

## INFINITELY DIVISIBLE GIBBS STATES

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**ABSTRACT.** A notion of infinite divisibility for Gibbs states which is based on the structure of the configuration space for lattice systems as a direct product of cyclic groups of order two is explored. Simple examples are provided to show that infinitely divisible probability measures do exist within the class of Gibbs states. This suggests a line of problems for which a few general results are obtained. It is also shown by example that passage to the thermodynamic limit may be required before infinite divisibility appears as a property of the state. Finally, it is shown that infinite divisibility is a sufficient condition for an interesting correlation inequality.

**1. Introduction.** Group analysis has previously been applied to the study of classical lattice systems in a wide variety of directions; see McKean [6] for a specific illustration, and Gruber, Hintermann, and Merlini [3] for the more comprehensive theory. Likewise the study of infinitely divisible probability measures on abstract group structures has been explored in several directions; see Parthasarathy [7] for example. However, the line of problems and examples given here appear to represent a first look at the extent to which the class of Gibbs states intersect the class of infinitely divisible probability measures. By no means are all of the issues resolved though. The motivation for such a study is based on the availability of the Levy-Khintchine formula for the characteristic function of infinitely divisible probability measures [7], the identification of  $n$ -point correlations of Gibbs states with their characteristic function, and the fundamental importance of  $n$ -point correlations in mathematical statistical mechanics.

Let  $Z^d$  denote the  $d$ -dimensional integer lattice and let  $\Omega = \{0, 1\}^{Z^d}$  be given the product topology and corresponding Borel sigma-field  $\mathcal{B}$ . The collection of Gibbs states on  $(\Omega, \mathcal{B})$  for a potential  $\Phi$  will be denoted by  $\mathcal{G}(\Phi)$ .

A configuration  $\eta \in \Omega$  will be represented as a subset  $X$  of  $Z^d$  in the usual way. Under symmetric difference  $\Delta$  we have that  $(\Omega, \Delta)$  is a compact abelian group with identity  $\phi$  and inverses  $X^{-1} = X$ ,  $X \in \Omega$ . The dual

group of  $\Omega$  is the collection  $\Omega_0$  of finite subsets of  $Z^d$  under  $\Delta$ . In particular each  $A \in \Omega_0$  corresponds to a character of  $\Omega$  of the form

$$(1.1) \quad \chi_A(X) = A(X) = (-1)^{|A \cap X|}, \quad X \in \Omega.$$

The characteristic function (Fourier transform) of a probability measure (state)  $\mu$  on  $(\Omega, \mathcal{B})$  is given by  $\hat{\mu}: \Omega_0 \rightarrow \mathbb{C}$  with

$$(1.2) \quad \hat{\mu}(A) = \int_{\Omega} (-1)^{|A \cap X|} \mu(dX).$$

In the mathematical physics literature  $\hat{\mu}(A)$  is referred to as an  $|A|$ -point correlation between particles in the configuration  $A$ .

A state  $\mu$  is infinitely divisible if and only if for each  $n \geq 1$  there is a probability measure  $\mu_n$  on  $(\Omega, \mathcal{B})$  such that

$$(1.3) \quad \hat{\mu} = \hat{\mu}_n^n \quad n = 1, 2, \dots$$

Equivalently, for each  $n \geq 1$ ,  $\mu$  may be represented as the  $n$ -fold convolution  $\mu_n^{*n}$  of  $\mu_n$  with itself where the convolution of states  $\mu, \nu$  is given by

$$(1.4) \quad \mu * \nu(B) = \int_{\Omega} \mu(B \Delta X) \nu(dX), \quad B \in \mathcal{B}.$$

Although we have adopted the lattice gas formalism, the Ising model for  $\pm$  spin magnetism will appear in the examples. In such cases a spin configuration  $\eta \in \{-1, 1\}^{Z^d}$  is identified with a subset  $X$  of  $Z^d$  according to the 'spin down' convention

$$(1.5) \quad X = \{i \in Z^d: \eta(i) = -1\}.$$

In the lattice gas framework  $\Delta$  corresponds to pointwise addition of configurations modulo two, while in the magnetic spin framework  $\Delta$  corresponds to ordinary pointwise multiplication of configurations.

For the most part the discussion will be restricted to finite range potentials  $\Phi$  although the statements of the various problems only require enough to insure that  $\mathcal{D}(\Phi)$  is non-empty; see Preston [8] for general existence conditions. In the case of the Ising models we find it to be more convenient to represent  $\Phi$  in terms of the usual coupling constants. In the case of the classical homogeneous Ising model with pairwise nearest neighbor interactions under zero external magnetic field and coupling constant  $J$  we have

$$(1.6) \quad \Phi(X) = \begin{cases} -4dJ, & X = \{i\} \\ 4J, & X = \{i, j\}, \quad |i - j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

For the treatment of the problems given here we also find the definition

of Gibbs states on  $(\Omega, \mathcal{B})$  given by Lanford and Ruelle [5] to be more convenient than the equivalent version in terms of conditional probabilities given by Dobrushin [2]. For this first note that the sigma field  $\mathcal{B}$  is generated by finite dimensional rectangles of the form

$$(1.7) \quad [Y, F] = \{X \in \Omega : X \cap F = Y\}, \quad Y \subset F \in \Omega_0.$$

For fixed  $F \in \Omega_0$  the rectangles  $[Y, F], Y \subset F$ , partition  $\Omega$  into  $2^{|F|}$  disjoint rectangles.

