q-1}, z]$. Next pick nodes $y_1 < y_2 < \cdots < y_{k-1}$ in (x_{q-1}, z) and let G consist of all $f \in F$ satisfying (2) for $1 \leqq i \leqq k-1$ with y_i in place of x_i. Then G is an unrestricted λ_k-family on $J = (y_{k-1}, b)$ where b is the right endpoint of I. Since $\lambda_k \leqq 4$ we may apply Theorem 4 to conclude that g is $(\lambda_k - 1, 1)$-*convex (or concave) on J. Next observe that if $y_k = z$ and $y_{k+1} = x_k$ or $y_k =$

$(y_{k-1}+z)/2$ and $y_{k+1} = z$, then $f^{(j)}(y_k) = g^{(j)}(y_k)$ for $j = 0. 1, \ldots, \lambda_k - 2$ and $f(y_{k+1}) = g(y_{k+1})$. It then follows that on (z, x_k) both $f \geqq g$ and $g \geqq f$ hold, and thus f and g are identical on $[x_{q-1}, x_k]$. That f and g must be identical on $[x_1, x_{q-1}]$ can be established similarly using Theorem 4 and the fact that $\lambda_1 \leqq 4$. The cases that $q = 1$ or $q = k + 1$ are resolved in the same manner noting that, for $q = k + 1$, $\lambda_{q-1} = \lambda_k \leqq 4$ and, for $q = 1$, $\lambda_q = \lambda_1 \leqq 4$. It thus follows that f and g are identical on $[\min\{z, x_1\}, \max\{z, x_k\}]$, and the result is proved.

In conclusion it should be observed that the hereditary assumption in Theorem 5 as well as Theorem 1 is stronger than is needed. In fact it is sufficient in these theorems to assume that the $\lambda(n, k)$-family F is "pointwise" hereditary, i.e., for each m, if $x_1 < x_2 < \cdots < x_{m-1} < x_{m+1} < x_{m+2} < \cdots < x_k$ are $k - 1$ points in I and if G consists of all $f \in F$ satisfying (1) for $i \neq m$, then G is a hereditary (λ_m)-family on (x_{m-1}, x_{m+1}). In particular by Theorem 1 if the $\lambda(n, k)$-family F is pointwise hereditary for $m = 1$ and $m = k$, and both $\lambda_1 \leqq 3$ and $\lambda_k \leqq 3$, then there is no distinction between convexity and *convexity.

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