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DIRECT LIMITS OF FINITE SPACES OF ORDERINGS

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Dedicated to the memory of Gus Efroymson

Direct limits of directed systems of finite spaces of orderings exist in the category of spaces of orderings. Let $\{(X_i, G_i): i \in I\}$ be a directed system of finite spaces of orderings and denote the direct limit by (X, G). G is the inverse limit of the groups $\{G_i: i \in I\}$. The natural map $\phi_i: G \to G_i$ induces a map on character groups $\phi_i^*: \chi(G_i) \to \chi(G)$ and X is the closure of $X_c: = \bigcup\{\phi_i^*(X_i): i \in I\}$ in $\chi(G)$.

Denote by W the Witt ring of (X, G) and by W_i the Witt ring of (X_i, G_i). W is the subring of the ring-theoretic inverse limit of the W_i consisting of compatible families with bounded anisotropic dimension.

The inverse limit topology on G (resp. W) is the weakest such that the function $\sigma: G \to \{\pm 1\}$ (resp. $\sigma: W \to \mathbf{Z}$) is continuous for all $\sigma \in X_c$. Further if $\sigma \in X$, then the following are equivalent: (i) $\sigma \in X_c$, (ii) $\sigma: W \to \mathbf{Z}$ is continuous and (iii) $\sigma: G \to \{\pm 1\}$ is continuous. Thus X_c depends only on the topology on W (resp. G) and not on the particular presentation of (X, G) as a direct limit of finite spaces. Also, the set $\mathscr{V}: = \{Y: Y$ is a finite subspace of X and $Y \subseteq X_c\}$ forms a directed system and (X, G) is the direct limit of this system. This is the "standard presentation" of (X, G) as a direct limit of finite spaces.

The stability index of (X, G) is the supremum of the stability indices of the spaces in the set $\mathscr{V}$. This is finite iff the following two conditions hold: (i) X_c, with the topology induced from X, is discrete and (ii) X is the Stone-Čech compactification of X_c. If the stability index of (X, G) is finite then a function $\psi: X_c \to \mathbf{Z}$ with finite image which satisfies the congruence $\sum_{\sigma \in V} \psi(\sigma) \equiv 0 \bmod |V|$ for all finite fans $V \in \mathscr{V}$ represents an element of W.

In the presence of certain finiteness conditions (e.g., finite stability will do) (X, G) is built up canonically from singleton spaces exactly as in the finite case, using sums and group extensions, except that now infinite sums are required. However, not all direct limits of finite spaces of orderings are built up in this way.

Since each finite space of orderings is realized as the space of orderings of a Pythagorean field, one can ask if the same is true for arbitrary spaces of orderings. By a construction involving ultrafilters one can show that if (X, G) is a direct limit of finite spaces, there exists a Pythagorean field K and morphisms $\alpha: (X_K, G_K) \to (X, G)$ and $\beta: (X, G) \to (X_K, G_K)$ such that $\alpha \circ \beta = 1$. Here, (X_K, G_K) denotes the space of orderings of K. In particular, this implies that (X, G) is both a subspace and a quotient space of (X_K, G_K). If (X, G) is any space of orderings, then the finite subspaces of (X, G) form a directed system and (X, G) is a quotient space of the direct limit. Combining this with the above, any space of orderings is a quotient of (X_K, G_K) for a suitably chosen Pythagorean field K.

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