Next define  $T_X: \Omega \rightarrow \Omega, X \in \Omega$ , by

$$(1.8) \quad T_X(Y) = X \Delta Y.$$

Since  $T_X: [Y, F] \rightarrow [(X \cap F) \Delta Y, F]$  is one-to-one and onto it follows immediately that  $T_X$  is measurable. Moreover  $T_X$  is continuous for the product topology on  $\Omega$ . The definition (1.9) follows Lanford and Ruelle [5].

DEFINITION 1.9. A probability measure  $\mu$  on  $(\Omega, \mathcal{B})$  belongs to  $\mathcal{D}(\Phi)$  if and only if for each  $F \in \Omega_0, X \subset F$ ,

$$(i) \quad \mu \circ T_X^{-1} \ll \mu \text{ on } [\phi, F]$$

$$(ii) \quad \frac{d\mu \circ T_X^{-1}(Y)}{d\mu} = \exp\left\{\beta \sum_{\substack{W \subset X \Delta Y \\ W \cap F \neq \emptyset}} \Phi(W)\right\} \text{ for } Y \in [\phi, F].$$

Define  $h_F^\phi, F \in \Omega_0$ , on  $\Omega$  by

$$(1.10) \quad h_F^\phi(X) = \exp\left\{\beta \sum_{\substack{W \subset X \\ W \cap F \neq \phi}} \Phi(W)\right\}.$$

The focus of the present paper is directed upon the following set of problems stated in their order of decreasing difficulty.

PROBLEM 1.11. Find conditions on  $\Phi$  under which the state  $\mu \in \mathcal{D}(\Phi)$  is infinitely divisible.

PROBLEM 1.12. Find conditions on  $\Phi$  for which  $\mu * \mu \in \mathcal{D}(\Phi')$  for some potential  $\Phi'$  given that  $\mu \in \mathcal{D}(\Phi)$ .

PROBLEM 1.13. Find conditions on  $\Phi$  such that for  $\mu \in \mathcal{D}(\Phi)$  and for all  $F \in \Omega_0, \mu * \mu \circ T_F^{-1} \ll \mu * \mu$  on  $[\phi, F]$ .

PROBLEM 1.14. Find conditions on  $\Phi$  such that for each  $X \in \Omega, \nu_X = \mu \circ T_X^{-1} \in \mathcal{D}(\Phi^X)$  for some potential  $\Phi^X$ .

The main problem is problem (1.11). The others are in the direction of providing a solution to (1.11). Problem (1.11) seems to be non-trivial indeed. At this stage we can do no more than verify infinite divisibility for certain examples and give some results for problems (1.12)–(1.14). The examples and proofs of infinite divisibility are given in section two.

The Levy-Khintchine measure is also calculated for these examples in section two.

Some results for problems (1.12)–(1.14) are given in section three. Problem (1.13) is a part of problem (1.12); namely the local absolute continuity condition (i) of (1.9). So a solution to (1.13) solves (1.12) up to the identification of the Radon-Nikodym derivative. Problem (1.14) is based on the observation in connection with (1.12) that  $\mu * \mu$  is the convex combination

$$(1.15) \quad \mu * \mu(\cdot) = \int_{\Omega} \mu \circ T_X^{-1}(\cdot) \mu(dX) = \int_{\Omega} \nu_X(\cdot) \mu(dX).$$

Although  $\mathcal{D}(\Phi)$  is convex (a Choquet simplex in fact) the potential for  $\nu_X$  will typically depend on  $X$  so that (1.14) does not completely resolve (1.11). However the solution to (1.14) can be used to solve (1.13).

Correlation inequalities of various sorts play an extremely important role in statistical mechanics. In section four it is shown that infinite divisibility is a sufficient condition for a correlation inequality first studied by Kelly and Sherman [4]. By means of a simple finite volume example, they show that this particular inequality cannot be expected to hold in general. However, in this connection it is of interest that we show (by example) that it may be necessary to pass to the thermodynamic limit for infinite divisibility to appear as a property of the state.

**2. Some examples.** In general the set of sites will be denoted by  $S$  although we take  $S = Z^d$  in most cases. In any case  $S$  is a countable (including finite) set.  $\Omega = \{0, 1\}^S$  is generally represented as the power set of  $S$  according to the usual convention that  $X = \{i \in S: \eta(i) = 1\}$ ,  $\eta \in \Omega$ , for lattice gas models. The ‘spin down’ convention (1.5) is used in the case of lattice magnets.

