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ON COMPLETE INTERSECTION REAL CURVES

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Dedicated to the memory of Gus Eforymson

Let $C \subset \mathbf{R}^n$ be an embedded smooth real algebraic curve. Let $\mathbf{R}[C]$ (resp. $\Gamma_C = \Gamma(C, \mathcal{O}_C)$) denote the ring of real polynomial functions (resp. of global regular functions) on C. If I_C (resp. J_C) is the ideal of C in $\mathbf{R}[X_1, \ldots, X_n]$ (resp. in $\Gamma_{\mathbf{R}^n} = \Gamma(\mathbf{R}^n, \mathcal{O}_{\mathbf{R}^n})$), then we have $\Gamma_C \simeq \Gamma_{\mathbf{R}^n}/J_c \simeq N^{-1}\mathbf{R}[C]$ where $N = \{s \in \mathbf{R}[C] \mid s(x) \neq 0 \ \forall x \in C\}$ and $\mathbf{R}[C] \simeq \mathbf{R}[X_1, \ldots, X_n]/I_c$.

Let $\mathscr{C}$ be the abstract curve of which C is a realization; to each embedding φ: $\mathscr{C} \to \mathbf{R}^N$ corresponds a f.g. $\mathbf{R}$-algebra $P = \mathbf{R}[\varphi(\mathscr{C})] = \mathbf{R}[X_1, \ldots, X_N]/I_{\varphi(\mathscr{C})}$ which is called an affine respresentation of $\mathscr{C}$ and has the property that $\Gamma_{\mathscr{C}} \simeq N_P^{-1}P$ where $N_P = \{s \in P \mid s(x) \neq 0$ for each $x \in \varphi(\mathscr{C})\}$. The various affine representations of $\mathscr{C}$ can be compared by introducing on the set of isomorphism classes of affine reprentations of $\mathscr{C}$ the following ordering relation $\prec$: "given two affine representations P, Q of $\mathscr{C}$ then $P \prec Q$ if there exists a homomorphism if $\mathbf{R}$-algebras $P \hookrightarrow Q$ which extends to an isomorphism $\Gamma_{\mathscr{C}} \simeq N_P^{-1}P \simeq N_Q^{-1}Q \simeq \Gamma_{\mathscr{C}}$. We assume that all the affine representations that we consider are regular as schemes (this is always possible). We say that P is "the canonical" affine representation if $P \prec Q$, for every other Q. Since to each affine representation corresponds a complexification (uniquely, up to isomorphism), we mix up the two notions. So, given two complexifications $\tilde{C}'$, $\tilde{C}''$, we say that $\tilde{C}' \prec \tilde{C}''$ if there is a real open immersion $\tilde{C}'' \hookrightarrow \tilde{C}'$.

In this setting the main result we know is the following (cf. [3]).

THEOREM 1. *A smooth affine real curve $\mathscr{C}$ has a canonical complexification if and only if it is either rational or embeddable as a non-compact subspace of* $\mathbf{R}^3$.

The above machinery can be used in order to find some results on the problem of complete intersection. We recall the following definition (cf. [2]): "an integral domain R which is a f.g. algebra over a field k, is called an abstract complete intersection (ACI) if there exists a polynomial ring

$S = k[X_1, \ldots, X_N]$ such that R is an epimorphic image of S with kernel a complete intersection ideal".

For a real smooth affine real curve $\mathscr{C}$, we prove the following theorem.

THEOREM 2. *If $\mathscr{C}$ has an embedding $\varphi: \mathscr{C} \to \mathbf{R}^n$ such that $J_{\varphi(\mathscr{C})}$ is a complete intersection ideal in $\Gamma_{\mathbf{R}^n}$, then there exists an affine representation which is an ACI.*

If $\mathscr{C}$ is an affine real curve, we say that $\mathscr{C}$ is an ACI if there exists an embedding $\phi: \mathscr{C} \to \mathbf{R}^n$ such that $\Gamma_{\mathscr{C}} \simeq N^{-1}P$ with P an ACI in the sense of [M. K.].

THEOREM 3. *An affine real curve $\mathscr{C}$ is an* ACI *if and only if the module of* 1*-differentials $\Omega^1_{\Gamma_{\mathscr{C}}/\mathbf{R}}$ has a free resolution of length $\leqq 1$. If $\mathscr{C}$ is smooth and* ACI *then any embedding in $\mathbf{R}^n$, $n \geqq 4$ is a complete intersection.*

Finally we have the following examples related to a question posed in [1].

EXAMPLE 1. A smooth real affine curve (of genus 2) whose canonical complexification is not a complete intersection.

EXAMPLE 2. A smooth compact real affine curve (of genus 3) which has a complexification which is a complete intersection and another which is not.

REFERENCES

1. J. Bochnak and W. Kucharz, (manuscript).

2. N. M. Kumar, *Complete intersections*, J. Math. Kyoto Univ. **17** (1977), 533–538.

3. M. G. Marinari, F. Odetti and M. Raimondo, *Affine curves over an algebraically non-closed field*, Pacific J. of Math. **107** (1983), 179–188.

4. A Tognoli, *Algebraic geometry and nash functions*, *Institutiones Math* III. Academic Press, London-New York, 1978.

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