**EXAMPLE 2.1. Ideal Gas.** In the case of the ideal gas with parameter  $\rho$ ,  $-\infty \leq \rho \leq \infty$ .

$$(2.2) \quad \begin{aligned} \Phi(X) &= 0 \text{ for } X \in \Omega_0, & \text{for } |X| > 1 \text{ or } X = \emptyset, \\ \Phi(X) &= \rho \text{ for } X = \{x\}. \end{aligned}$$

In this case  $\mathcal{D}(\Phi)$  is a singleton consisting of the Bernoulli product measure  $\mu_p$ ,  $p = e^{\beta\rho}/2 \cosh(\beta\rho)$ , given by

$$(2.3) \quad \mu_p[Y, F] = p^{|Y|} q^{|F| - |Y|}, \quad Y \subset F \in \Omega_0,$$

where  $q = 1 - p$ . In particular,

$$(2.4) \quad \begin{aligned} \mu_p * \mu_p([x, x]) &= 2pq & \text{and} \\ \mu_p * \mu_p([\phi, x]) &= p^2 + q^2, & q = 1 - p. \end{aligned}$$

Note that in the case  $p = 1/2$  (i.e.,  $\rho = 0$ )  $\mu_{1/2}$  is normalized Haar measure for the group  $(\Omega, \Delta)$ . The characteristic function of  $\mu_p$  is given by

$$(2.5) \quad \begin{aligned} \hat{\mu}_p(A) &= \int_{\mathcal{O}} (-1)^{|A \cap X|} \mu(dX) \\ &= (q - p)^{|A|}. \end{aligned}$$

It follows that  $\mu_p$  is infinitely divisible for  $p \leq 1/2$  (i.e.,  $\rho \leq 0$ ). In particular if  $p_n = (1 - \sqrt[2n]{1 - 2p})/2$ , then

$$(2.6) \quad \mu = \mu_{p_n}^{*n}, \quad n = 1, 2, \dots$$

In the special case  $p = 1/2$ , note that

$$(2.7) \quad \hat{\mu}_{1/2}(A) = \begin{cases} 1, & A = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Moreover  $\mu_{1/2}$  is idempotent (i.e.,  $\mu_{1/2} = \mu_{1/2} * \mu_{1/2}$ ).

EXAMPLE 2.8. Mixed Phase Ideal Gas. Let  $\mu = p\mu_{p_1} + q\mu_{p_2}$  where  $0 < p < 1$ ,  $p + q = 1$ , and  $\mu_{p_1}, \mu_{p_2}$  are Bernoulli product measures with parameters  $p_1, p_2$ , respectively. Certainly if  $0 \leq p_1 = p_2 \leq 1/2$  then  $\mu$  is infinitely divisible. For a real mixture we restrict the example to  $p_1 \neq p_2$ , say  $0 \leq p_1 < p_2 \leq 1/2$ .

In the case of real valued random variables with the ordinary group structure it is generally not to be expected that mixtures of infinitely divisible laws are again infinitely divisible although there are special exceptions (see Steutel, [9]). One special case here is the case  $0 \leq p_1 < p_2 = 1/2$ . Since Haar measure  $\mu_{1/2}$  is idempotent  $\mu^{*n} = p^n \mu_{p_1}^{*n} + (1 - p^n) \mu_{1/2}$ . It is therefore apparent how to construct  $\mu_n, n \geq 1$ , for (1.3). In general note that for  $0 \leq p_1 < p_2 \leq 1/2$

$$\begin{aligned} \hat{\mu}(A) &= p\hat{\mu}_{p_1}(A) + q\hat{\mu}_{p_2}(A) \\ &= \lambda_1^{|A|}(p + q(\lambda_2/\lambda_1)^{|A|}) \end{aligned}$$

where  $\lambda_1 = 1 - 2p_1 > 0$ , and  $\lambda_2 = 1 - 2p_2 \geq 0$ . It follows that

$$\mu = \mu_{p_1} * (p\mu_0 + q\mu_{p_3})$$

where

$$p_3 = \frac{1}{2} \left( 1 - \frac{\lambda_2}{\lambda_1} \right) = \frac{p_2 - p_1}{1 - 2p_1} \leq \frac{1}{2}.$$

Since the class of infinitely divisible laws is closed under convolution it suffices to consider the case  $\mu = p\mu_0 + q\mu_{p_2}$ . Note that  $\mu_0 = \delta_{\{\emptyset\}}$  and  $\mu_{p_2} \perp \mu_0$  for  $p_2 > 0$ . In this case we have

$$\begin{aligned} \hat{\mu}(A) &= p\hat{\mu}_0(A) + q\hat{\mu}_{p_2}(A) \\ &= p + q(1 - 2p_2)^{|A|}. \end{aligned}$$

It can be shown from here that  $\mu$  is not infinitely divisible unless  $p_2 = 0$  or  $p_2 = 1/2$ ; an easy way to do this is to assume the Levy-Khintchine representation (2.19) noting that  $H = \{\emptyset\}$  for  $0 < p_2 < 1/2$  and arrive at a contradiction of the form  $F(\{\{x, y\}, \{x, y\}\}) < 0$ .

EXAMPLE 2.9. One dimensional Ising ferromagnet.

The one dimensional Ising ferromagnet with zero external field is given by a parameter  $J \geq 0$  with

$$(2.10) \quad \Phi(X) = \begin{cases} -4J, & X = \{i\} \\ 4J, & X = \{i, j\}, \quad |i - j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

In this case  $\mathcal{D}(\Phi)$  is a singleton consisting of a stationary  $\pm 1$  valued (double-ended) Markov chain  $\mu_J$  on  $Z$  with  $\mu_J([\phi, 0]) = \mu_J([0, 0]) = 1/2$  and stationary transition law

$$(2.11) \quad P = \begin{matrix} & -1 & +1 \\ -1 & \left[ \begin{array}{cc} \frac{e^{\beta J}}{2 \cosh(\beta J)} & \frac{e^{-\beta J}}{2 \cosh(\beta J)} \\ \frac{e^{-\beta J}}{2 \cosh(\beta J)} & \frac{e^{\beta J}}{2 \cosh(\beta J)} \end{array} \right] \\ +1 & \end{matrix}$$

In general it is not the case that the sum of two independent Markov chains is a Markov chain. The following lemma further illustrates the richness of the structure of the one-dimensional Ising ferromagnet.

LEMMA 2.12. Let  $\mu_{J_1}$  and  $\mu_{J_2}$  denote one-dimensional Ising ferromagnets with coupling constants  $J_1 > 0$  and  $J_2 > 0$ , respectively. Then  $\mu_{J_1} * \mu_{J_2}$  is a one-dimensional Ising ferromagnet with coupling constant  $J'$  given by

$$(2.13) \quad \tanh(\beta J') = \tanh(\beta J_1) \cdot \tanh(\beta J_2).$$

PROOF. Let  $\{\eta(i) : i \in Z\}$  and  $\{\sigma(i) : i \in Z\}$  denote the respective Markov chains with states  $\mu_{J_1}$  and  $\mu_{J_2}$ , respectively. Also let  $\xi(i) = \eta(i)\sigma(i)$ ,  $i \in Z$ . Since  $\sigma$  and  $\eta$  are independent, we have upon taking  $\beta = 1$ ,

$$(2.14) \quad \begin{aligned} P(\xi(j+1) = 1 \mid \eta(j) = \epsilon_j, \sigma(j) = \theta_j) &= \frac{e^{J_1\epsilon_j}e^{J_2\theta_j} + e^{-J_1\epsilon_j}e^{-J_2\theta_j}}{(e^{J_1} + e^{-J_1})(e^{J_2} + e^{-J_2})} \\ &= \frac{e^{J_1+J_2} e^{-J_1-J_2}}{(e^{J_1} + e^{-J_1})(e^{J_2} + e^{-J_2})} \text{ if } \epsilon_j \cdot \theta_j = 1 \\ &= \frac{e^{J_1-J_2} e^{J_2-J_1}}{(e^{J_1} + e^{-J_1})(e^{J_2} + e^{-J_2})} \text{ if } \epsilon_j \cdot \theta_j = -1. \end{aligned}$$

In particular  $P(\xi(j + 1) = 1 \mid \eta(j), \sigma(j))$  is a function of  $\xi(j) = \eta(j) \cdot \sigma(j)$ . So, for any real-valued function  $f$

$$\begin{aligned} E[f(\xi(j + 1)) \mid \xi(1), \dots, \xi(j)] &= E\{E[f(\xi(j + 1)) \mid (\sigma(1), \eta(1)) \cdots (\sigma(j), \eta(j))] \mid \xi(1), \dots, \xi(j)\} \\ &= E\{E[f(\xi(j + 1)) \mid (\sigma(j), \eta(j))] \mid \xi(1), \dots, \xi(j)\} \\ &= E[f(\xi(j + 1)) \mid \xi(j)]. \end{aligned}$$

So the Markov property for  $\{\xi(j): i \in Z\}$  is established. To see that the transition probabilities (2.14) may be expressed in the Ising model form (2.11) for some  $J'$  let

$$\begin{aligned} \frac{e^{J'}}{2 \cosh(J')} &= \frac{e^{J_1+J_2} + e^{-J_1-J_2}}{(e^{J_1} + e^{-J_1})(e^{J_2} + e^{-J_2})} \\ \frac{e^{-J'}}{2 \cosh(J')} &= \frac{e^{J_1-J_2} + e^{J_2-J_1}}{(e^{J_1} + e^{-J_1})(e^{J_2} + e^{-J_2})}. \end{aligned}$$

Equation (2.13) is an algebraic equivalent. Note that equation (2.13) has a solution  $J' > 0$  since  $J_1$  and  $J_2$  are positive.

Infinite divisibility now follows in the case of the one-dimensional Ising ferromagnet from Lemma (2.12). Specifically since  $0 \leq \tanh(\beta J) \leq 1$  for  $J > 0$ , for each  $n \geq 1$  there is a  $J_n > 0$  such that

$$(2.15) \quad \tanh(\beta J) = [\tanh(\beta J_n)]^n.$$

From the lemma we get that

$$(2.16) \quad \mu_J = \mu_{J_n}^{\ast n}, \quad n \geq 1.$$

REMARK 2.17. A simple calculation of the two-point correlation function for the finite volume Ising model on the torus (i.e., periodic boundary conditions) makes it clear that this example is not infinitely divisible. However, according to the above we do get infinite divisibility in the thermodynamic limit.

EXAMPLE 2.18. Poisson (Group) Sums. Suppose that  $X_1, X_2, \dots$  are independent and identically distributed as the state  $\nu$ . Let  $N_\lambda$  be a Poisson random variable independent of  $X_1, X_2, \dots$  and consider

$$X = \bigtriangleup_{i=0}^{N_\lambda} X_i,$$

with  $X_0 = \phi$  with probability one. Let  $\mu$  denote the state of  $X$ . Then,

$$\begin{aligned} \hat{\mu}(A) &= E(\hat{\nu}(A))^{N_\lambda} \\ &= \exp\{\lambda[\hat{\nu}(A) - 1]\}. \end{aligned}$$

So,  $\mu$  is infinitely divisible regardless of the state  $\nu$ .

The characteristic function of an infinitely divisible state  $\mu$  on  $(\Omega, \Delta)$  admits the Levy-Khintchine representation

$$(2.19) \quad \hat{\mu}(A) = \hat{\lambda}_H(A) \exp \left\{ \int [(-1)^{|A \cap X|} - 1] dF(X) \right\}, \quad A \in \Omega_0,$$

where  $\lambda_H$  is the Haar measure of a compact subgroup of  $\Omega$  and  $F$  is a  $\sigma$ -finite measure which has finite mass outside every neighborhood of  $\phi$ ; see Parathasarathy [7]. Also,

$$(2.20) \quad \int [1 - (-1)^{|A \cap X|}] dF(X) < \infty \text{ for all } A \in \Omega_0.$$

Finally, since every element of the compact group  $\Omega$  is of order two the representation is unique (see [7] p. 112). The simple form of the Levy-Khintchine formula (2.19) is related to the fact that  $\Omega$  is totally disconnected for the product topology; in particular note that  $[x, x]$  and  $[\phi, x]$  are disjoint open sets which decompose the space  $\Omega$ .

The calculations of the Levy-Khintchine representation for the examples given above are rather straightforward so we omit the details and leave the verification of the following results to the reader.

*Ideal Gas.* In this example  $H = \Omega$  for  $p = 1/2$ . In the case  $p < 1/2$   $H = \{\phi\}$  and  $F_p$  is concentrated on the singleton configurations; i.e., configurations  $\sigma_i \in \Omega$  of the form

$$(2.21) \quad \sigma_i(j) = \begin{cases} 1, & j = i \\ 0, & j \neq i. \end{cases}$$

That is,  $\sigma_i = \{i\}$ . Moreover for any finite set  $D \subset Z^d$ ,

$$(2.22) \quad F_p(\{\sigma_i: i \in D\}) = |D| \log((1 - 2p)^{-1/2}), \quad 0 \leq p < \frac{1}{2}.$$

*One-Dimensional Ising Ferromagnet.* In this example  $H = \{\phi, Z\} = \{\sigma^+, \sigma^-\}$  where  $\sigma^+, \sigma^-$  are the pure phases given by

$$(2.23) \quad \begin{aligned} \sigma^+(i) &= 1 \text{ for all } i \in Z \\ \sigma^-(i) &= -1 \text{ for all } i \in Z. \end{aligned}$$

The Levy-Khintchine measure  $F_J$  is concentrated on the set of configurations  $\{\eta_i: i \in Z\}$  of the form

$$(2.24) \quad \eta_i(j) = \begin{cases} -1, & j \leq i \\ 1, & j > i. \end{cases}$$

That is,  $\eta_i = \{ \dots, -2, -1, 0, 1, \dots, i - 1, i \}$ . Moreover,  $F_J$  is given by



$$(2.25) \quad F_f(\gamma_i) = \log((\tanh(J))^{-1/2}).$$

Note also that

$$(2.26) \quad \lambda_H = \frac{1}{2} \delta_{\sigma^+} + \frac{1}{2} \delta_{\sigma^-}.$$

*Poisson (Group) Sum.* In this example  $H = \{\phi\}$  and  $F = \lambda\nu$ . Note that  $F$  is a totally finite measure in this model.

**3. General results.** Partial solutions to the problems stated in the introduction are given in this section. First we consider problem (1.14). Although the result shows that it is always the case that  $\nu_Y = \mu \circ T_Y^{-1}$ ,  $Y \in \Omega$ , is a Gibbs state, the result needs improvement in the direction of identifying the structure of the potentials in the various interesting models of statistical physics. However, the intent of this section is only to provide some general results in the direction of the basic problem (1.11).

The theorem which solves (1.14) for finite range potentials will be preceded by two lemmas which have well-known versions in terms of Gibbs specifications in the sense of Dobrushin, see Preston, [8]. It is not hard to deduce the versions given below from this.

LEMMA 3.1. *Let  $\Phi$  be a finite range potential. Then  $\mu \in \mathcal{D}(\Phi)$  if and only if for each  $x \in S$*

(i)  $\mu \circ T_x^{-1} << \mu$  on  $[\phi, x]$

(ii)  $\frac{d\mu \circ T_x^{-1}}{d\mu}(Y) = \exp\{\beta \sum_{\substack{W \subset Y \Delta x \\ x \in W}} \Phi(W)\}$  for  $Y \in [\phi, x]$ .

LEMMA 3.2. *Suppose that  $h_x: \Omega \rightarrow R$  is  $\mathcal{B}$  measurable and depends on only finitely many coordinates. Then there is a finite range potential  $\Phi$  such that*

$$h_x(X \Delta x) = \exp\{\beta \sum_{\substack{W \subset Y \Delta x \\ x \in W}} \Phi(W)\}$$

*if and only if  $h_x(X \Delta x)h_y(X \Delta x \Delta y) = h_y(X \Delta y)h_x(X \Delta x \Delta y)$  for all  $x, y \in S$  and  $X \in \Omega$  such that  $x \notin X$  and  $y \notin X$ .*

THEOREM 3.3. *Let  $\mu \in \mathcal{D}(\Phi)$  for a finite range potential  $\Phi$ . Then for  $Y \in \Omega$ ,  $\nu_Y = \mu \circ T_Y^{-1} \in \mathcal{D}(\Phi^Y)$  for a finite range potential  $\Phi^Y$ . Moreover,*

$$h_x^{\Phi^Y}(X \Delta x) = \frac{h_x^\Phi(X \Delta Y \Delta x)}{h_x^\Phi(X \Delta Y)}$$
 for  $X \in \Omega$ .

PROOF. Let  $f$  be an arbitrary continuous function on  $\Omega$ . Then for  $X \in [\phi, x]$  we have

$$\begin{aligned} \int_{[x, x]} f(X \Delta x) d\nu_Y(X) &= \int_{[x, x]} f(X \Delta x) d\mu \circ T_Y^{-1}(X) \\ &= \int_{[(Y \cap X) \Delta x, x]} f(X \Delta Y \Delta x) d\mu(X) \\ &= \begin{cases} \int_{[\phi, x]} f(X \Delta Y \Delta x) d\mu(X) & \text{if } x \in Y \\ \int_{[x, x]} f(X \Delta Y \Delta x) d\mu(X) & \text{if } x \notin Y. \end{cases} \end{aligned}$$

We treat the two cases  $x \in Y$  and  $x \notin Y$  separately. First if  $x \notin Y$  then

$$\begin{aligned} \int_{[x, x]} f(X \Delta x) d\nu_Y(X) &= \int_{[x, x]} f(X \Delta Y \Delta x) d\mu(X) \\ &= \int_{[\phi, x]} f(X \Delta Y) h_x^\phi(X \Delta x) d\mu(X) \\ &= \int_{[\phi, x]} f(X) h_x^\phi(X \Delta Y \Delta x) d\nu_Y(X) \end{aligned}$$

since  $T_Y$  maps  $[\phi, x]$  onto  $[\phi, x]$  for  $x \notin Y$ . Since  $f$  is arbitrary it follows that  $d\nu_Y \circ T_x^{-1} / d\nu_Y(X) = h_x^\phi(X \Delta Y \Delta x)$  on  $[\phi, x]$ . In the second case  $x \in Y$  simply note that

$$\frac{d\nu_Y \circ T_x^{-1}}{d\nu_Y}(X) = \frac{d\nu_{Y \Delta x}}{d\nu_{Y \Delta x} \circ T_x^{-1}}(X) = \frac{1}{h_x^\phi(X \Delta Y \Delta x \Delta x)} \text{ on } [\phi, x]$$

by case 1 above. Therefore,

$$\frac{d\nu_Y \circ T_x^{-1}}{d\nu_Y}(X) = \frac{1}{h_x^\phi(X \Delta Y)} \text{ on } [\phi, x] \text{ for } x \in Y.$$

Since for any potential  $\phi$  we have  $h_x^\phi(X) = 1$  if  $x \notin X$ , it follows that in either case we have

$$\frac{d\nu_Y \circ T_x^{-1}}{d\nu_Y}(X) = \frac{h_x^\phi(X \Delta Y \Delta x)}{h_x^\phi(X \Delta Y)} \text{ for } X \in [\phi, x].$$

Define  $h_x$  on  $\Omega$  by

$$h_x(X) = \frac{h_x^\phi(X \Delta Y)}{h_x^\phi(X \Delta Y \Delta x)}, \quad X \in \Omega.$$

Note that  $h_x(X) = 1$  if  $x \notin X$ . Also,  $h_x$  depends on only finitely many coordinates and  $h_x(X \Delta x) = d\nu_Y \circ T_x^{-1} / d\nu_Y(X)$ . Also, the consistency condition required by Lemma (3.2) is easily verified for  $h_x$  since it holds for  $h_x^\phi$ .

REMARK 3.4. Note that if  $\mu$  is an infinitely divisible probability measure then for  $Y \neq \phi$ ,  $\nu_Y = \mu * \delta_{(y)}$  is generally not infinitely divisible, although

$\mu_{1/2} = \mu_{1/2} * \delta_{\{y\}}$  certainly is infinitely divisible. However, this is a minor technicality which can be dealt with by amending the definition of infinite divisibility (as is done in Parthasarathy [7]).

For an example illustrating the change in the potential take the Bernoulli field  $\mu_p$  and  $Y = \{y\}$ . Since  $\nu_r$  is  $\mu_p$  with a ‘flip at  $y$ ’ we get

$$(3.5) \quad \Phi^{(y)}(X) = \begin{cases} -\Phi(y), & X = \{y\} \\ \Phi(x), & X = \{x\}, x \neq y \\ 0, & \text{otherwise.} \end{cases}$$

We now proceed to a solution of problem (1.13). The theorem which solves this problem is a consequence of (3.3) and the following lemma.

LEMMA 3.5. *Let  $\{\nu_r\}$  and  $\{\mu_r\}$  be two families of measures indexed by a measure space  $(R, \rho)$  such that the maps  $r \rightarrow \nu_r(A)$  and  $r \rightarrow \mu_r(A)$  are measurable for measurable sets  $A$ . If  $\nu_r << \mu_r$  for  $r \in R$  then  $\int_R \nu_r(\cdot) \rho(dr) << \int_R \mu_r(\cdot) \rho(dr)$ .*

PROOF. If  $\int_R \mu_r(A) \rho(dr) = 0$  then by non-negativity of the measures  $\mu_r(A) = 0$  for  $\rho - a.e. r \in R$ . So  $\nu_r(A) = 0$  for  $\rho - a.e. r \in R$  and the result follows.

THEOREM 3.6. *Let  $\mu \in \mathcal{D}(\Phi)$  with  $\Phi$  finite range. Then  $\mu * \mu \circ T_x^{-1} << \mu * \mu$  on  $[\phi, x]$  for all  $x \in S$ .*

PROOF. First note that  $(\mu * \mu) \circ T_x^{-1} = \mu * (\mu \circ T_x^{-1})$ . Therefore,

$$\begin{aligned} \mu * \mu \circ T_x^{-1} &= \int_{\Omega} \mu \circ T_x^{-1} \circ T_Y^{-1}(\cdot) d\mu(Y) \\ &= \int_{\Omega} \mu \circ T_Y^{-1} \circ T_x^{-1}(\cdot) d\mu(Y). \end{aligned}$$

Since  $\mu \in \mathcal{D}(\Phi)$  it follows that  $\mu \circ T_Y^{-1} \in \mathcal{D}(\Phi^Y)$ . In particular  $\mu \circ T_Y^{-1} \circ T_x^{-1} << \mu \circ T_Y^{-1}$  on  $[\phi, x]$ . The result now follows by an application of Lemma 3.5.

**4. A correlation inequality.** The inequality appearing in Proposition 4.1 below was first studied by Kelly and Sherman [4]. Since they were able to construct an example of a finite volume state for which the inequality does not hold, it follows from Proposition 4.1 that neither is this state infinitely divisible; perhaps the inequality will prove useful for ruling out infinite divisibility in other cases as well. However, in this connection it should also be noted that by (2.16) and (2.17) the breakdown of the inequality in finite volumes need not imply non-infinite divisibility in the thermodynamic limit.

PROPOSITION 4.1. *Let  $\mu$  be an infinitely divisible finite volume state on*

$\Omega$  with  $S = \{1, 2, \dots, N\}$  such that  $H = \{\phi\}$ ; i.e.,  $\hat{\mu}(A) > 0, A \in \Omega_0$ . Then

$$\prod_{A:|A \cap B| \text{ even}} \hat{\mu}(A) \geq \prod_{A:|A \cap B| \text{ odd}} \hat{\mu}(A) \quad \text{for each } B \subset S.$$

PROOF. Since  $H = \{\phi\}$  the Levy-Khintchine representation is given by

$$\hat{\mu}(A) = \exp\left\{ \sum_{B \subset S} [(-1)^{|A \cap B|} - 1] F(B) \right\}$$

so that

$$\log \hat{\mu}(A) = \sum_{B \subset S} (-1)^{|A \cap B|} F(B) - \sum_{B \subset S} F(B).$$

Let  $F(S) = \sum_{B \subset S} F(B) = \log f(S)$ . Then

$$\log[f(S)\hat{\mu}(A)] = \sum_{B \subset S} (-1)^{|B \cap A|} F(B).$$

Apply Fourier inversion to get for  $A \neq \emptyset$ ,

$$\begin{aligned} 0 \leq F(B) &= 2^{-N} \sum_{A \subset S} (-1)^{|A \cap B|} \log[f(S)\hat{\mu}(A)] \\ &= 2^{-N} \sum_{A \subset S} (-1)^{|A \cap B|} \log \hat{\mu}(A) + 2^{-N} \log f(S) \sum_{A \subset S} (-1)^{|A \cap B|} \\ &= 2^{-N} \left[ \sum_{A:|A \cap B| \text{ even}} \log \hat{\mu}(A) - \sum_{A:|A \cap B| \text{ odd}} \log \hat{\mu}(A) \right]. \end{aligned}$$

So,

$$\log\left( \prod_{A:|A \cap B| \text{ even}} \hat{\mu}(A) \right) \geq \log\left( \prod_{A:|A \cap B| \text{ odd}} \hat{\mu}(A) \right),$$

and the result follows.

In closing this section we shall make one final observation with regard to the role of the Levy-Khintchine representation in the Ising model. For this consider the finite volume Ising ferromagnet with coupling constants  $\{J_A: A \subset S\}$ ,  $S = \{1, 2, \dots, N\}$ . Namely, the state  $\mu$  is given by

$$(4.2) \quad \mu(\sigma) = Z^{-1} \exp\left\{ \beta \sum_{A \subset S} J_A \sigma_A \right\}$$

where we use the conventional  $\pm$  spin representation of configurations  $\sigma \in \{-1, 1\}^S$  and

$$(4.3) \quad \sigma_A = \prod_{i \in A} \sigma(i).$$

Now observe that

$$\begin{aligned}
 (4.4) \quad \mu(\sigma) &= Z^{-1} e^{\beta \sum_{A \subseteq S} J_A} \exp\left\{ \sum_{A \subseteq S} [\sigma_A - 1] J_A \right\} \\
 &= \exp\{-\beta\psi(S)\} \exp\left\{ \sum_{A \subseteq S} [\sigma_A - 1] F(A) \right\}
 \end{aligned}$$

where

$$(4.5) \quad \psi(S) = - \sum_A J_A + \frac{1}{\beta} \log Z$$

is the sum of the ground state energy (for ferromagnetism) plus the free energy, and  $F$  has point mass at  $\{A\}$  given by

$$(4.6) \quad F(A) = \beta J_A \geq 0.$$

The point here is that the second factor appearing in (4.4) is the Levy-Khintchine representation of the characteristic function of an infinitely divisible random field  $Q$ . So,

$$(4.7) \quad \mu(\sigma) = \exp\{-\beta\psi(S)\} \cdot \hat{Q}(X)$$

where

$$(4.8) \quad X = \{i \in S: \sigma_i = -1\}.$$

The random field  $Q$  can be identified by the so-called ‘finite sum expansion’ method after first doing a Fourier inversion in (4.7) to get  $Q$  in terms of the characteristic function (i.e., correlations) of  $\mu$ . Doing the expansion we get the following representation of  $Q$  which has already appeared in other contexts; see Kelly and Sherman [4]. Let  $B_1, B_2, \dots, B_h$ ,  $h = 2^N$ , be an enumeration of the subsets of  $S$  and define independent Bernoulli (i.e., two-valued) random variables  $X_1, \dots, X_h$  by

$$\begin{aligned}
 \text{Prob}(X_i = B_i) &= \frac{\tanh(\beta J_{B_i})}{1 + \tanh(\beta J_{B_i})} \leq \frac{1}{2} \\
 \text{Prob}(X_i = \emptyset) &= \frac{1}{1 + \tanh(\beta J_{B_i})}, \quad i = 1, 2, \dots, h.
 \end{aligned}$$

Then

$$(4.9) \quad Q(X) = \text{Prob}(S_h = X)$$

where

$$(4.10) \quad S_h = X_1 \Delta X_2 \Delta \dots \Delta X_h.$$

For an exploitation of the probabilistic as well as the graph theoretic structure of  $Q$  as represented by (4.9) see Aizenman [1]. It would be very interesting to determine the extent to which (4.7) has an infinite volume analog; perhaps a local analog in terms of Fourier transforms of the Radon-Nikodym derivatives which describe the state locally.

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ADDED IN PROOF. The author has recently shown that regardless of the temperature, there is always at least one infinitely divisible Ising ferromagnet on the Bethe lattice. However methods different from those used in Lemma 2.12 are required in this setting.

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